Topological arguments for automata-theoretic hierarchies

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Infinite computations

- Büchi (1960) and Rabin (1969) used the concept of infinite computations of finite automata to establish the decidability results in logic.
- D. Muller (1960) used similar concepts to analyse asynchronous digital circuits.
- Since 1980s, computer scientists study infinite computations in context of verification of computing systems (reactive, concurrent, open, . . . ). Non-termination is an expected behaviour.
- Mathematicians have been playing infinite games since the 1930s (Banach–Mazur, later Gale–Stewart, . . . )
Automata theory classifies automata along several axes

- working on infinite words or trees,
- in deterministic, non-deterministic, or alternating mode,
- with a certain acceptance condition and, specifically, a Rabin–Mostowski index.

All this relates to the complexity of decision problems.

Sets of infinite words or trees can be also classified by hierarchies of set-theoretical topology (Borel, projective, Wadge).

Topological hardness is often behind the hierarchy results.

But sometimes the two complexities diverge…
Topics of the talk

• Strictness of the Rabin-Mostowski index hierarchy
  — when and how topology helps

• Wadge reductions of game languages

• When the two complexities diverge
  — new reading of an old proof by Rabin

• Ambiguous classes and inseparability of game languages
Classical definability theory

1900 Borel, Baire, Lebesgues

1917 Lusin, Suslin

1929 Tarski, Kuratowski

1940 Mostowski, Kleene
For $R \subseteq \omega^k \times (\{0, 1\}^\omega)^\ell$, let

$\exists^0 R = \{\langle m, \alpha \rangle : (\exists n) R(m, n, \alpha)\}$

$\exists^1 R = \{\langle m, \alpha \rangle : (\exists \beta) R(m, \alpha, \beta)\}$

\begin{align*}
\Sigma^0_0 &= \text{recursive relations} \\
\Pi^0_0 &= \{\overline{R} : R \in \Sigma^0_0\} \\
\Sigma^0_{n+1} &= \{\exists^0 R : R \in \Pi^0_n\} \\
\Delta^0_n &= \Sigma^0_n \cap \Pi^0_n \\
\Sigma^1_0 &= \text{arithmetical relations} \\
\Pi^1_0 &= \{\overline{R} : R \in \Sigma^0_0\} \\
\Sigma^1_{n+1} &= \{\exists^1 R : R \in \Pi^1_n\} \\
\Delta^1_n &= \Sigma^1_n \cap \Pi^1_n
\end{align*}
Arithmetical hierarchy

\[ \Sigma^0_0 = \text{recursive relations} \]
\[ \Pi^0_n = \{ \overline{R} : R \in \Sigma^0_n \} \]
\[ \Sigma^0_{n+1} = \{ \exists^0 R : R \in \Pi^0_n \} \]
\[ \Delta^0_n = \Sigma^0_n \cap \Pi^0_n \]

Analytical hierarchy

\[ \Sigma^1_0 = \text{arithmetical relations} \]
\[ \Pi^1_n = \{ \overline{R} : R \in \Sigma^0_n \} \]
\[ \Sigma^1_{n+1} = \{ \exists^1 R : R \in \Pi^1_n \} \]
\[ \Delta^1_n = \Sigma^1_n \cap \Pi^1_n \]

Relativized (boldface) hierarchies

For \( \beta \in \{0, 1\}^\omega \), let \( R[\beta] = \{ \langle m, \alpha \rangle : R(m, \alpha, \beta) \} \).

\[ \Sigma^i_n = \{ R[\beta] : R \in \Sigma^i_n, \beta \in \{0, 1\}^\omega \} \]
\[ \Pi^i_n = \{ R[\beta] : R \in \Pi^i_n, \beta \in \{0, 1\}^\omega \} \]

\[ \Sigma^0_1 = \text{open} \]
\[ \Pi^0_1 = \text{closed} \]
\[ \Delta^1_1 = \text{Borel} \]
Büchi automata on infinite words

\[ \mathcal{A} = \langle \Sigma, Q, q_I, Tr, F \rangle \]

where \( Tr \subseteq Q \times \Sigma \times Q \), \( F \subseteq Q \).

\[ (a + b)^* b \omega \]

\[ (a + b)^* a \omega \]

The second one cannot be recognized by a deterministic automaton.
So \((a + b)^* a^\omega\) cannot be recognized by a deterministic automaton.

**But this also follows by a topological argument!**

We assume the **Cantor** topology on \(X^\omega\), induced by the metric

\[
d(u, v) = 2^{-\min\{m : u_m \neq v_m\}}
\]

(or 0, if \(u = v\)).

If \(A\) is **deterministic** then \(\text{run} : \Sigma^\omega \rightarrow Q^\omega\) is a function which continuously reduces \(L(A)\) to \((Q^* F)^\omega\):

\[
L \ni u \iff \text{run}(u) \in (Q^* F)^\omega
\]

But

- \((Q^* F)^\omega\) is \(\Pi_2^0 (G_\delta)\),
- \((a + b)^* a^\omega\) is **complete** in \(\Sigma_2^0 (F_\sigma)\), a **contradiction**! 
Parity automata

\[ \mathcal{A} = \langle \Sigma, Q, q_I, Tr, \text{rank} \rangle \]

where \( \text{rank} : Q \to \{0, 1, \ldots, k\} \).

\[ \limsup_{i \to \infty} \text{rank}(q_i) \text{ is even} \]

\[ (a + b)^* a^\omega \]

The Rabin-Mostowski index of a parity automaton \( \mathcal{A} \) is

\[ (\min \text{rank}(Q), \max \text{rank}(Q)) \]

We can assume \( \min \text{rank}(Q) \in \{0, 1\} \).
The McNaughton Theorem (1966)

A nondeterministic Büchi automaton can be simulated by a deterministic parity automaton of some index \((i, k)\).

The minimal index \((i, k)\) may be arbitrarily high (Wagner 1979, Kaminski 1985).

Again, it can be inferred by a topological argument.

Let

\[
M_{i,k} = \{ u \in \{i, \ldots, k\}^\omega : \limsup_{\ell \to \infty} u_\ell \text{ is even} \}
\]
No continuous reduction down the hierarchy.
Wadge game $G(A, B)$

<table>
<thead>
<tr>
<th>Spoiler</th>
<th>Duplicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0 \in \Sigma$</td>
<td>$b_0 \in \Sigma$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>$b_{12}$</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>wait</td>
</tr>
<tr>
<td>$a_{14}$</td>
<td>wait</td>
</tr>
<tr>
<td>$a_{15}$</td>
<td>$b_{13}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Here $A, B \subseteq \Sigma^\omega$ ($\Sigma$ finite).

Duplicator wins if $a_0 a_1 a_2 \ldots \in A \iff b_0 b_1 b_2 \in B$.

Fact

Duplicator has a winning strategy iff there is a continuous $f : \Sigma^\omega \to \Sigma^\omega$ s.t. $A = f^{-1}(B)$,

in symbols, $A \leq_w B$. 
Spoiler’s strategy, e.g., in $G(M_{0,5}, M_{1,6})$

<table>
<thead>
<tr>
<th>Spoiler</th>
<th>Duplicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Note

$i$ If a deterministic automaton of index $(1, 6)$, $i-1$ accepted $M_{0,5}$ there would be a continuous reduction of $M_{0,5}$ to $M_{1,6}$

$\text{wait } u \mapsto \text{rank} \circ \text{run}(u)$. Contradiction!
Parity tree automata

\[ \mathcal{A} = \langle \Sigma, Q, q_I, Tr, rank \rangle \]

where \( Tr \subseteq Q \times \Sigma \times Q \times Q \), \( rank : Q \rightarrow \{0, 1, \ldots, k\} \).
Parity tree automata cont’d

A run of $\mathcal{A}$ on a tree $t : \{l, r\}^* \rightarrow \Sigma$ is a tree $\rho : \{l, r\}^* \rightarrow Q$, such that,
\[
\langle \rho(w), t(w), \rho(wl), \rho(wr) \rangle \in Tr, \text{ for each } w \in \text{dom} (\rho)
\]

The run is accepting if, for each path $P = p_0p_1 \ldots \in \{l, r\}^\omega$,
\[
\limsup_{k \rightarrow \infty} \text{rank}(\rho(p_0p_1 \ldots p_k)) \text{ is even}.
\]
Example

\[
\begin{align*}
\text{rank}(a) &= 0 \\
\text{rank}(b) &= 1
\end{align*}
\]

recognizes the set of trees where, on each branch, \( b \) appears only finitely often.

The complement can be recognized by

\[
\begin{align*}
\text{rank}(a) &= 1 \\
\text{rank}(b) &= \text{rank}(\text{skip}) = 2
\end{align*}
\]
Example cont’d

\[
\begin{align*}
&a/b, a \\
&a \\

&a/b, b \\
&b \\

rank(a) &= 0 \\
rank(b) &= 1
\end{align*}
\]

This set cannot be recognized by a Büchi automaton (i.e., of index \((1,2)\)), Rabin 1970.
Again, a topological argument could be used instead, as this set is $\Pi^1_1$ complete, while the Büchi automata can recognize only $\Sigma^1_1$ sets.
The following witness the strictness of the non-deterministic index hierarchy.

\[ T_{i,k} = \{ t \in \{i, \ldots, k\}\{l,r\}^* : \text{each branch is in } M_{i,k} \} \] (N 1986)

Here, a topological argument cannot be used, as all the sets \( T_{i,k} \) are \( \Pi^1_1 \) complete, hence Wadge–equivalent (except for \( T_{0,0}, T_{1,1}, T_{1,2} \)).

**Game tree languages**

Alphabet: \( \{\exists, \forall\} \times \{i, \ldots, k\} \).

\[
\begin{align*}
\text{Eve}: & \quad \exists, j \quad \exists, j \\
\text{Adam}: & \quad \forall, j \quad \forall, j
\end{align*}
\]

Eve wins an infinite play \((x_0, i_0), (x_1, i_1), (x_2, i_2), \ldots \ (x_\ell \in \{\exists, \forall\})\) iff \( \limsup_{\ell \to \infty} i_\ell \) is even.

The set \( W_{i,k} \) consists of all trees such that Eve has a winning strategy.
Conventions about duality: \((1, n)\) is **dual** to \((0, n - 1)\).

\((i, k)\) denotes the dual of \((i, k)\).

Up to renaming: \(W_{i,k} \approx \overline{W_{i,k}}\).
**Topological argument:** The sets $W_{i,k}$ form a strict hierarchy w.r.t. the Wadge reducibility (Arnold & N, 2008).

From this, the strictness of the alternating index hierarchy easily follows.
An input tree $t \in \Sigma_{\{l,r\}}^*$ induces a computation tree over states, $\text{comp}(t)$. Composing with the function $\text{rank} : Q \to \{i, \ldots, k\}$, we have

$$t \in T(A) \iff \text{rank} \circ \text{comp}(t) \in W_{i,k}$$

Hence,

$$T(A) \leq_w W_{i,k}$$

In particular, if an alternating automaton $A$ of index $(i, k)$ accepted $W_{\overline{i,k}}$, we would have $W_{\overline{i,k}} \leq_w W_{i,k}$, a contradiction.
Sketch of proof that $W_{i,k} \not\leq_w W_{i,k} \approx W_{i,k}$

By Banach Fixed-Point Theorem, there is no contracting reduction of $L$ to $\overline{L}$

$$x_{fix} \in L \iff f(x_{fix}) \in \overline{L} \iff x_{fix} \in \overline{L}$$

**Main Lemma** If $f$ reduces $W_{i,k}$ to some $L$ then there is a mapping $h : \{i, \ldots, k\}\{l,r\}^* \rightarrow \{i, \ldots, k\}\{l,r\}^*$ (padding), such that

- $h$ reduces $W_{i,k}$ to itself,

- $f \circ h$ is **contracting**.

About $h$: For $W_{0,k}$, it “stretches” the original tree completing by the nodes labeled by $(\forall, 0)$. For $W_{1,k}$, by $(\exists, 1)$. 
Witnesses of the index hierarchies

deterministic
Wadge hierarchy

non-deterministic
Wadge equivalent

alternating
Wadge hierarchy
When the two complexities diverge...

If a recognizable set of trees is Büchi recognizable (equivalently $\nu \mu, \exists S2S$) then it is $\Sigma^1_1$.

**The converse does not hold.**

Let

$$H! = \text{binary trees over } \{a, b\} \text{ where } b \text{ appears infinitely often on exactly one branch.}$$

By Lusin Theorem ([Kechris, Thm. 18.11]), $H!$ is $\Pi^1_1$ (complete).

Hence $H!$ is $\Sigma^1_1$.

But it is **not** Büchi recognizable!
Rabin’s proof works…
Note

$H^!$ is non-ambiguous, i.e., can be recognized by a non-ambiguous parity tree automaton (exactly one accepting run).

Questions

• Are all non-ambiguous languages $\Pi^1_1$?  
  (It is so for deterministic languages.)

• Is it decidable, if a given tree language is non-ambiguous?  
  (It is so for determinism.)

• What is the expressive power of non-ambiguous automata?
Fact

No non-ambiguous automaton can recognize the set of binary trees over \( \{a, b\} \) such that \( b \) appears at least once (elementary proof: Carayol & L"oding 2007).

\( \iff \) (N. & Walukiewicz 1996) The S2S formula

\[ X = \emptyset \lor y \in X \]

cannot be made functional \( (X \mapsto y) \).

Consequently, S2S is not uniformizable (Gurevich & Shelah 1983).

In contrast to S1S…

Rabin 1970 showed that if both $L$ and $\overline{L}$ are Büchi recognizable then $L$ is definable in weak monadic second-order logic ($\text{WS2S}$).

\[ \leftrightarrow \]

is recognized by weak alternating automaton (Muller, Schupp, Saoudi 1986).

\[ \leftrightarrow \]

is definable in the alternation-free $\mu$-calculus (Arnold & N. 1991).

Does this generalize to higher levels?
Classes \textit{Comp}

\begin{align*}
\text{Comp} \mathcal{T} &= \text{closure of } \mathcal{T} \text{ under composition} \\
\mu \mathcal{T} &= \text{closure of } \mathcal{T} \text{ under } \mu \text{ and composition} \\
\nu \mathcal{T} &= \text{closure of } \mathcal{T} \text{ under } \nu \text{ and composition} \\
\Sigma^\mu_1(\mathcal{F}) &= \mu \mathcal{F} \quad \Pi^\mu_1(\mathcal{F}) = \nu \mathcal{F} \\
\Sigma^\mu_{n+1}(\mathcal{F}) &= \mu \Pi^\mu_n(\mathcal{F}) \quad \Pi^\mu_{n+1}(\mathcal{F}) = \nu \Sigma^\mu_n(\mathcal{F}) \\
\text{Büchi} &= \Pi^\mu_2(\{\cup, f, \ldots\}) = \Pi^\mu_2(\{\cup, \cap, f, \ldots\}) \\
\Sigma^\mu_2 \cap \Pi^\mu_2 &= \text{Comp} (\Sigma^\mu_1 \cup \Pi^\mu_1)
\end{align*}

Santocanale and Arnold 2005 showed that this fails for \( n \geq 3 \).
Separability

On positive side, Santocanale and Arnold 2005 generalized Rabin’s result, by showing that if both $L$ and $\overline{L}$ are recognizable by nondeterministic automata of level $\Pi^\mu_{n+1}$ then $L$ is in $\text{Comp} (\Sigma^\mu_n \cup \Pi^\mu_n)$ (collapse property).

They showed in fact a stronger result, that any disjoint sets $L$ and $M$ recognized by nondeterministic automata of level $\Pi^\mu_{n+1}$, for $n \geq 1$, can be separated by a language $C \in \text{Comp} (\Sigma^\mu_n \cup \Pi^\mu_n)$, i.e., $L \subseteq C \subseteq \overline{M}$ (separation property).

Note: separation $\implies$ collapse, but in general not vice versa.

S & A 2005 also showed that the collapse property fails for nondeterministic automata in $\Sigma^\mu_n$ classes, for $n \geq 3$.

We show that separation fails for $\Sigma^\mu_2$ (co-Büchi).
Let $W'_{0,1}$ be obtained from $W_{0,1}$ by interchanging $\exists \leftrightarrow \forall$ and $0 \leftrightarrow 1$.

That is, in $W'_{0,1}$ Adam has a strategy to force that there is only finitely many 0’s.

**Proposition** $W_{0,1}$ and $W'_{0,1}$ are both in $\Sigma^\mu_2$, but not separable by any Borel set.

So, *a fortiori* they are not separable by any $\text{Comp} (\Sigma^\mu_1 \cup \Pi^\mu_1)$ set.

(Hummel, Michalewski, N., STACS 2009.)
Lemma For any Borel set $B \subseteq \{0, 1\}^\omega$, there is a continuous reduction

$$f : \{0, 1\}^\omega \rightarrow \mathcal{P}\{0, 1\} \times \{\exists, \forall\}$$

such that

- $u \in B \Rightarrow f(u) \in W_{0,1}$
- $u \notin B \Rightarrow f(u) \in W'_{0,1}$

If there were a Borel set $C$ s.t. $W_{0,1} \subseteq C \subseteq \overline{W'_{0,1}}$, we would have

- $u \in B \Rightarrow f(u) \in C$
- $u \notin B \Rightarrow f(u) \in \overline{C}$

Hence any Borel set would be a continuous inverse image of $C$, which is impossible, since the Borel hierarchy is strict.
Proof of the lemma. For a clopen set $B$ it is enough to take

\[ B \ni x \mapsto t_1 \in W_{0,1} \]
\[ B \not\ni x \mapsto t_2 \in W'_{0,1} \]

Suppose $B = \bigcup B_n$ and we have suitable reductions $F_{B_n}$. We let $F_B(x) =$

The case $B = \bigcap B_n$ follows from symmetry.
The following pair has been well known in descriptive set theory.

\[ WF = \{ t \in T_{\{0,1\}} : \text{on each path there are finitely many } 1's \} \]
\[ UB = \{ t \in T_{\{0,1\}} : \text{there is exactly one path with } \infty \text{ } 1's \} \]

Both languages are in $\Pi^1_1$, recognizable by parity tree automata, and the pair $(WF, UB)$ is Borel inseparable.

But it was not useful for us, since $UB$ is not co-Büchi.

The pair $(WF, UB)$ is known to be universal for $\Pi^1_1$ inseparable pairs.

**Conjecture** The pair $(W_{0,1}, W'_{0,1})$ is universal, too.
Separation property

Class \( \mathcal{L} \) has first separation property if any two disjoint sets \( L, M \in \mathcal{L} \) are separated by some set \( K \) in

\[
\Delta = \mathcal{L} \cap \overline{\mathcal{L}}
\]

with

\[
\overline{\mathcal{L}} = \{ X : \bar{X} \in \mathcal{L} \}
\]

For example, first separation property holds for \( \Sigma_1^1 \) but fails for \( \Pi_1^1 \).
**Conjecture** First separation property holds for classes $\Pi_n^\mu$, and fails a classes $\Sigma_n^\mu$. In terms of the index hierarchy, it holds for the class $(i, k)$ iff $k$ is even.
Decidability issues

For deterministic tree automaton $A$, it is decidable

- if $T(A)$ is Büchi recognizable (Urbański 2000),
- the exact index $(i, k)$ of $T(A)$ in the non-deterministic hierarchy,
- if $T(A)$ is Borel (N., Walukiewicz 2003),
- the exact place of $T(A)$ in the Borel hierarchy (Murlak 2005).

Moreover, Murlak 2006 showed that, for deterministic $A$ and $B$, it is decidable if

$$T(A) \leq_w T(B).$$

and described the whole Wadge hierarchy of deterministic tree languages.

The non-deterministic case remains open.
Conclusion

Topological complexity often, but not always, underlines the automata-theoretic complexity.

Topological arguments appear to work better for deterministic or alternating, rather than for non-deterministic automata (?).

Decidability issues: we can almost completely classify the topological complexity of a language, but only if the input automaton is deterministic. Are the non-ambiguous automata the next step?

Separation problems make a link between automata theory and descriptive set-theory. But they may lead to a more general \textit{definability theory}, subsuming both.
Wadge reducibility—decidability issues

Fact (Büchi & Landweber 1969). For Büchi automata on infinite words:

1. If $L(A) \leq_w L(B)$ then there exists a finite–state transducer reducing $L(A)$ to $L(B)$.

2. It is decidable if $L(A) \leq_w L(B)$.

For trees, (1) does not hold. Nevertheless, Murlak 2006 shows

Fact. It is decidable if $T(A) \leq_w T(B)$, for deterministic tree automata $A$, $B$.

Rather than comparing two automata “from scratch”, one computes, for each deterministic automaton $A$, its place in the hierarchy, i.e., an ordinal and a canonical automaton equivalent to $A$.

Construction of canonical automata is the core of the proof.

Again, the non-deterministic case remains open.