Reachability analysis of first-order definable pushdown systems

(= pushdown systems in sets with atoms)

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joint work with Lorenzo Clemente

builds on previous joint work with:
Mikołaj Bojańczyk, Bartek Klin, Joanna Ochremiak, Szymon Toruńczyk

Computability in Europe, Bucharest, 2015.07.02

Outline

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- Re-interpreting models of computation in FO definable sets

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- FO definable PDA
- Well-behaved case: oligomorphic and homogeneous atoms
- Reachability in FO definable PDA over oligomorphic atoms
- Ill-behaved case: time atoms

Atoms

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$x_1 = x_2 \neq x_3 \vee x_1 \neq x_2 = x_3$

$x_1 < x_2 \leq x_3$

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Example:

$\{ (x_1, x_2, x_3) : x_1 < x_2 \leq x_3 \} \cup \{ (x_1, x_2) : x_1 \neq x_2 \}$

different dimensions

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Option: quantifier-free definable sets.

Simple idea

Relax finiteness to... FO definability

Instantiate widely accepted symbolic approach: instead of enumerating sets, represent them and process symbolically.

FO definable NFA

[Bojańczyk, Klin, L. 2011, 2014]

- alphabet A
- states Q
- transitions $\delta \subseteq Q \times A \times Q$
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DFA:

- $\delta : Q \times A \rightarrow Q$

input alphabet: $A = \mathbb{A}$

language: "exactly two different atoms appear"

states:

transitions:

initial state:

accepting states:

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number of registers may vary from one location to another

states: $Q = \mathbb{A}^0 \cup \mathbb{A}^0 \cup \mathbb{A}^1 \cup \mathbb{A}^2$

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transitions: $\delta : Q \times A \to Q$

δ(init, a) =	(a)	a atom

if in state init atom a is read, goto state (a)

initial state: init

accepting states: $\mathbb{A}^2$

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$\delta(\text{init}, a) =$	(a)	a atom
$\delta((a), b) =$	(ab)	$a \neq b$

if in state (a), atom b ≠ a is read, goto state (ab)

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$\delta((ab), c) =$	reject	$c \neq a, b$

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Over equality atoms, FO definable NFA slightly generalize register automata (aka finite-memory automata) of [Francez, Kaminsky 1994]:

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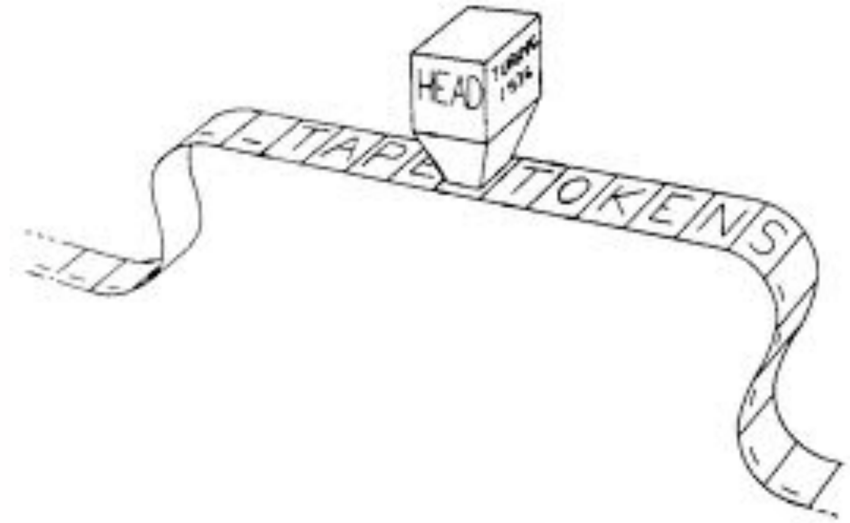
- number of registers may vary from one control state to another
- alphabet letters may contain more than one atom
- arbitrary FO constraints on register valuations and transitions
- instead of (finite set) $\times\ \mathbb{A}$, disjoint union $\mathbb{A} \cup \mathbb{A} \cup \ldots$

FO definable Turing machines

[Bojańczyk, Klin, L., Toruńczyk 2013]
[Klin, L., Ochremiak, Toruńczyk 2014]



- tape alphabet A
- states Q
- transitions $\delta \subseteq Q \times A \times Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$
- $I, F \subseteq Q$

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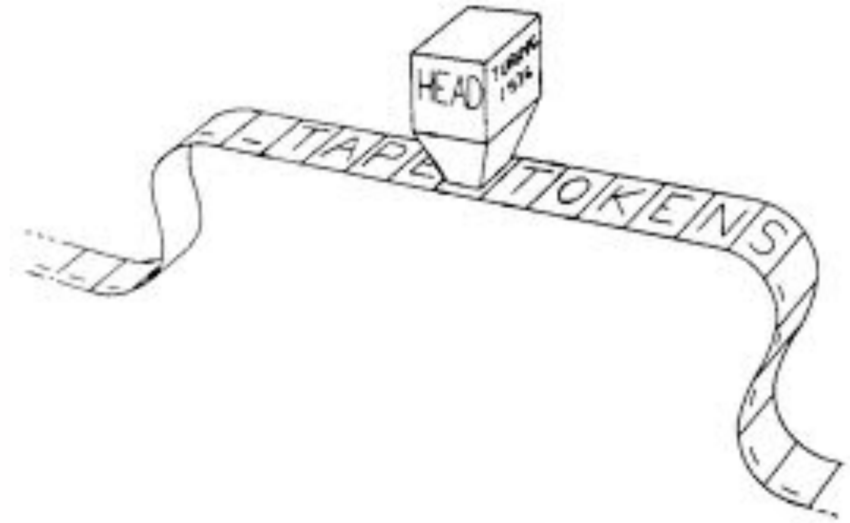

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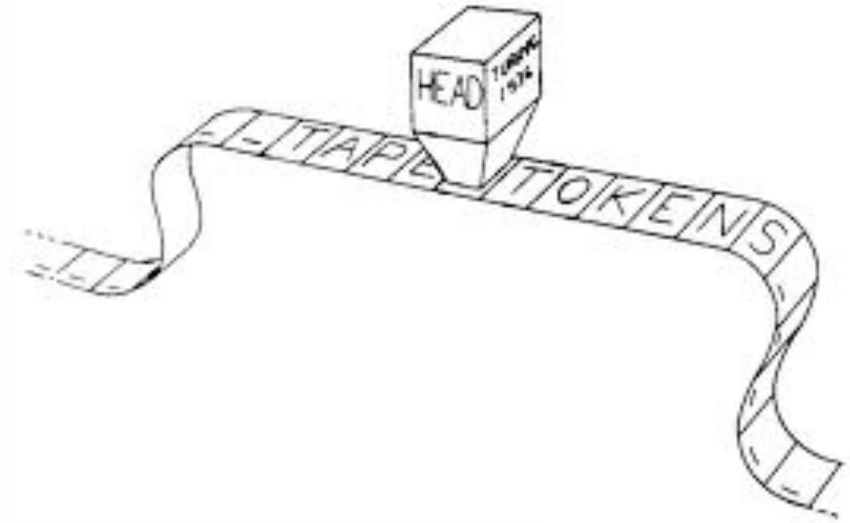

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Acceptance defined as for classical Turing machines.

Finite presentation

FO definable NFA, Turing machines, PDA, etc. can be finitely presented.

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- Re-interpreting models of computation in FO definable sets
- **FO definable PDA**
- Well-behaved case: oligomorphic and homogeneous atoms
- Reachability in FO definable PDA over oligomorphic atoms
- Ill-behaved case: time atoms

FO-definable PDA

- alphabet A
- states Q
- stack alphabet S
- $\rho \subseteq Q \times S \times (A \cup \{\varepsilon\}) \times Q \times S^*$
- $I, F \subseteq Q$

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FO definable sets instead of finite ones

Acceptance defined as for classical PDA, e.g. configurations = $Q \times S^*$

input alphabet: $A = \mathbb{Q}$

language: "ordered palindromes"

states:

stack alphabet:

transitions:

initial state:

accepting state:

input alphabet: $A = \mathbb{Q}$

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states: $Q = \{$init, finish, acc$\}$

stack alphabet:

transitions:

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input alphabet: $A = \mathbb{Q}$

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states: $Q = \{init, finish, acc\}$

stack alphabet: $S = \mathbb{Q} \cup \{\bot\}$

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initial state: init

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init, ⊥, a	init, a⊥	a atom

if in state init, ⊥ is topmost on the stack and atom a is read, stay in state init and push a on the stack

initial state: init

accepting state: acc

input alphabet: $A = \mathbb{Q}$

language: "ordered palindromes"

states: $Q = \{init, finish, acc\}$

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init, ⊥, a	init, a⊥	a atom
init, b, c	init, cb	b < c

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Pushdown register automata?

Over equality atoms, FO definable PDA slightly generalize pushdown register automata of [Murawski, Ramsay, Tzevelekos 2014], exactly like FO definable NFA slightly generalize register automata.

FO-definable context-free grammars

- symbols S
- terminal symbols $A \subseteq S$
- an initial symbol
- $\rho \subseteq (S-A) \times S^*$

} FO definable sets instead of finite ones

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Under what assumptions on atoms:

- are context-free grammars as expressive as PDA?
- is equivalence of two PDAs decidable?
- is reachability problem decidable for PDA?

Expressiveness

Theorem: [Bojańczyk, Klin, L. 2014]

The following models recognize the same languages:

- FO definable context-free grammars
- FO definable PDA
- FO definable prefix rewriting systems,

when $\mathbb{A}$ is oligomorphic

Equivalence-checking

Theorem: [Murawski, Ramsay, Tzevelekos 2015]
Bisimulation equivalence is undecidable for FO definable PDA over equality atoms.

Reachability

Assumption: From now on assume that FO satisfiability problem in $\mathbb{A}$ is decidable.

Given: an FO formula over the vocabulary of $\mathbb{A}$

Question: is the formula satisfiable in $\mathbb{A}$?

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This is necessary but far not enough!

Fact: The reachability problem for FO definable NFA over dense-time atoms $(\mathbb{Q}, <, +1)$ is undecidable.

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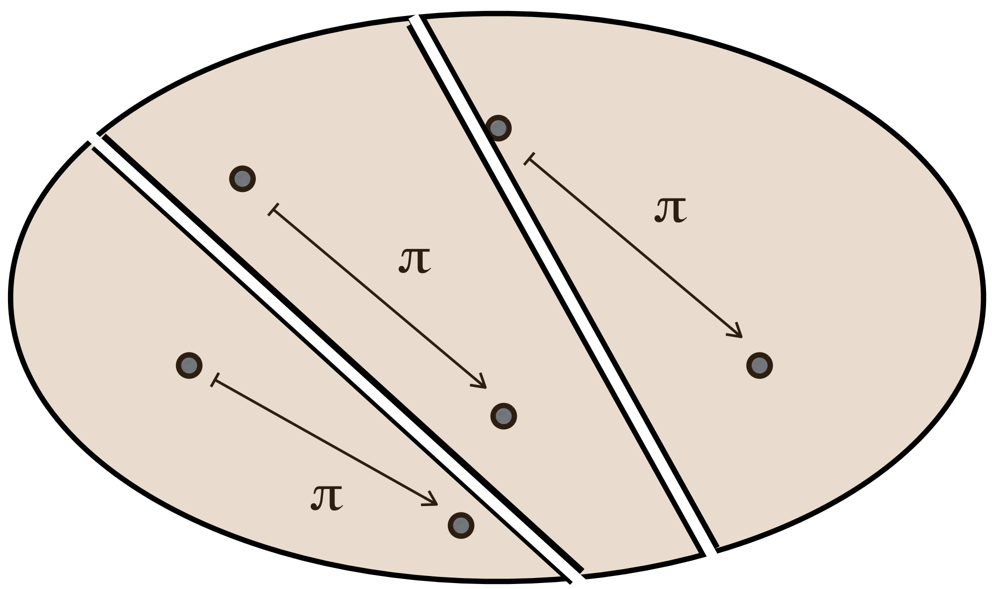
total order atoms $(\mathbb{Q}, <)$	monotonic bijections
dense-time atoms $(\mathbb{Q}, <, +1)$	monotonic bijections preserving integer differences
discrete-time atoms $(\mathbb{Z}, <, +1)$	translations
equivalence atoms $(\mathbb{A}, R, =)$	equivalence-preserving bijections
random graph $(\mathbb{V}, E, =)$	random graph automorphisms
…	…

Orbits

Atom automorphisms π act on $\mathbb{A}^n$ thus splitting it into orbits.



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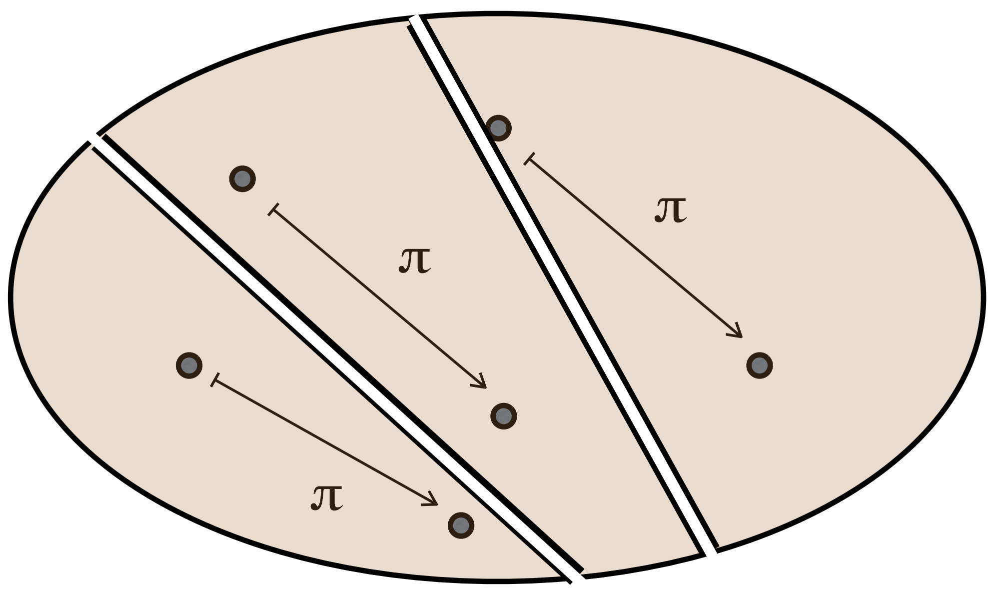

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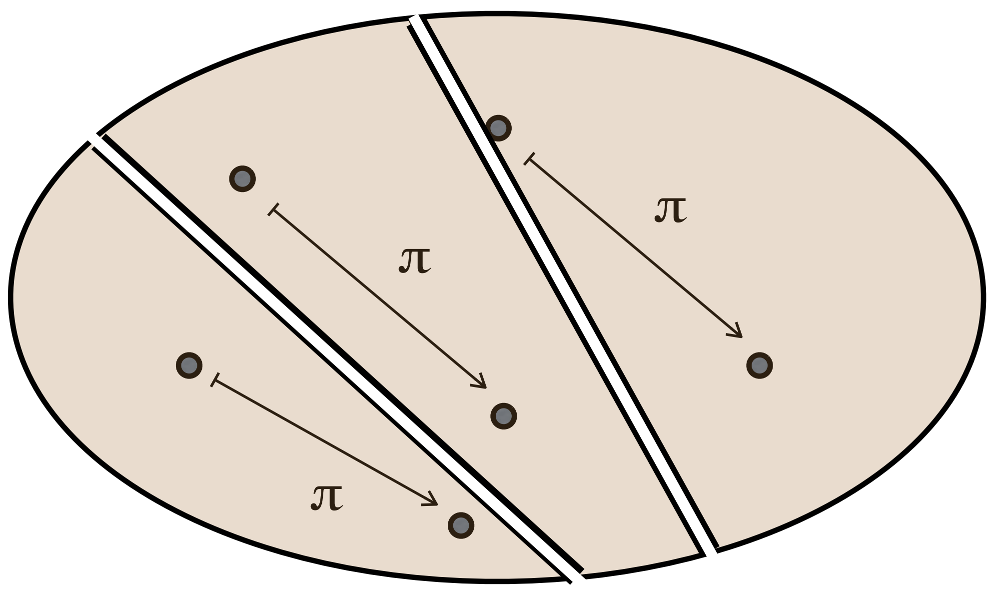

Examples:

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Non-examples:

$x_1 = x_2 \neq x_3 \vee x_1 \neq x_2 = x_3$

$x_1 < x_2 \leq x_3$

$x_1 < x_2 \leq x_3 \leq x_1+1+1$

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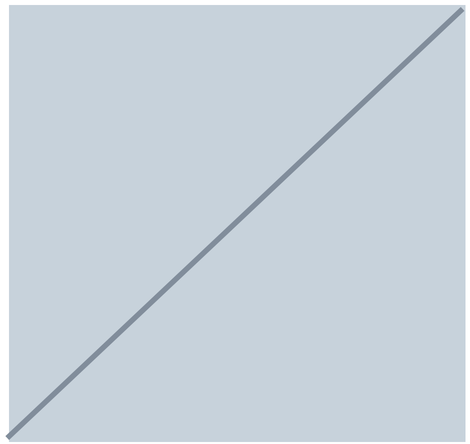
Example: $(\mathbb{Q}, <)$

$\mathbb{Q}^2$ has 3 orbits:

- $\{ (x, y) : x < y \}$
- $\{ (x, y) : x = y \}$
- $\{ (x, y) : x > y \}$



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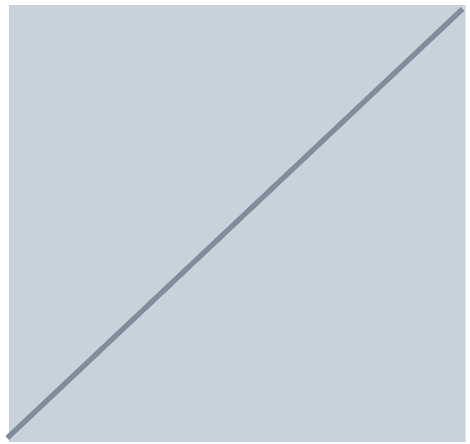
- $\{ (x, y) : x = y \}$
- $\{ (x, y) : x > y \}$



$\mathbb{Q}^3$ has 13 orbits

Homogeneous structures

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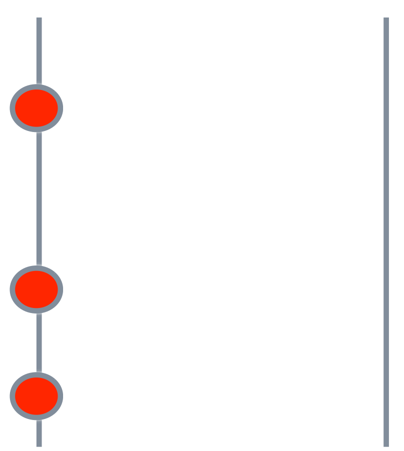

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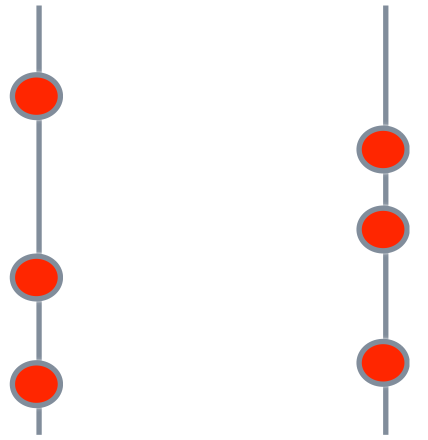

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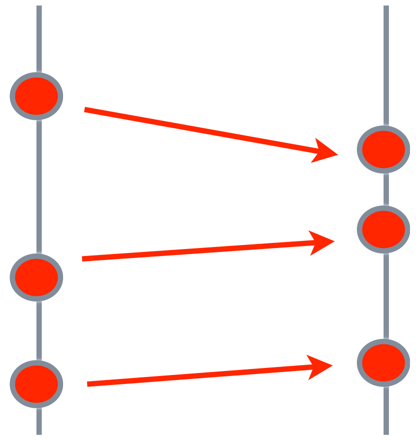

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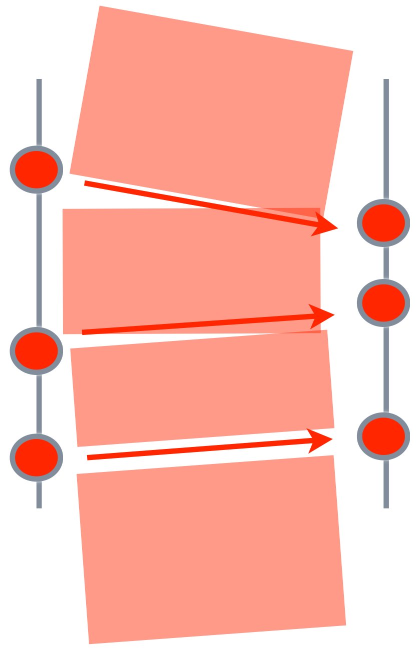

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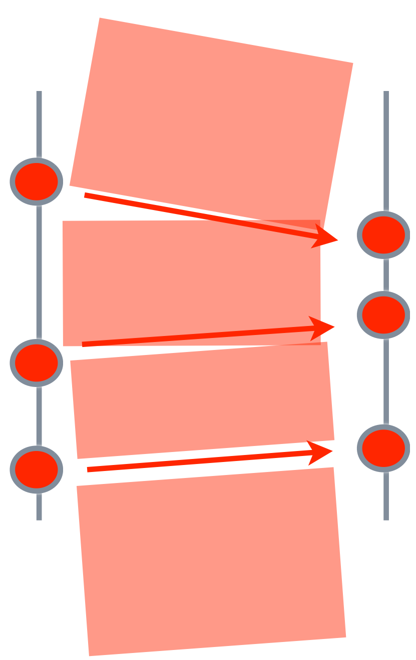
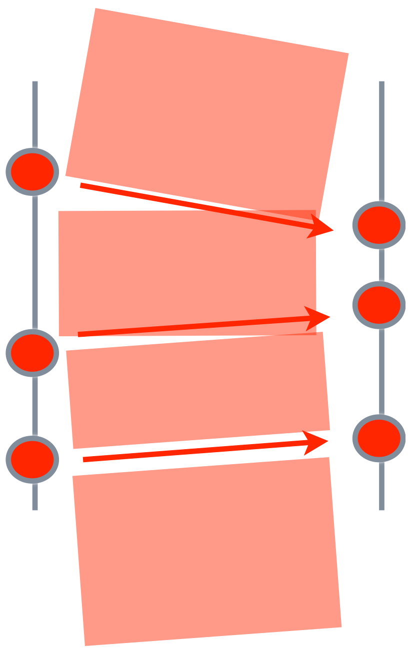


Theorem: [Freisse 1953]

A homogeneous structure is uniquely determined by its finite induced substructures (age).

Homogeneous structures



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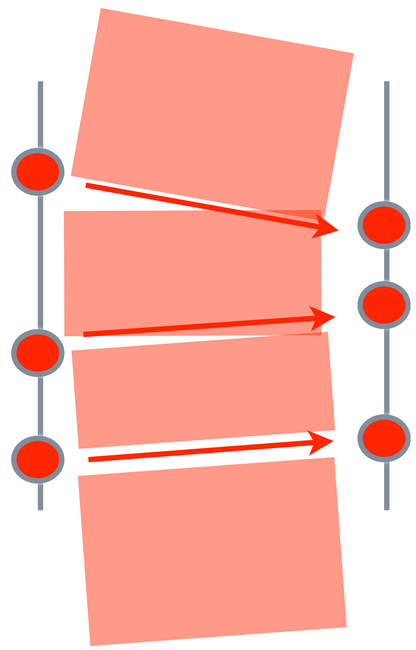

equality atoms $(\mathbb{A}, =)$
total order atoms $(\mathbb{Q}, <)$
dense-time atoms $(\mathbb{Q}, <, +1)$

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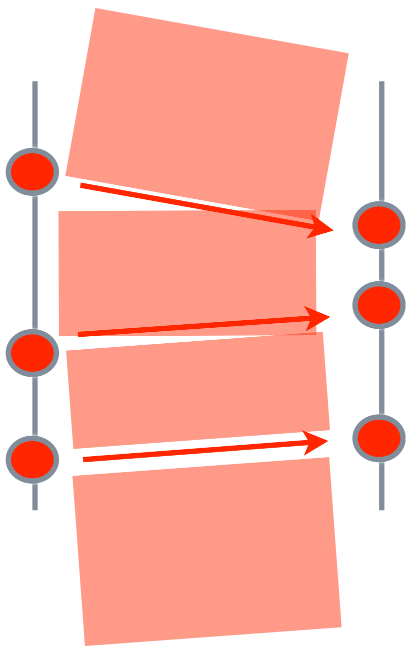

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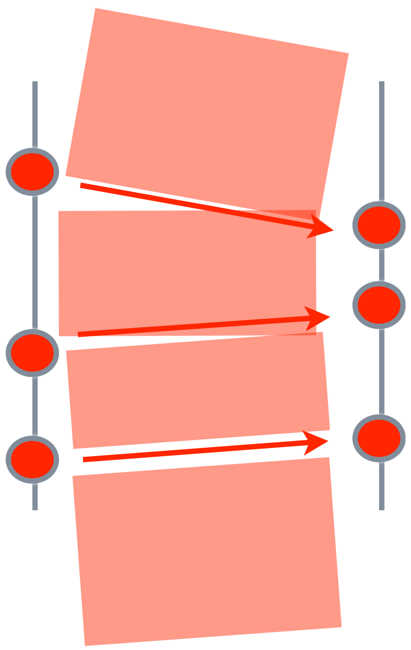

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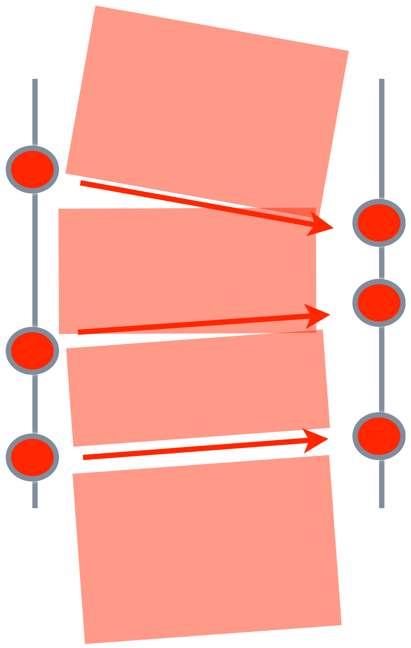

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equivalence atoms
universal (random) graph
universal partial order
universal directed graph
universal tournament
...

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Theorem: Every homogeneous relational structure is oligomorphic

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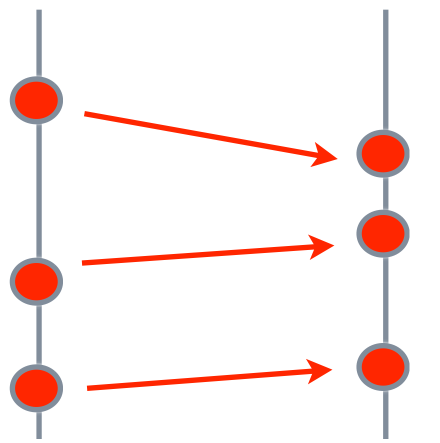
Proof:



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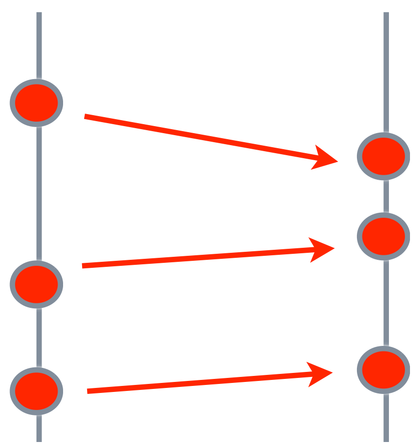

Theorem: Homogeneous = oligomorphic + quantifier elimination

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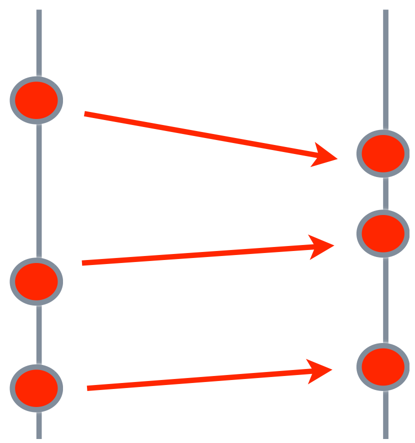

Theorem: Homogeneous = oligomorphic + quantifier elimination

Corollary: When $\mathbb{A}$ is a homogeneous structure,
FO definable = quantifier-free definable

Outline

- Re-interpreting models of computation in FO definable sets
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- Well-behaved case: oligomorphic and homogeneous atoms
- **Reachability in FO definable PDA over oligomorphic atoms**
- Ill-behaved case: time atoms

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Ignore input alphabet:

$$\rho(q, s, q', s's'') \quad \text{iff} \quad \exists a\, \rho(q, s, a, q', s's'') \;\vee\; (q, s, \varepsilon, q', s's'')$$

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Wlog. assume that transitions of PDA partition into:

$$push \subseteq Q \times S \times Q \times S^2 \quad \text{and}$$

$$pop \subseteq Q \times S \times Q$$

Oligomorphic atoms: decidability

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Theorem: Pre*(regular set) is regular for FO definable PDA, and may be effectively computed

Corollary: Configuration-to-configuration reachability of FO definable PDA is decidable

No proof idea!

Saturate transitions $\delta \subseteq Q \times S \times Q$ of NFA A:

$\delta' := \delta \cup pop$
repeat
  $\delta' := \delta \cup forced(\delta')$
until $forced(\delta') \subseteq \delta'$

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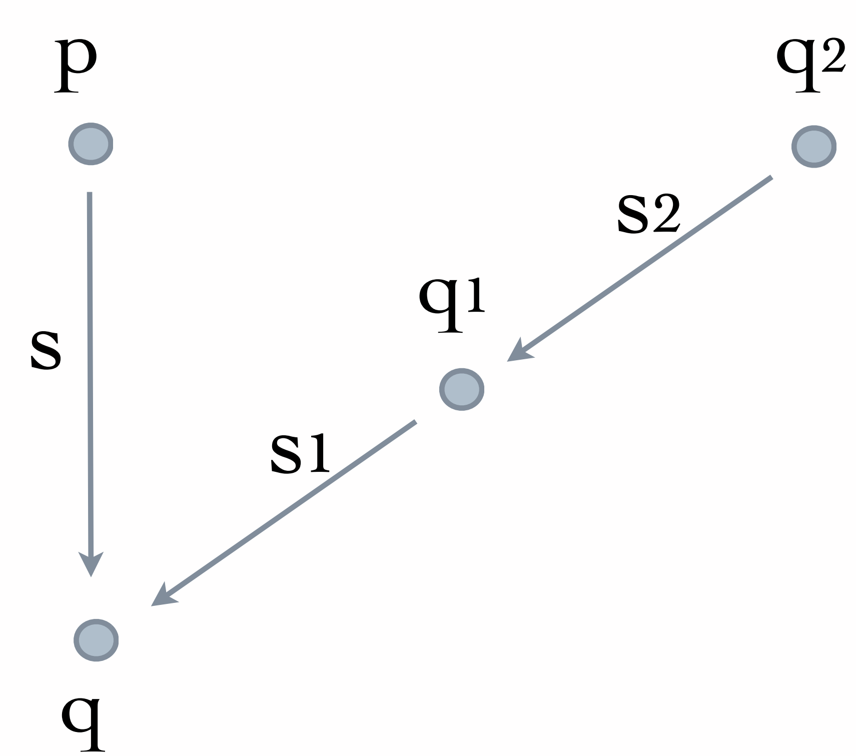

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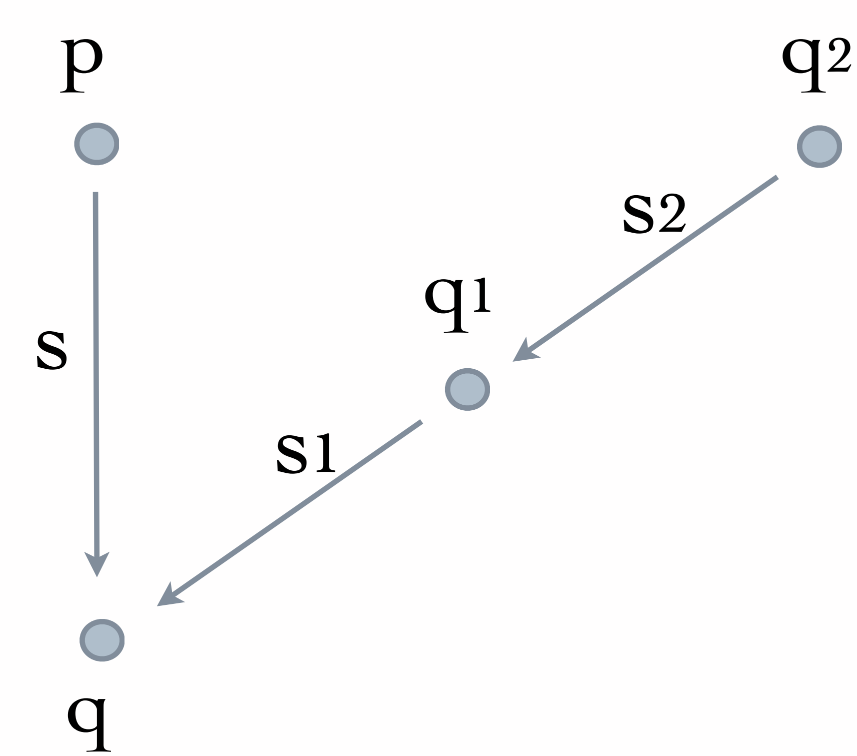

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termination due to oligomorphicity!

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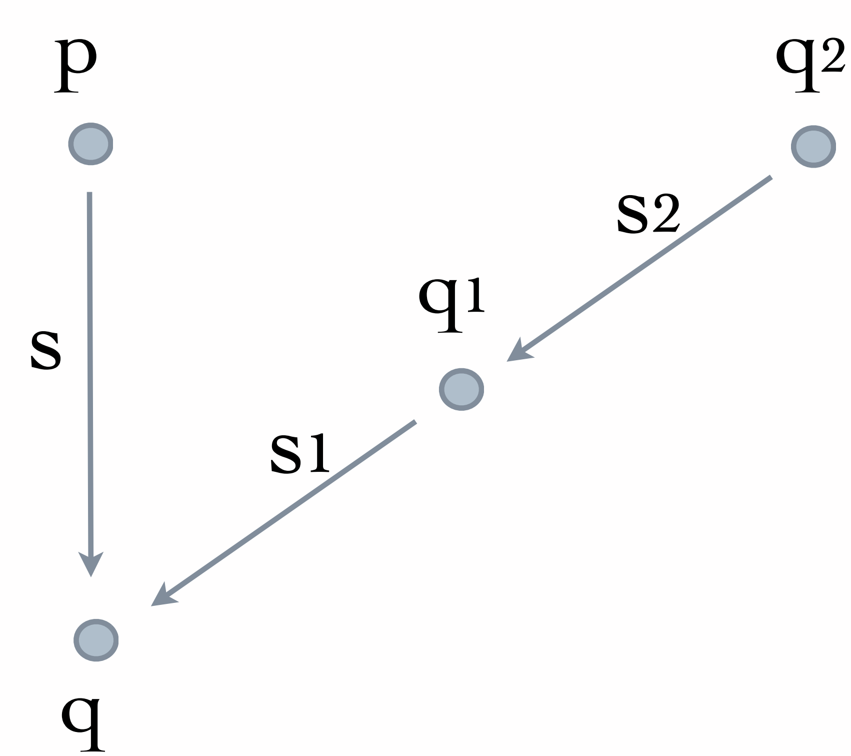

Further assumptions

From now on assume that

- the induced substructure problem for $\mathbb{A}$ is decidable
- $\mathbb{A}$ is homogeneous

Given: a finite relational structure over the vocabulary of $\mathbb{A}$

Question: is the structure an induced substructure of $\mathbb{A}$?

(Does the structure belong to age of $\mathbb{A}$?)

Homogeneous atoms: complexity

Theorem: Reachability problem for FO definable PDA is EXPTIME-complete, roughly speaking

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greatest number n of vars $\phi(x_1, x_2, \ldots, x_n)$

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Corollary: Reachability problem for FO definable PDA is fixed-parameter tractable wrt. the dimension

Theorem: [Murawski, Ramsay, Tzevelekos 2014]
Reachability problem for pushdown register automata is EXPTIME-complete.

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Reachability problem for pushdown register automata is EXPTIME-complete.

We generalize EXPTIME-completeness to arbitrary homogeneous atoms whose induced substructure problem is in polynomial time.

Arbitrarily high complexity

Theorem: Even when $\mathbb{A}$ is homogeneous, the reachability problem for FO definable PDA can have arbitrary high complexity.

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The result applies to various structures of atoms:

- equality atoms
- total-order atoms
- equivalence atoms $(\mathbb{A}, R, =)$, isomorphic to the wreath product

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- nested equality atoms $(\mathbb{A}, R_1, R_2, R_3, \ldots, =)$
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Potential application to infinite-state abstractions in analysis of recursive program.

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- **Ill-behaved case: time atoms**

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Dense-time atoms are ill-behaved:

- non-oligomorphic: $\mathbb{Q}^2$ is orbit-infinite
- definable sets are not necessarily orbit-finite
- reachability is undecidable already for FO definable NFA

Patch for time atoms?

- alphabet A
- states Q
- stack alphabet S
- $\rho \subseteq Q \times S \times Q \times S^*$
- I, F $\subseteq$ Q

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This works for NFA [Bojańczyk, L. 2012], but not for PDA:

Theorem: Reachability problem is still undecidable

Another attempt

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} FO definable

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orbit-finite?

FO definable

Too strong restriction! Span of transitions is bounded

Right choice: orbit-finite PDA

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orbit-finite (alphabet, states, stack alphabet)

orbit-finite ($Q \times S$)

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FO definable

Theorem: Reachability problem is in NEXPTIME

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orbit-finite (alphabet A, states Q, stack alphabet S)

orbit-finite ($Q \times S$)

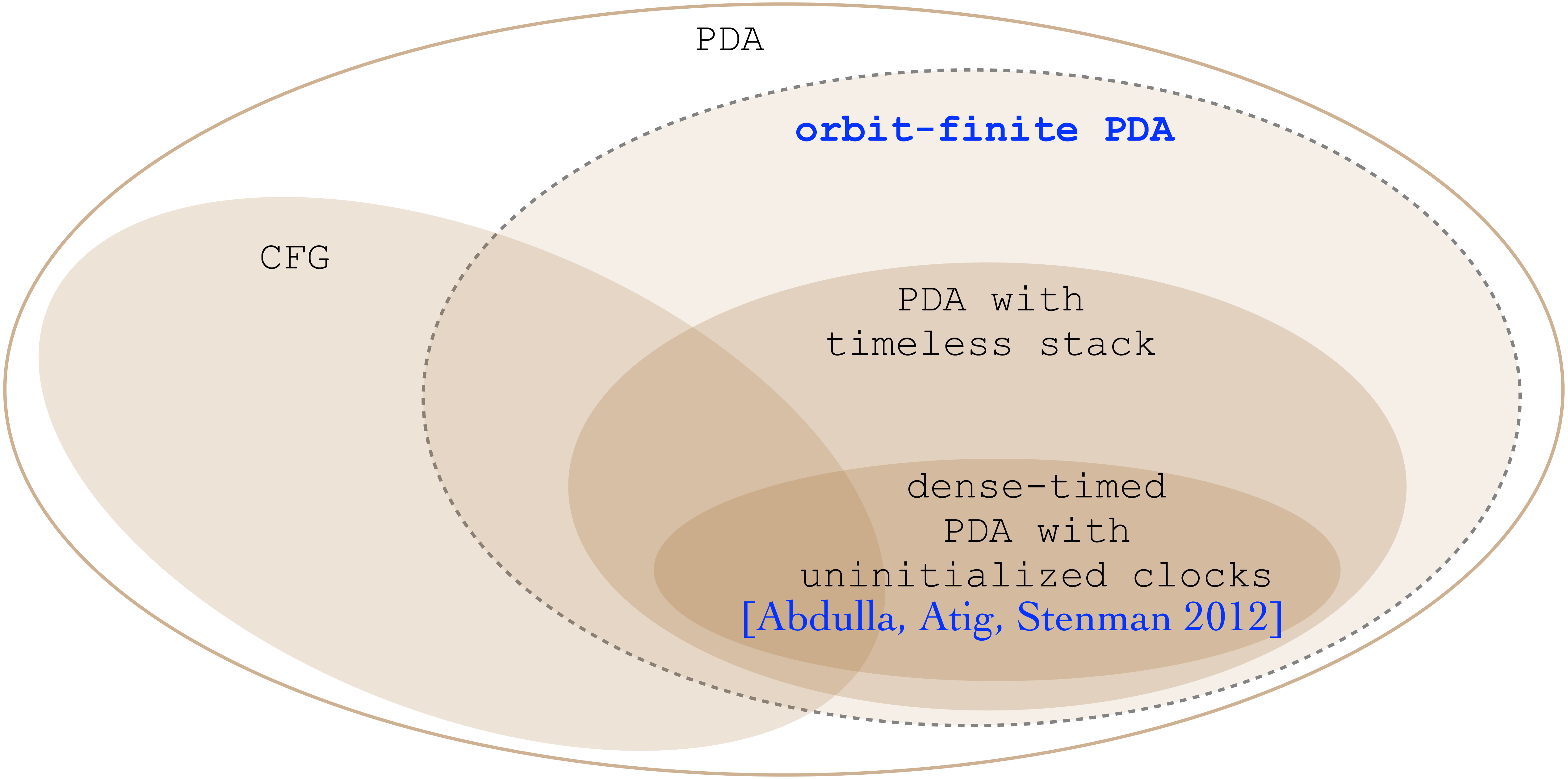
orbit-finite ($Q \times S^*$)

FO definable (all of the above)

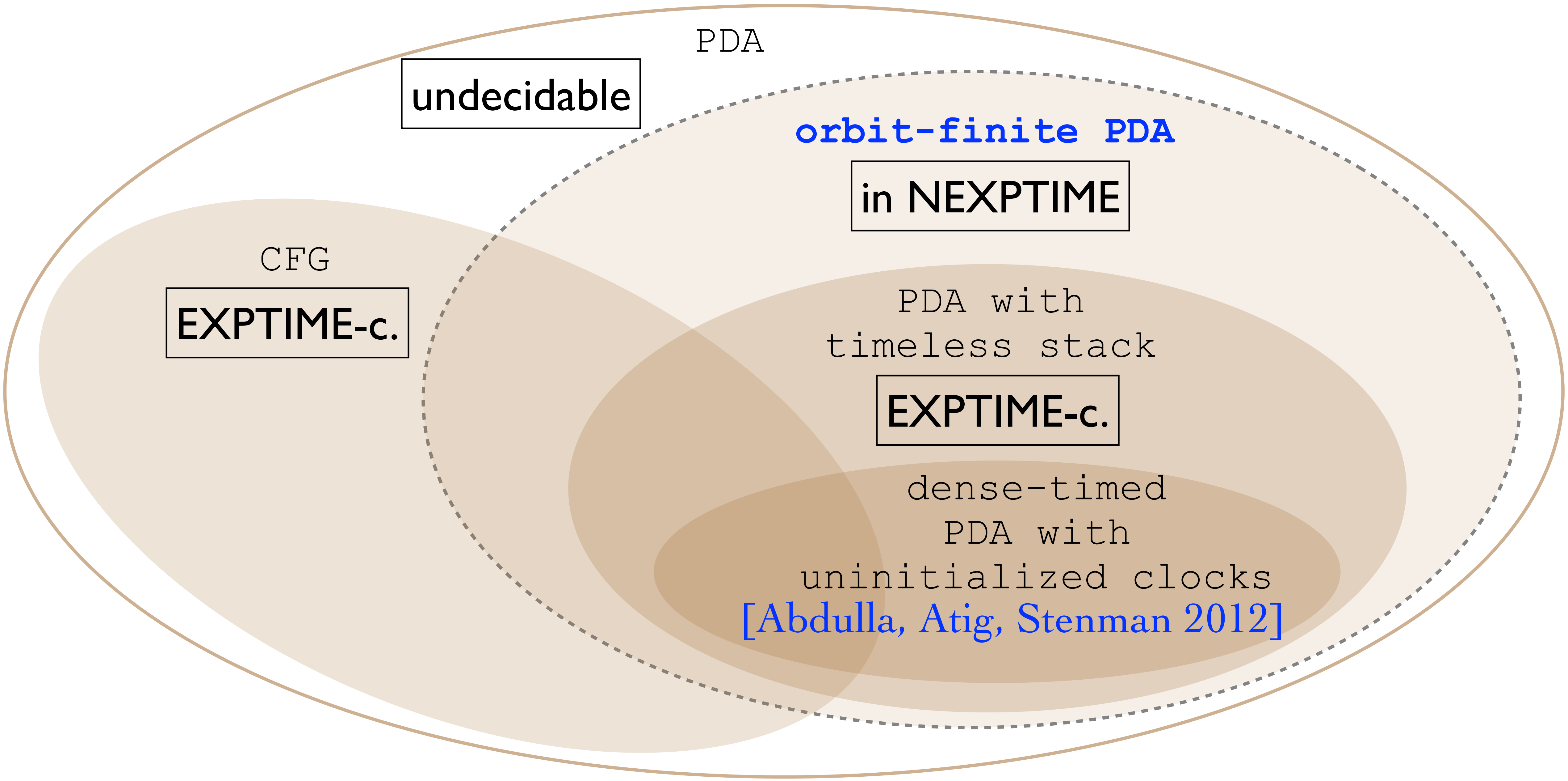
Theorem: Reachability problem is in NEXPTIME

Proof idea: Reduction to equations over sets of integers.

Expressiveness



Complexity of reachability



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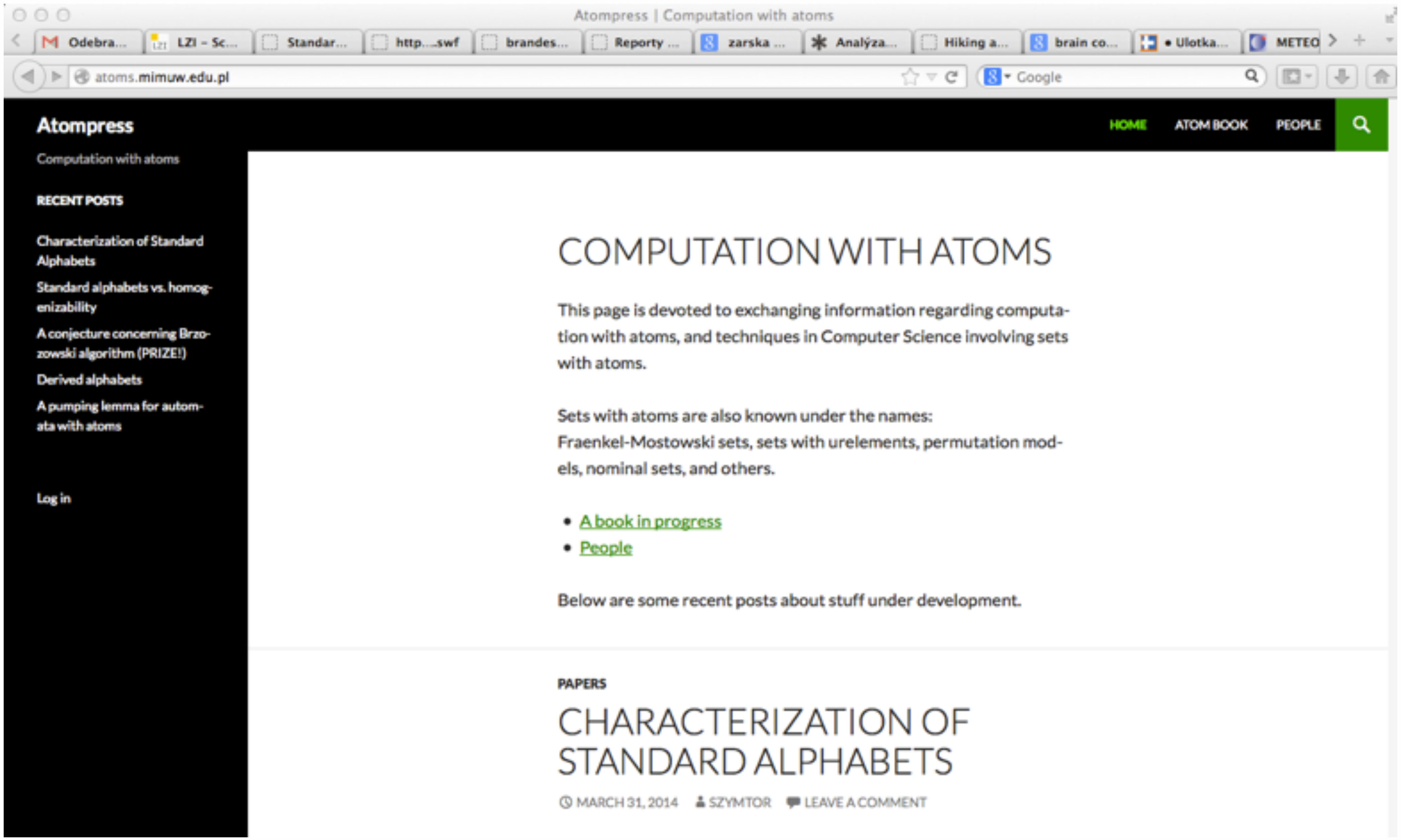

thank you!