Solvability of orbit-finite systems of linear equations

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Fix <u>atoms</u> $\mathbb{A} = (\{1,2,\dots\}, =)$

orbit-finite sets $\stackrel{\text{def.}}{=}$ **sets finite up to automorphisms of $\mathbb{A}$ ***

$\|$

finite unions of orbits

<u>Examples:</u>

- $\mathbb{A}^2$ is the union of 2 orbits:

$$\{ aa : a \in \mathbb{A} \}$$

$$\mathbb{A}^{(2)} = \{ ab : a,b \in \mathbb{A}, \ a \neq b \}$$

- $\mathbb{A}^3$ is the union of 5 orbits
- 2-sets $\binom{\mathbb{A}}{2}$ is 1 orbit

* up to automorphisms of $\mathbb{A}$ that fix some finite subset $S \subseteq \mathbb{A}$

$\mathbb{A} = (\{1,2,\dots\}, =)$

hereditarily

orbit-finite sets **=** **definable sets**

Examples:

- $\{\{a,\{a,b\}\} : a,b \in \mathbb{A},\ a \neq b\}$
- $\{a : a \in \mathbb{A},\ a \neq 1\}$
- $\{a : a \in \mathbb{A},\ a = 1 \vee a = 2\}$

sets with atoms [Bojańczyk, Klin, L. 2011]

nominal sets [Gabbay, Pitts 1999]

[Pistore 1999]

Fraenkel-Mostowski sets [Fraenkel 1922]

Orbit-finite vectors

$\mathbb{A} = (\{1, 2, \ldots\}, =)$

$v : B \longrightarrow \mathbb{F}$ ← a ring

orbit-finite index set

orbit-finite function

Examples:

$B = \mathbb{A}^{(2)}$ $\quad \mathbb{F} = \mathbb{Q}$

orbit-finite $\quad v : \begin{array}{ll} (1, b) \longmapsto 0.5 & \\ (a, b) \longmapsto 0 & a \neq 1 \end{array}$

finite $\quad v : \begin{array}{ll} (1, 2) \longmapsto 0.5 & \\ (a, b) \longmapsto 0 & (a, b) \neq (1, 2) \end{array}$

Orbit-finite systems

$\mathbb{A}$ and $\mathbb{F}$ fixed

$$\text{row index is orbit-finite}\left\{ \overbrace{\begin{bmatrix} \quad A \quad \end{bmatrix}}^{\text{column index is orbit-finite}} \times \begin{bmatrix} x \end{bmatrix} \right\}\text{set of variables} = \begin{bmatrix} t \end{bmatrix}$$

matrix and rhs vector are orbit-finite

(orbit-)finite solvability problem:

given a system decide, whether it has an (orbit-)finite solution

(unrestricted solvability)

Example:

$$\mathbb{A} = (\{1,2,\dots\}, =)$$

- variables $\quad ab \in \mathbb{A}^{(2)}$

column index $= \mathbb{A}^{(2)}$

$\boxed{\text{atom dimension } 2}$

- equations

$$\{ab + bc + ca = 0 : a,b,c \in \mathbb{A},\ a \neq b \neq c \neq a\}$$

row index $= \binom{\mathbb{A}}{3}$

$\boxed{\text{atom dimension } 3}$

$$\left\{\begin{array}{lllllllll}
12 & & + 23 & + 31 & & & & & = 0 \\
 & 21 & & & + 13 & + 32 & & & = 0 \\
 & & 23 & & & & + 34 & + 42 & = 0 \\
\vdots & & \vdots & & & & & &
\end{array}\right.$$

rows/equations are finite

columns are not

Motivation

- Data-enriched models:
  - Petri nets with data
  - weighted register automata
    [Bojańczyk, Klin, Moerman 2021]

- Linear algebra in orbit-finite dimension

Previous work

[Hofman, Leroux, Totzke 2015]

[Hofman, Różycki 2021]

restrictions on row index

$$\left[\;A\;\right]$$

columns finite

rows finite

[Klin, Kopczyński, Ochremiak, Toruńczyk 2015]

finite solvability over $\mathbb{Q}$, $\mathbb{Z}$

unrestricted solvability over finite fields

Our results

orbit-finite solvability

⇓

finite solvability

⇓

classical solvability of finite systems

blow up exponential in atom dimension

over any fixed <u>effective</u> commutative ring $\mathbb{F}$, e.g.:

$\mathbb{Q}$, $\mathbb{Z}$,
finite commutative rings,
algebraic numbers

Complexity:

- EXPTIME
- PTIME in fixed atom dimension

Key technical result

Let B be a fixed orbit-finite set.
The set of all orbit-finite vectors $v : B \to \mathbb{F}$
has an orbit-finite base.

Example :

$B = \mathbb{A}^{(2)}$ $\mathbb{F} = \mathbb{Q}$

The subspace of all vectors

$$v : \mathbb{A}^{(2)} \to \mathbb{Q} \quad \text{s.t.} \quad v(a,b) = -v(b,a)$$
$$\forall (a,b) \in \mathbb{A}^{(2)}$$

has no orbit-finite base.

Further work

- unrestricted solvability
- computation / representation of solution sets
- richer atoms, e.g. $\mathbb{A} = (\mathbb{Q}, \leq)$
- inequalities :

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \times \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} t \\ \\ \\ \end{bmatrix}$$

still decidable over $\mathbb{Q}$

undecidable over $\mathbb{Z}$