

1

Solvability of orbit-finite systems of linear equations

ARKA
GHOSH

PIOTR
HOFMAN

STAWOMIR
LASOTA

University of Warsaw

Fix atoms $A = (\{1, 2, \dots\}, =)$

orbit-finite sets $\stackrel{\text{def.}}{=} \begin{array}{l} \text{sets finite up to} \\ \text{automorphisms of } A^* \\ \parallel \\ \text{finite unions of orbits} \end{array}$

Examples:

- A^2 is the union of 2 orbits:
 $\{aa : a \in A\}$
 $A^{(2)} = \{ab : a, b \in A, a \neq b\}$
- A^3 is the union of 5 orbits
- 2-sets $\binom{A}{2}$ is 1 orbit

* up to automorphisms of A that fix some finite subset $S \subseteq A$

$$\mathbb{A} = (\{1, 2, \dots\}, =)$$

hereditarily
orbit-finite
sets

=

definable
sets

- Examples:
- $\{ \{ a, \{ a, b \} \} : a, b \in \mathbb{A}, a \neq b \}$
 - $\{ a : a \in \mathbb{A}, a \neq 1 \}$
 - $\{ a : a \in \mathbb{A}, a = 1 \vee a = 2 \}$

sets with atoms

[Bojańczyk, Klin, L. 2011]

nominal sets

[Gabbay, Pitts 1999]

[Pistore 1999]

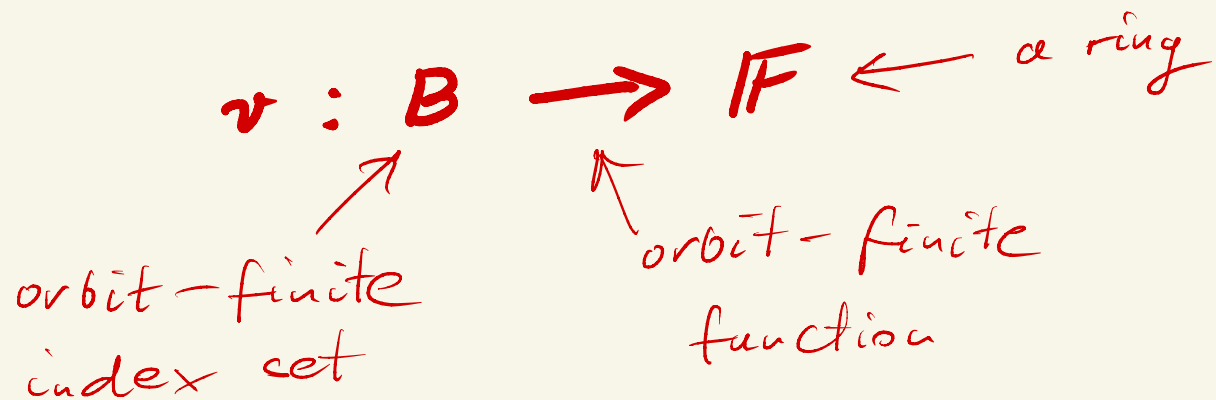
Fraenkel-Mostowski sets

[Fraenkel 1922]

Orbit - finite vectors

$$A = (\{1, 2, \dots\}, =)$$

4



Examples:

$$B = A^{(2)}$$

$$\mathbb{F} = \mathbb{Q}$$

orbit - finite

$$v : (1, b) \mapsto 0.5$$

$$(a, b) \mapsto 0 \quad a \neq 1$$

finite

$$v : (1, 2) \mapsto 0.5$$

$$(a, b) \mapsto 0 \quad (a, b) \neq (1, 2)$$

Orbit - finite systems

A and t fixed

column index is orbit-finite

row index
is
orbit-finite

$$\left[\begin{array}{c} \text{row index} \\ \text{is} \\ \text{orbit-finite} \end{array} \right] \left[\begin{array}{c} \text{column index is orbit-finite} \\ A \\ \end{array} \right] \times \left[\begin{array}{c} \text{set of variables} \\ x \\ \end{array} \right] = \left[\begin{array}{c} t \\ \end{array} \right]$$

matrix and rhs vector
are orbit-finite

(orbit-)finite solvability problem:

given a system decide, whether it has an
(orbit-)finite solution

(unrestricted solvability)

Example :

$$\mathbb{A} = (\{1, 2, \dots\}, =)$$

• variables $ab \in \mathbb{A}^{(2)}$

column index = $\mathbb{A}^{(2)}$

atom dimension 2

• equations

$$\{ab + bc + ca = 0 : a, b, c \in \mathbb{A}, a \neq b \neq c \neq a\}$$

row index = $\binom{\mathbb{A}}{3}$

atom dimension 3

$$\left. \begin{array}{l} 12 + 23 + 31 \\ 21 + 13 + 32 \\ 23 + 34 + 42 \\ \vdots \end{array} \right\} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ \end{array}$$

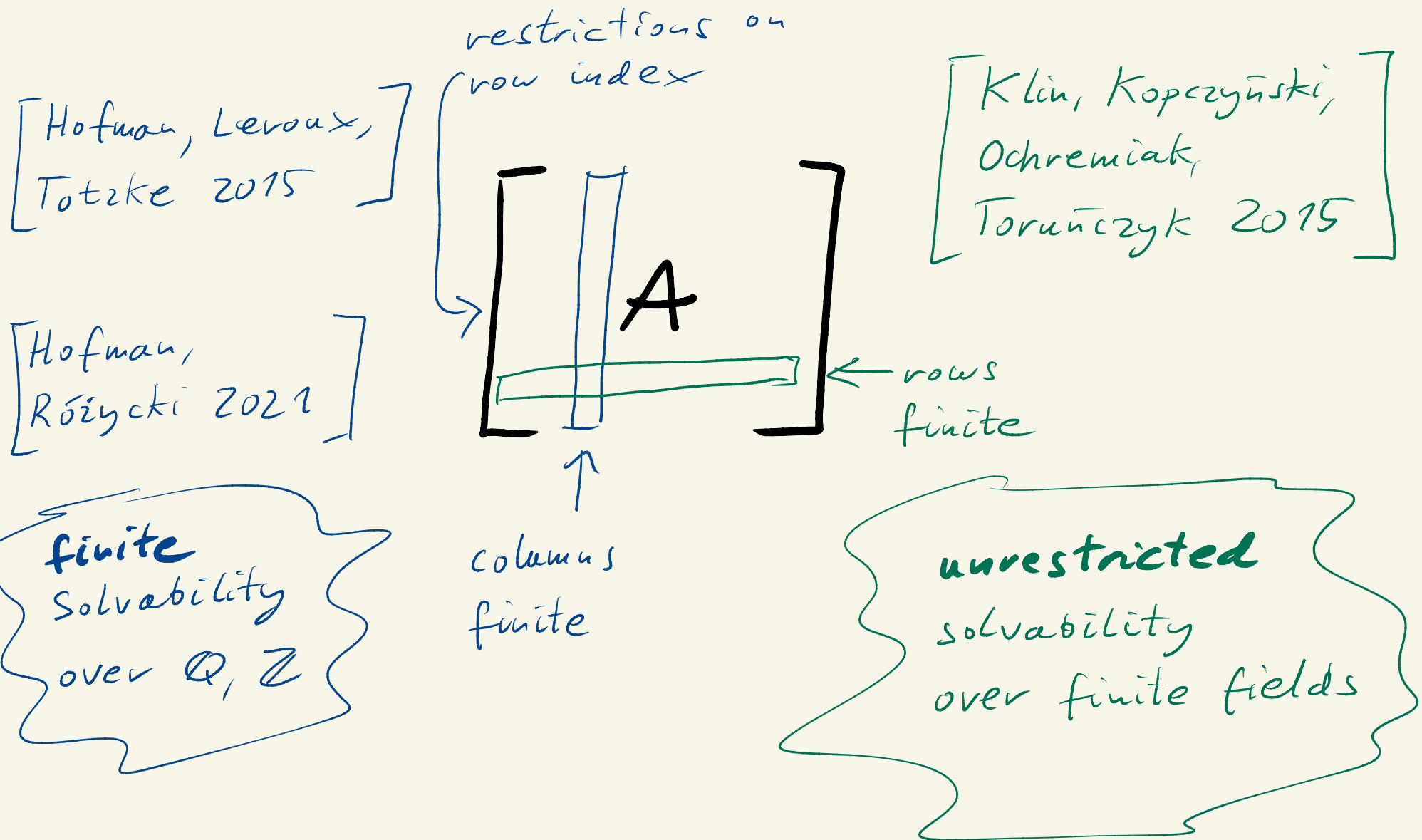
rows/equations are finite
columns are not

Motivation

- Data-enriched models:
 - Petri nets with data
 - weighted register automata

[Bojańczyk, Klin, Moerman 2021]
- Linear algebra in orbit-finite dimension

Previous work



Our results

orbit-finite solvability



finite solvability



classical
solvability of
finite systems

blow up
exponential
in
atom dimension

over any fixed
effective

commutative ring R ,

e.g. :

\mathbb{Q}, \mathbb{Z} ,

finite commutative
rings,

algebraic numbers

Complexity :

- EXP TIME
- PTIME in fixed atom dimension

Key technical result

Let B be a fixed orbit-finite set.

The set of all orbit-finite vectors $v : B \rightarrow \mathbb{F}$ has an orbit-finite base.

Example : $B = A^{(2)}$ $\mathbb{F} = \mathbb{Q}$

The subspace of all vectors

$$v : A^{(2)} \rightarrow \mathbb{Q} \text{ s.t. } v(a, b) = -v(b, a) \\ \forall (a, b) \in A^{(2)}$$

has no orbit-finite base.

Further work

- unrestricted solvability
- computation / representation of solution sets
- richer atoms, e.g. $A = (\mathbb{Q}, \leq)$
- inequalities:

$$\begin{bmatrix} A \end{bmatrix} \times \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} t \end{bmatrix}$$

still decidable over \mathbb{Q}

undecidable over \mathbb{Z}