Algorithmic aspects of game theory: assignments

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Solutions are due to **15 May 2019**.

Solutions to problems 1 and 2 should be sent to Dr. Michał Przybyłek:
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They can be written in English or in Polish, but a solution to one problem should not mix up the two languages. For each problem, we indicate the number of **points** one can get for the solution. You may send answers to any selection of problems.

1 Flow games

Let $\mathcal{G} = \langle V, E\rangle$ be a "doubly-weighted" graph, with weights (natural numbers) assigned to both vertices V and edges $E \subseteq V \times V$. We shall denote the weighting function as $w \colon V \sqcup E \to \mathcal{N}$. We say that a vertex $a \in V$ is *firing* with respect to a set of edges $R \subseteq E$ if the cumulative weight of all input edges of a that are in R is at least the weight of a, i.e.

$$w(a) \leq \sum_{(x,a)\in R} w(x, a)$$

Let us denote the set of all such vertices by $\mathit{fire}_{\mathcal{G}}(R)$. We say that graph $\mathcal{G}$ is *legal* if its every vertex is firing with respect to the set of all edges, i.e.: $V = \mathit{fire}_{\mathcal{G}}(E)$. Moreover, we shall assume that loops are not allowed and between any two nodes there is at most one edge:

$$\forall_{(a,b)\in E} \;\; (b, a) \notin E$$

We distinguish zero, one, and two-player flow games.

1.1 Zero-player game (1.5 point + 1.5 point)

Let $\mathcal{G} = \langle V, E\rangle$ be a legal graph and $R \subseteq E$ any set of edges in $\mathcal{G}$. We shall define a sequence of moves in a zero-player game starting from configuration $\mathcal{G}_0 = \mathcal{G}, R_0 = R$ recursively:

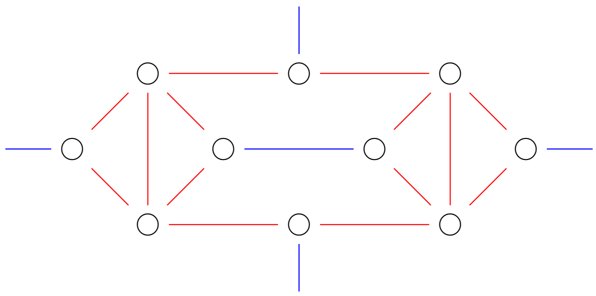
$$\begin{aligned}
R_{k+1} &= \{(x, a) \in (E \setminus \bigcup_{i=0}^{k} R_i) \colon a \in \mathit{fire}_{\mathcal{G}_k}(\bigcup_{i=0}^{k} R_i)\} \\
E_{k+1} &= (E_k \setminus R_{k+1}) \cup R_{k+1}^{op} \\
\mathcal{G}_{k+1} &= \langle V, E_{k+1}\rangle
\end{aligned}$$

where $(x, y) \in R^{op}_{k+1} \Leftrightarrow (y, x) \in R_{k+1}$ and the weights are unchanged.

Consider the following problem: given a legal graph $\mathcal{G} = \langle V, E\rangle$, a set of edges $R \subseteq E$ and an edge $e \in E$, does there exist a sequence of moves that reverses e?

1. Show that the above problem is P-complete.
2. Show that it remains P-complete on planar graphs (HINT: consider crossover gadget from Figure 1)
3. Show that it remains P-complete if we restrict to graphs whose vertices have degrees at most 3 and the possible weights of both vertices and edges are $\{1, 2\}$.

Fig. 1. The crossover gadget. Red arrows have weight 1, whereas blue arrows have weight 2. All nodes have weight 2.



There is also an unbounded version of the game, with the following moves:

$$R_{k+1} = \{(x, a) \in E_k \colon a \in \mathit{fire}_{\mathcal{G}_k}(R_K) \oplus (a, x) \in R_k\}$$
$$E_{k+1} = (E_k \setminus R_{k+1}) \cup R^{op}_{k+1}$$
$$\mathcal{G}_{k+1} = \langle V, E_{k+1}\rangle$$

where $\oplus$ is the XOR operator, i.e.: $\phi \oplus \psi \leftrightarrow (\phi \vee \psi) \wedge (\neg\phi \vee \neg\psi)$. In this game, during the $k{+}1$-th step we reverse all edges that not been reversed in the previous step and whose targets are firing wrt. R_k. We also reverse back the edges that have been reversed in the previous step, but whose targets are not firing now.

Consider the following unbounded problem: given a legal graph $\mathcal{G} = \langle V, E\rangle$, a set of edges $R \subseteq E$ and an edge $e \in E$, does there exist a sequence of moves that reverses e?

1. Show that the unbounded problem is PSpace-complete.

1.2 One-player game (1.5 point)

Let $\mathcal{G} = \langle V, E\rangle$ be a legal graph. A move in a one-player game in $\mathcal{G}$ is a reversal of a single edge, such that the resulting graph is legal.

Consider the following problem: given a legal graph $\mathcal{G} = \langle V, E \rangle$ and an edge $e \in E$, does there exist a sequence of moves that reverse edge e?

1. We proved that if we restrict the above problem to the case where every edge can be reversed at most once, then the problem is NP-complete (reduction from 3CNF)
2. Show that the above problem is PSpace-complete.

1.3 Two-player game (2 points)

Let $\mathcal{G} = \langle V, E \rangle$ be a legal graph, whose edges are partitioned onto disjoint sets $E = E_\exists \sqcup E_\forall$. The idea behind such partition is that set $E_\exists$ is controlled by one player (the first player) and set $E_\forall$ is controlled by another player (the second player). A play on graph $\mathcal{G} = \langle V, E \rangle$ with a chosen edge $e \in E$ consists of a sequence of moves taken alternatively by the players. The first player wins if e is reversed during the play. Otherwise, the second player wins (therefore, if any of the players cannot move then the second player wins).

Consider the following problem: given a legal graph $\mathcal{G} = \langle V, E \rangle$ and an edge $e \in E$, does the first player have a winning strategy?

1. We proved that if we restrict the above problem to the case where every edge can be reversed at most once, then the problem is PSpace-complete (reduction from QBF)
2. (*) Show that the above problem is EXP-complete

2 Definable games (1 point)

An *alternating graph* is a graph $\mathcal{G} = \langle V, E \rangle$ whose vertices are partitioned onto disjoint sets $V = V_\exists \sqcup V_\forall$. A move in an alternating graph from a vertex $v \in V$ is a vertex $w \in V$ such that $E(v, w)$. If $v \in E_\exists$ then the first of the players chooses a move, otherwise (i.e. $v \in E_\forall$) the second player chooses a move. The reachability game consists of an alternating graph $\mathcal{G} = \langle V, E \rangle$ together with two vertices $s, k \in V$. A play in such a game consists of a finite sequence of moves that starts from s. The first player wins if k has been chosen during the play. Otherwise, the second player wins. Consider the following problem: given a reachability game $\mathcal{G}, v, k$ does the first player have a winning strategy?

1. We have learnt that the above problem can be solved on finite graph in linear time (prof. Niwinski during the third lecture).

Let $\mathcal{N}$ be an infinite, countable set interpreted as a structure over empty first-order signature with equality. We shall say that a set $A \subseteq \mathcal{N}^k$ is *definable* if there exists a first-order formula ϕ over the empty signature, such that:

$$A = \{\langle x_1, x_2, \cdots, x_k \rangle \in \mathcal{N}^k : \phi(x_1, x_2, \cdots, x_k)\}$$

Here are some examples of definable sets:

$$\emptyset = \{x \in \mathcal{N} \colon \bot\}$$
$$\mathcal{N}^k = \{\langle x_1, x_2, \cdots, x_k\rangle \in \mathcal{N}^k \colon \top\}$$
$$\mathcal{N}^{(2)} = \{\langle x, y\rangle \in \mathcal{N}^2 \colon x \neq y\}$$

A *definable relation* between definable sets is a relation between sets that is definable as a set itself.

1. Prove that the existence of a winning strategy for the first player in a reachability game on definable graphs is decidable.

3 ω-regular games (1.5 point)

Let $\mathcal{G} = \langle V = V_0 \sqcup V_1, E, v_I, \lambda, L\rangle$ be an ω-regular game, i.e. a game on a graph where the objective $L \subseteq \Gamma^\omega$ is a regular set of infinite words.

We have shown that ω-regular games are determined under finite memory strategies. Show that deciding whether the first player has a winning strategy is:

- EXP-complete, if L is given by a non-deterministic parity automaton,
- and 2EXP-complete, if L set is given by an alternating parity automaton.

4 Multi-parity games (1.5 point)

Let G be a game on graph where the labelling function $\lambda \colon V \to [0, d] \times [0, d]$ assigns pairs of priorities. We say that the first player wins if the play satisfies the parity condition on both coordinates.

1. Is the game (positionally) determined?
2. Show that deciding whether the first player has a winning strategy is coNP-complete.

5 Multi-reachability game (2 points)

In reachability games there is a set of vertices which the first player wants to reach. In multi-reachability games (MRG) there is a family of sets of vertices and the first player wins if every set in the family has been visited at least once.

1. Show that MRG game is PSpace-complete.
2. Show that one-player MRG is NP-complete.
3. Show that if the sets are singletons then MRG is P-complete.

6 Closed games (1.5)

Let Γ be an alphabet, i.e. a finite set. Let $w, u \in \Gamma^\omega$ be two infinite words. We say that w, u are in distance 2^{-n}, denoted $\mathrm{d}(u, v) = 2^{-n}$, if their longest common prefix is of length n. For instance, $\mathrm{d}(a^\omega, b^\omega) = 1$, $\mathrm{d}(a^\omega, a^4 b^\omega) = 2^{-4}$, and $\mathrm{d}(a^\omega, a^n b^\omega) = 2^{-n}$.

We say that a word $w \in \Gamma^\omega$ is the limit of a sequence of infinite words u_n, $n \geq 0$, (or that u_n converges to w) if the sequence of distances $\mathrm{d}(u_n, w)$ converges to 0. The closure $cl(S)$ of a set of infinite words S is the set of all limits of all its sequences, i.e. $cl(S) = \{w \in \Gamma^\omega \mid \exists (u_n).$ the sequence u_n converges to $w\}$.

Set of infinite words $S \subseteq \Gamma^\omega$ is called closed if $S = cl(S)$. A set of infinite words is called open, if its complement is closed.

1. Show that if a game on graphs has a closed winning objective, then one of the players has a winning strategy.