Information theory, 2017/2018
Problem for independent solving. Deadline: 27.01.2018.

Please send solutions in pdf format to my address niwinski@mimuw.edu.pl
Solution can be graded up to 1.5 points.

A pawn lands in a chessboard 8×8, and calls for help. Could he use 6 bits, he would be able to describe his position exactly. Suppose however that the pawn can only send $k < 6$ bits. Fortunately, he will be saved if the rescue arrives (at least) to some neighbor position (possibly in diagonal). Find k, such that the pawn will be saved for sure. For $k-1$ estimate the probability that the pawn will be saved.

We now make the above informal description more precise. The position of the pawn at the chessboard is described by a random variable A taking values in $\{1, \ldots, 8\}^2$. The call for help is described by a random variable B taking values in $\{0,1\}^k$, for some $k < 6$. Finally, the rescue, which arrives to help the pawn, is described by a random variable C, taking values again in $\{1, \ldots, 8\}^2$.

We assume that whatever the rescue agency knows about the actual position of pawn, comes from his call for help. This is captured by the following condition.

(*) $I(A; C|B) = 0$ (equivalently, A i C are conditionally independent[1] given B).

Whenever A takes value (i, j), and C value (i', j') then **the pawn is saved**, if

$$\max(|i' - i|, |j' - j|) \leq 1.$$

The tasks in the homework are as follows.

1. Find a minimal k such that, for all A, there exist B and C satisfying (*), and such that the pawn is saved with probability 1. **Argue for minimality**.
   (A correct answer to this question guarantees 1 point.)

2. If k is the value found in the previous task, consider $k' = k - 1$, so that B now assumes values in $\{0,1\}^{k-1}$. Assume that A has uniform distribution. Give a lower bound (as good as you can) to the maximal probability of the event that the pawn is saved, where the maximum is over all pairs B, C, satisfying the condition (*).

3. This is "problem with star" (difficult). Give the upper bound to the value described in the previous point, better than 1.

[1] See https://www.mimuw.edu.pl/~niwinski/Info/2017-info.pdf, Sect. 1.6.