Complexity of infinite tree languages

when automata meet topology

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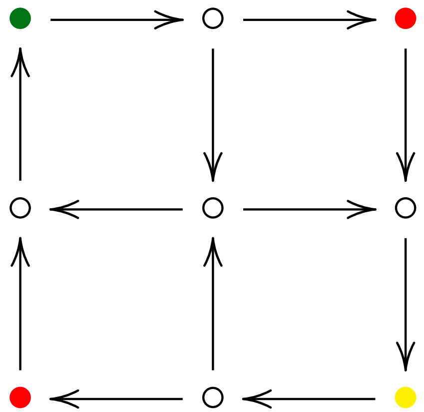
Example 1

Is there a path that meets ● and ● infinitely often, but

● only finitely often ?



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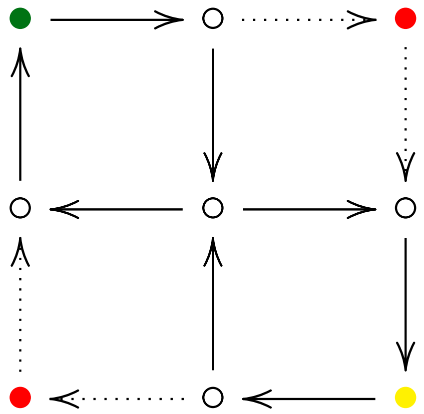
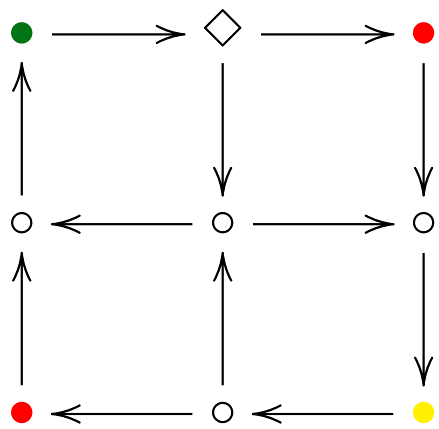


Example 2

Is there a strategy for ◯ against ◇ to achieve the same condition ?



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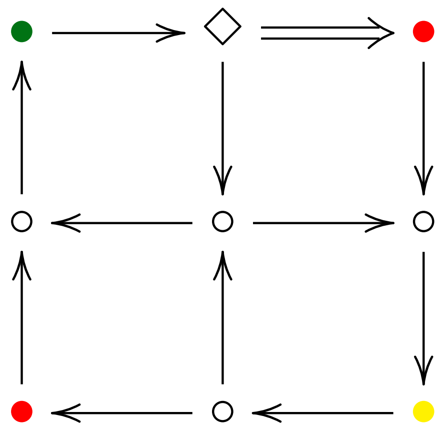
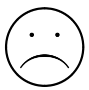
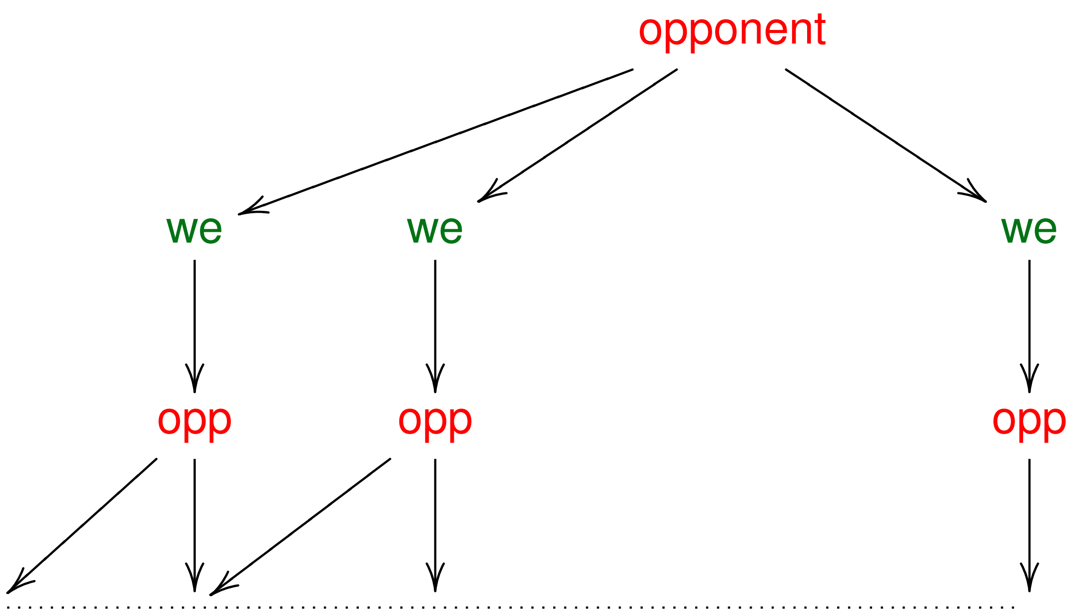




General method



Strategies are (possibly infinite) trees.

Construct an automaton accepting the winning strategies, and test it for non-emptiness.

Infinite computations

- Büchi (1960) and Rabin (1969) considered infinite computations of finite automata in their proofs of decidability of the theories S1S and S2S.
- D. Muller(1960) used similar concepts to analyse asynchronous digital circuits.
- Since 1980s, computer scientists study infinite computations in context of verification of computing systems (reactive, concurrent, open, . . .).

  Non-termination is an expected behaviour.
- Mathematicians have been playing infinite games since the 1930s: Banach–Mazur, later Gale–Stewart (1953), . . .

Automata can be classified along several axes

- working on infinite words or trees,
- in deterministic, non-deterministic, or alternating mode,
- with a certain acceptance condition and, specifically, a Rabin–Mostowski index.

All these features relate to the complexity of the non-emptiness problem.

Sets of infinite words or trees can be also classified by hierarchies of set-theoretical topology (Borel, projective, Wadge).

Topological hardness is often behind the hierarchy results.

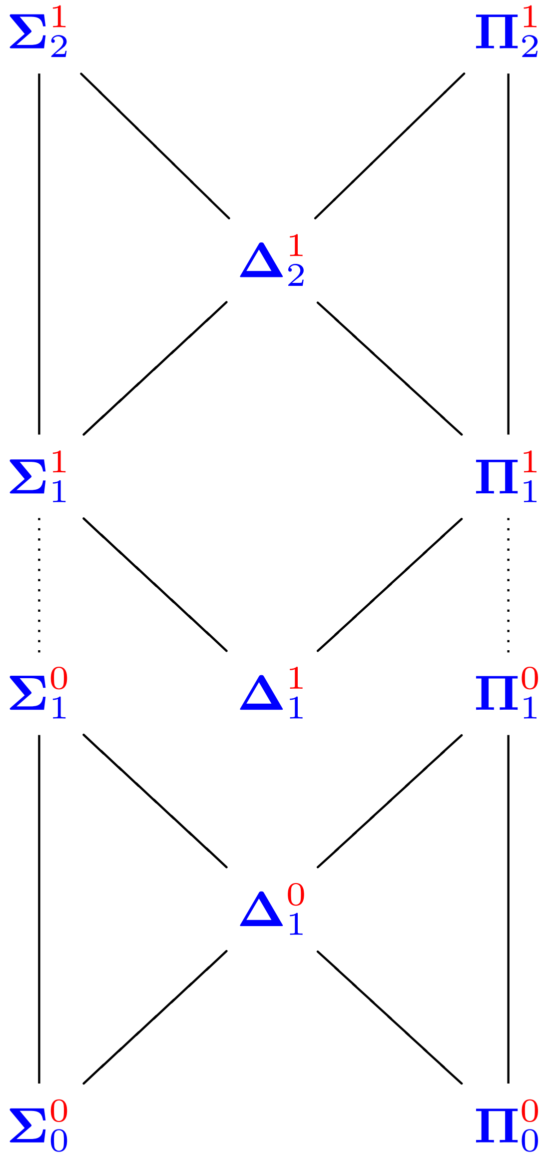
Classical definability theory

1900 Borel, Baire, Lebesgues

1917 Lusin, Suslin

1929 Tarski, Kuratowski

1940 Mostowski, Kleene



,,Easy can be hard.”

We will see

- regular sets of infinite trees beyond the **Borel hierarchy**,
- **inseparable** pair of regular sets.

For $R \subseteq \omega^k \times (\{0,1\}^\omega)^\ell$, let

$$\exists^0 R = \{\langle \mathbf{m}, \alpha \rangle : (\exists n)\, R(\mathbf{m}, n, \alpha)\}$$

$$\exists^1 R = \{\langle \mathbf{m}, \alpha \rangle : (\exists \beta)\, R(\mathbf{m}, \alpha, \beta)\}$$

Arithmetical hierarchy			Analytical hierarchy		
Σ^0_0	$=$	recursive relations	Σ^1_0	$=$	arithmetical relations
Π^0_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$	Π^1_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$
Σ^0_{n+1}	$=$	$\{\exists^{\mathbf{0}} R : R \in \Pi^0_n\}$	Σ^1_{n+1}	$=$	$\{\exists^{\mathbf{1}} R : R \in \Pi^1_n\}$
Δ^0_n	$=$	$\Sigma^0_n \cap \Pi^0_n$	Δ^1_n	$=$	$\Sigma^1_n \cap \Pi^1_n$

Arithmetical hierarchy

$$\Sigma^0_0 = \text{recursive relations}$$
$$\Pi^0_n = \{\overline{R} : R \in \Sigma^0_n\}$$
$$\Sigma^0_{n+1} = \{\exists^0 R : R \in \Pi^0_n\}$$
$$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$$

Analytical hierarchy

$$\Sigma^1_0 = \text{arithmetical relations}$$
$$\Pi^1_n = \{\overline{R} : R \in \Sigma^0_n\}$$
$$\Sigma^1_{n+1} = \{\exists^1 R : R \in \Pi^1_n\}$$
$$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$$

Relativized (boldface) hierarchies

For $\beta \in \{0,1\}^\omega$, let $R[\beta] = \{\langle \mathbf{m}, \alpha \rangle : R(\mathbf{m}, \alpha, \beta)\}$.

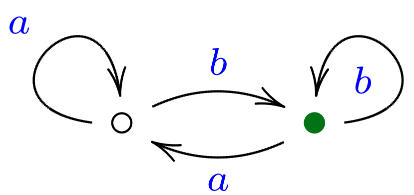
$$\boldsymbol{\Sigma}^i_n = \{R[\beta] : R \in \Sigma^i_n, \beta \in \{0,1\}^\omega\} \qquad \boldsymbol{\Delta}^i_n = \boldsymbol{\Sigma}^i_n \cap \boldsymbol{\Pi}^i_n$$
$$\boldsymbol{\Pi}^i_n = \{R[\beta] : R \in \Pi^i_n, \beta \in \{0,1\}^\omega\} \qquad i \in \{0,1\}$$

$$\boldsymbol{\Sigma}^0_1 = \mathit{open}$$
$$\boldsymbol{\Pi}^0_1 = \mathit{closed}$$
$$\boldsymbol{\Delta}^1_1 = \mathit{Borel}$$

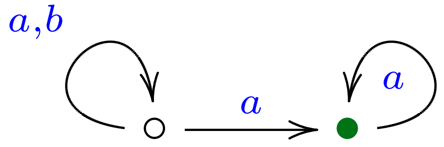
Büchi automata on infinite words

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, F \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q$, $F \subseteq Q$.



$$((a+b)^*b)^\omega$$



$$(a+b)^*a^\omega$$

The second one cannot be recognized by a deterministic automaton.

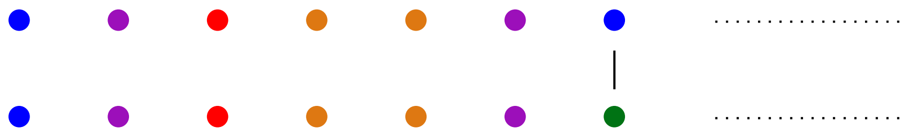
$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\;\ldots\ldots\;\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\;\ldots$$

So $(a+b)^* a^\omega$ cannot be recognized by a deterministic automaton.

But this also follows by a topological argument!

We assume the Cantor topology on X^ω, induced by the metric

$$d(u,v) = 2^{-\min\{m \,:\, u_m \neq v_m\}}$$



If A is **deterministic** then the mapping

$$\Sigma^\omega \ni u \quad \mapsto \quad run(u) \in Q^\omega$$

satisfies $$L(A) \ni u \quad \Longleftrightarrow \quad run(u) \in (Q^* F)^\omega,$$

hence **continuously reduces** $L(A)$ to $(Q^* F)^\omega$.

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But

- $(a+b)^*a^\omega \in \mathbf{\Sigma}_2^0 - \mathbf{\Pi}_2^0$,
- $(Q^*F)^\omega \in \mathbf{\Pi}_2^0$.

Hence $(a+b)^*a^\omega$ **cannot** be recognized by a deterministic automaton.

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- $(Q^*F)^\omega \in \mathbf{\Pi}_2^0$.

Hence $(a+b)^*a^\omega$ **cannot** be recognized by a deterministic automaton.

photo P.Jahr

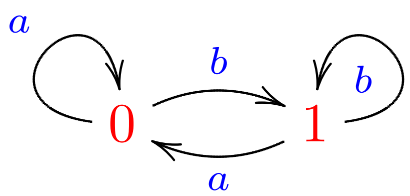
?

Parity automata

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \mathit{rank} \rangle$$

where $\mathit{rank} : Q \longrightarrow \{0, 1, \ldots, k\}$.

A run is accepting if $\limsup_{i \to \infty} \mathit{rank}(q_i)$ is even.



$$(a + b)^* a^\omega$$

The Rabin-Mostowski **index** of a parity automaton $\mathcal{A}$ is

$$(\min \mathit{rank}, \max \mathit{rank})$$

We can assume $\min \mathit{rank} \in \{0, 1\}$.

The McNaughton Theorem (1966)

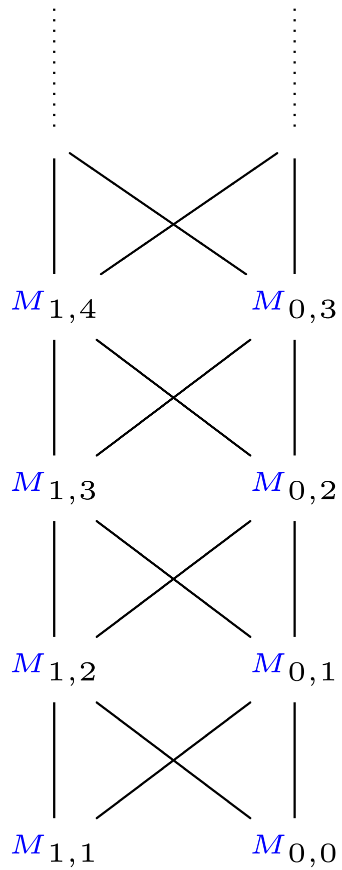
A nondeterministic Büchi automaton can be simulated by a deterministic parity automaton of some index (i, k).

The minimal index (i, k) may be arbitrarily high (Wagner 1979, Kaminski 1985).

Again, it can be inferred by a topological argument.

Let

$$M_{i,k} = \{u \in \{i, \ldots, k\}^{\omega} : \limsup_{\ell \to \infty} u_{\ell} \text{ is even}\}$$



No continuous reduction down the hierarchy.

Wadge game $G(A, B)$

Spoiler	Duplicator
$a_0 \in \Sigma$	$b_0 \in \Sigma$
a_1	b_1
a_2	b_2
$\vdots$	$\vdots$
a_{12}	b_{12}
a_{13}	wait
a_{14}	wait
a_{15}	b_{13}
$\vdots$	$\vdots$

Here $A, B \subseteq \Sigma^\omega$ (Σ finite).

Duplicator wins if $a_0 a_1 a_2 \ldots \in A \iff b_0 b_1 b_2 \in B$.

Fact

Duplicator has a winning strategy iff there is a continuous $f : \Sigma^\omega \to \Sigma^\omega$ s.t. $A = f^{-1}(B)$,

in symbols, $A \leq_w B$.

Spoiler 's strategy, e.g., in $G(M_{0,5}, M_{1,6})$

Spoiler	Duplicator
0	4
3	5
4	1
⋮	⋮
	i
$i-1$	
⋮	⋮
	wait
0	
⋮	⋮

Note

If a deterministic automaton of index $(1, 6)$, accepted $M_{0,5}$ there would be a continuous reduction of $M_{0,5}$ to $M_{1,6}$

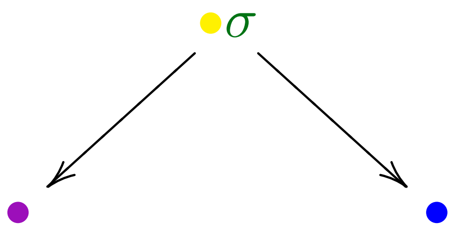
$u \mapsto \mathit{rank} \circ run(u)$.

Contradiction!

Parity tree automata

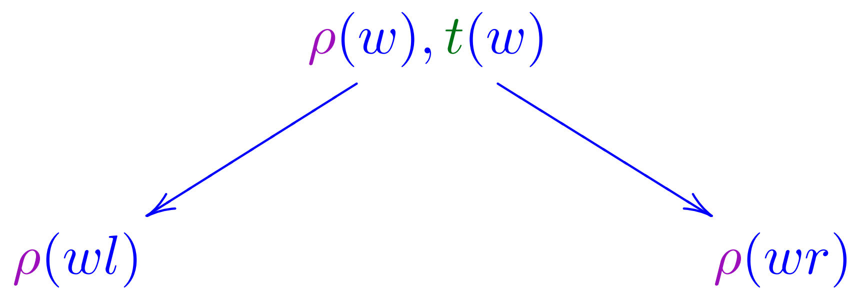
$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q \times Q$, $\textit{rank} : Q \to \{0, 1, \ldots, k\}$.



Parity tree automata cont'd

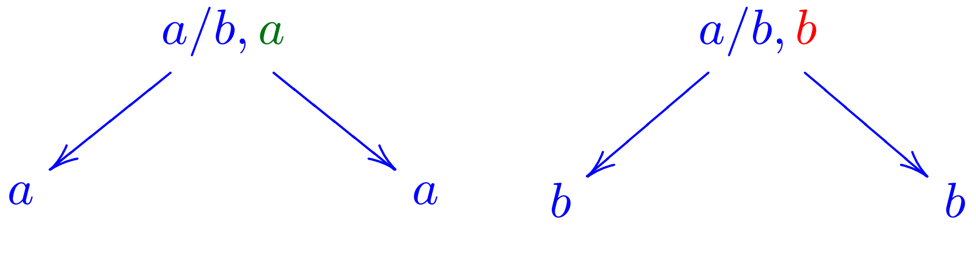
A run of $\mathcal{A}$ on a tree $t : \{l, r\}^* \longrightarrow \Sigma$ is a tree $\rho : \{l, r\}^* \longrightarrow Q$, such that, $\langle \rho(w), t(w), \rho(wl), \rho(wr) \rangle \in Tr$, for each $w \in$ dom (ρ)



The run is accepting if, for each path $P = p_0 p_1 \ldots \in \{l, r\}^\omega$,

$$\limsup_{k \to \infty} \textit{rank}(\rho(p_0 p_1 \ldots p_k)) \text{ is even.}$$

Example

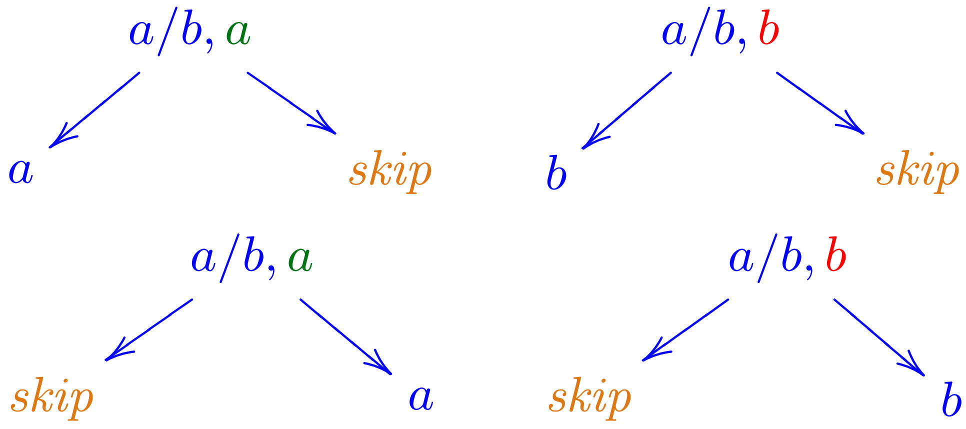


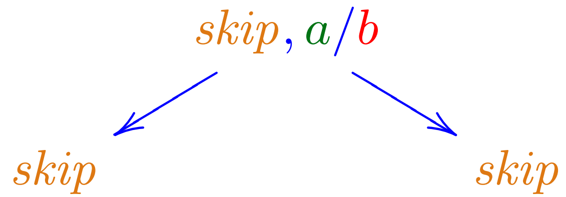
$rank(a) = 0$

$rank(b) = 1$

recognizes the set of trees where, on each branch, b appears only finitely often.

The complement can be recognized by

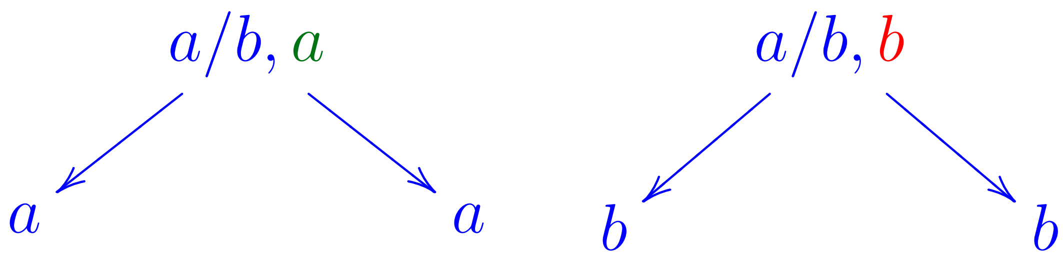




$rank(a) = 1$

$rank(b) = rank(skip) = 2$

Example cont'd



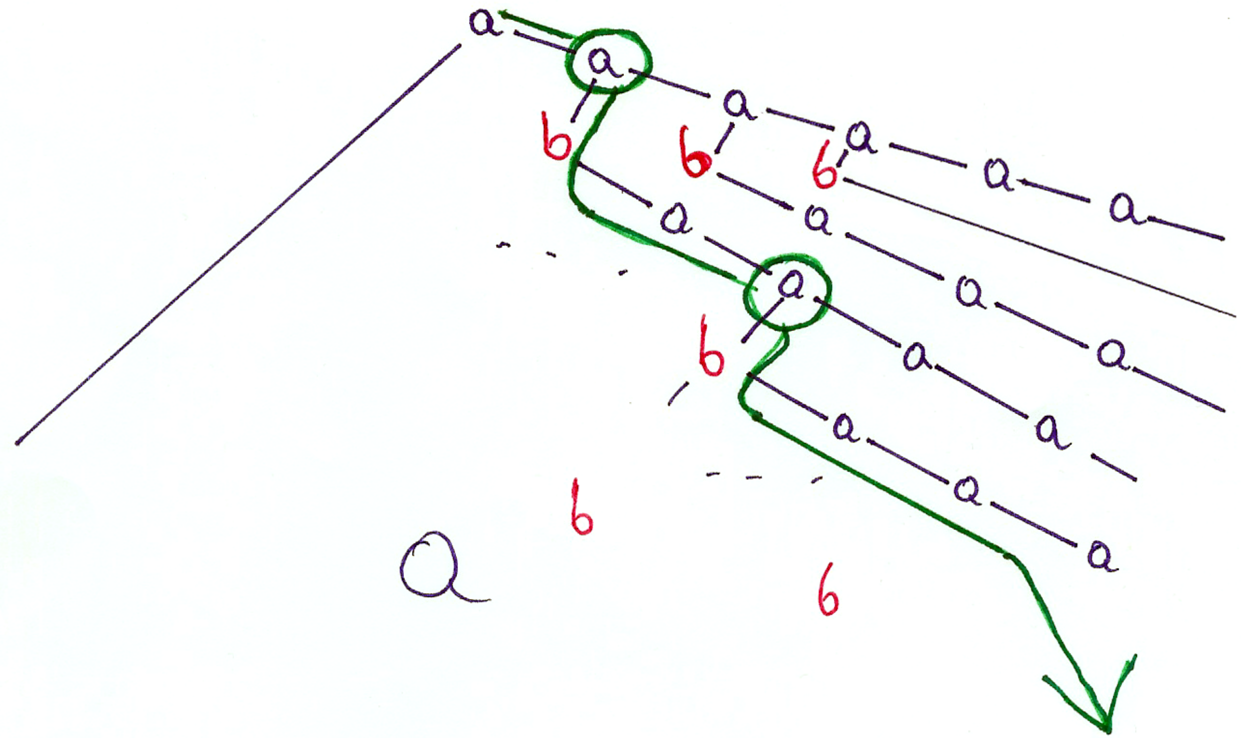
$\textit{rank}(a) = 0$

$\textit{rank}(b) = 1$

This set cannot be recognized by a Büchi automaton

(i.e., of index (1,2)), Rabin 1970.

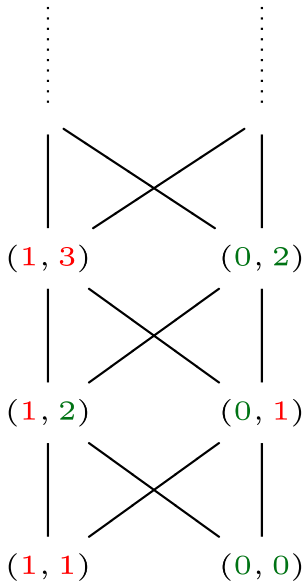
Rabin's proof



Again, a topological argument could be used instead, as this set is Π^1_1 complete, while the Büchi automata can recognize only Σ^1_1 sets.

Rabin's example generalizes to

$$T_{i,k} = \{t \in \{i, \ldots, k\}^{\{l,r\}^*} : \text{ each branch is in } M_{i,k} \}$$



These sets witness the strictness of the non-deterministic index hierarchy (N 1986).

A topological argument is not helpful here, as all the sets $T_{i,k}$ are $\mathbf{\Pi}^1_1$ complete, hence Wadge–equivalent (except for $T_{0,0}$, $T_{1,1}$, $T_{1,2}$).

But topology comes back in the proof of the strictness of the alternating index hierarchy (Bradfield 1996, 1998, another proof by Arnold 1998).

Parity games

$V_\exists$	positions of Eve
$V_\forall$	positions of Adam (disjoint)
$\longrightarrow \subseteq V \times V$	possible moves (with $V = V_\exists \cup V_\forall$)
$p_1 \in V$	initial position
rank : $Q \to \omega$	the ranking function.

An infinite play $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots$ is won by Eve iff $\limsup_{n\to\infty}$ *rank*(v_n) is even.

Parity games enjoy *positional determinacy*

(Emerson and Jutla 1991, Mostowski 1991).

Game tree languages

Alphabet : $\{\exists, \forall\} \times \{i, \ldots, k\}$.

Eve : $\exists, j$ $\qquad$ $\exists, j$

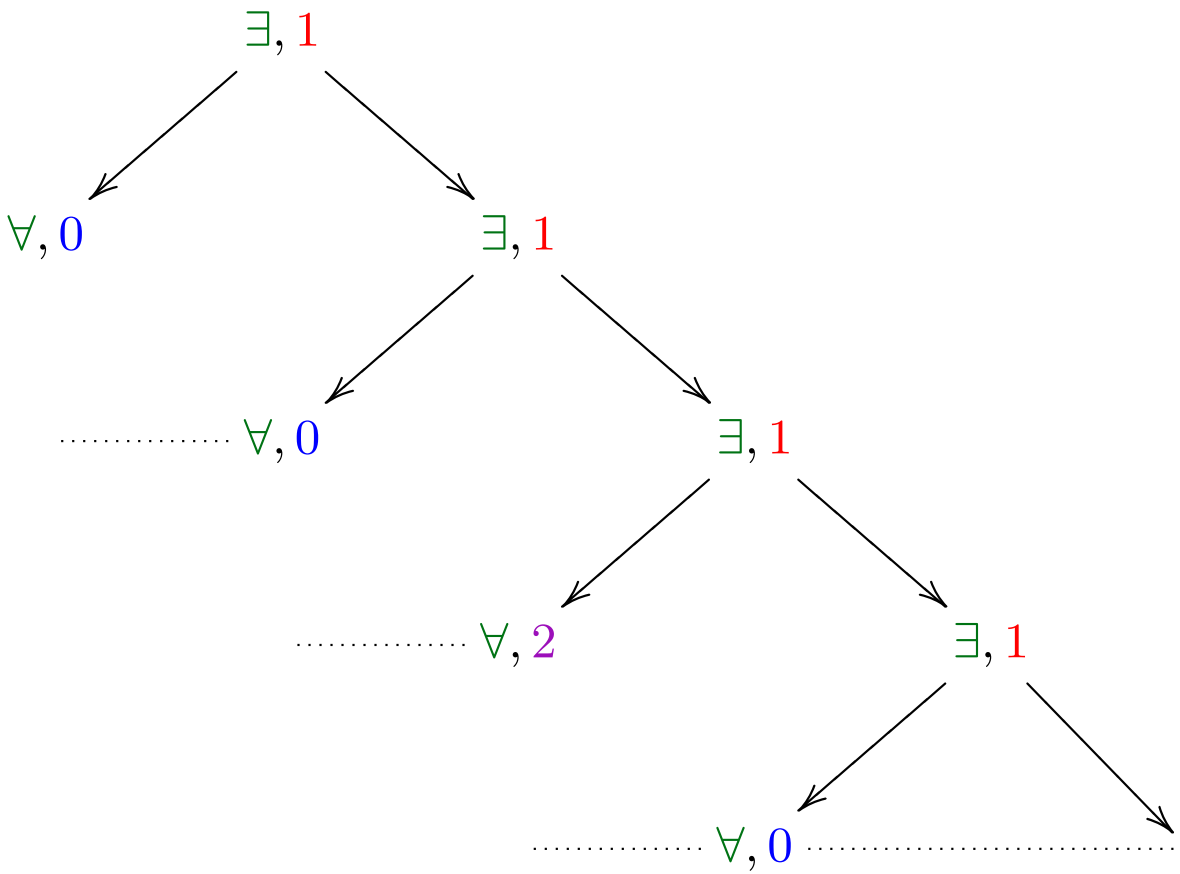
Adam : $\forall, j$ $\qquad$ $\forall, j$

Eve wins an infinite play $(x_0, i_0),\ (x_1, i_1),\ (x_2, i_2), \ldots \quad (x_\ell \in \{\exists, \forall\})$

iff $\limsup_{\ell \to \infty} i_\ell$ is even.

The set $W_{i,k}$ consists of all trees such that Eve has a winning strategy.

Example



Alternating parity tree automata

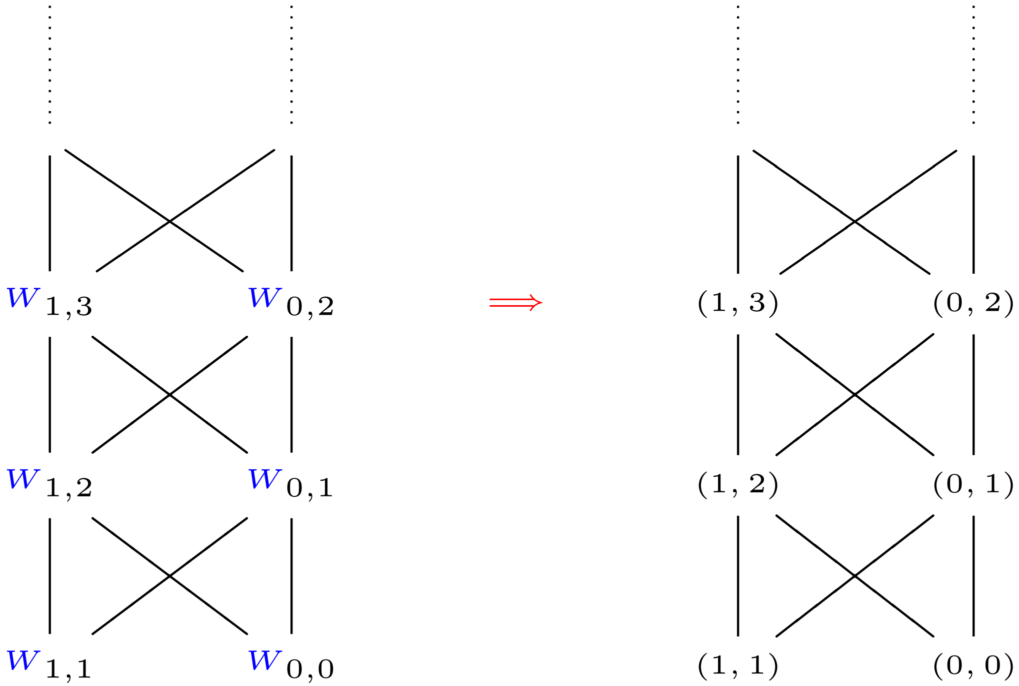
$$\mathcal{A} = \langle \Sigma, Q, Q_\exists, Q_\forall, q_I, Tr, \textit{rank} \rangle$$

where $Q = Q_\exists \dot\cup Q_\forall$,
$Tr \subseteq Q \times \Sigma \times \{1, 2, \varepsilon\} \times Q$,
rank : $Q \to \{0, 1, \ldots, k\}$.

An input tree t is accepted by $\mathcal{A}$ iff Eve has a winning strategy in the parity game

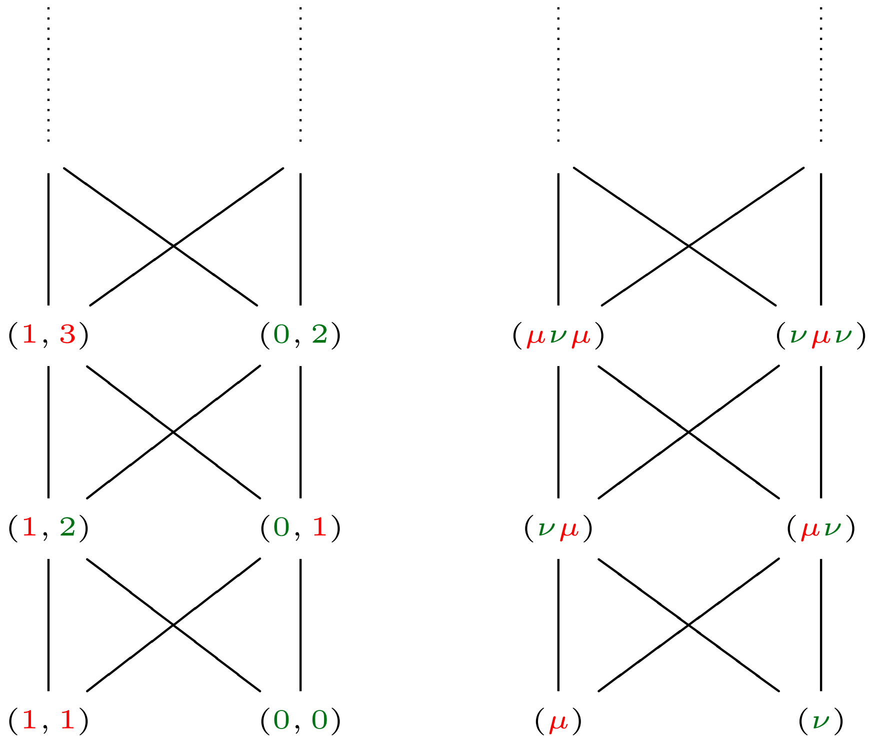
$Q_\exists \times \{1, 2\}^*$,
$Q_\forall \times \{1, 2\}^*$,
(q_0, ε),
$\mathrm{Mov} = \{((p, v), (q, vd)) : v \in \mathrm{dom}(t),\ (p, t(v), d, q) \in Tr\}$
$\textit{rank}(q, v) = \mathrm{rank}(q)$.

Topological argument: The sets $W_{i,k}$ form a strict hierarchy w.r.t. the Wadge reducibility (Arnold & N, 2008).



This gives an alternative proof of the strictness of the alternating index hierarchy (Bradfield 1998).

The importance of the alternating index hierarchy follows from its relation to the μ-calculus.



An idea of the proof:

Given an alternating automaton $\mathcal{A}$ of index (i, k),
an input tree t induces a **full tree of the game** $G(\mathcal{A}, t)$.

The mapping $t \mapsto G(\mathcal{A}, t)$ is continuous and satisfies

$$L(\mathcal{A}) \ni t \quad \Longleftrightarrow \quad G(\mathcal{A}, t) \in W_{i,k}$$

hence it **reduces** $L(\mathcal{A})$ to $W_{i,k}$.

Therefore any automaton recognizing $W_{i,k}$ must have an index **at least** (i, k).

Sketch of proof that $W_{\overline{i,k}} \not\leq_w W_{i,k}$:

Up to renaming,

$$W_{\overline{i,k}} \approx \overline{W_{i,k}}$$

By Banach Fixed-Point Theorem, there is no contracting reduction of L to $\overline{L}$

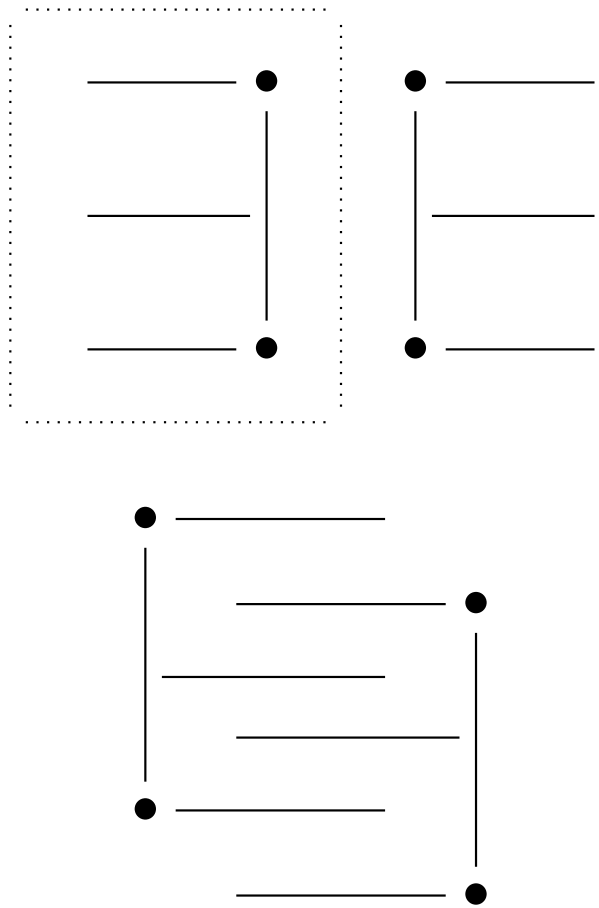
$$x_{\mathit{fix}} \in L \iff f(x_{\mathit{fix}}) \in \overline{L} \iff x_{\mathit{fix}} \in \overline{L}$$

Main Lemma If f reduces $W_{i,k}$ to some L then there is a mapping $h : \{i, \ldots, k\}^{\{l,r\}^*} \to \{i, \ldots, k\}^{\{l,r\}^*}$ (padding), such that

- h reduces $W_{i,k}$ to itself,
- $f \circ h$ is contracting.

About h: For $W_{0,k}$, it "stretches" the original tree completing by the nodes labeled by $(\forall, 0)$. For $W_{1,k}$, by $(\exists, 1)$.

Separation problem



Given disjoint sets A and B, find a *simple* set C, such that $A \subseteq C \subseteq \overline{B}$

We are interested in separation of sets of trees

photo M.Bojańczyk

Lusin separation theorem.

Any two disjoint $\mathbf{\Sigma}_1^1$ sets are separable by a Borel set.

There exist two disjoint $\mathbf{\Pi}_1^1$ sets **not** separable by a any Borel set.

We show that these sets can be chosen as regular sets of trees.

Let $W'_{0,1}$ be obtained from $W_{0,1}$ by interchanging $\exists \leftrightarrow \forall$ and $0 \leftrightarrow 1$.

That is, in $W'_{0,1}$ Adam has a strategy to force that there is only finitely many 0's.

Note that $W'_{0,1}$ is a *copy* of $W_{0,1}$ included in $\overline{W_{0,1}}$.

Fact (Hummel, Michalewski, N. 2009)

$W_{0,1}$ and $W'_{0,1}$ are not separable by any Borel set.

Consequently, $W_{0,1}$ and $W'_{0,1}$ are not separable by any set definable in **weak** monadic second-order logic (WS2S)

$$\Updownarrow$$

of index simultaneously $(1,2)$ and $(0,1)$ (Büchi and co-Büchi) (Rabin 1970)

$$\Updownarrow$$

recognized by **weak** alternating automaton (Muller, Schupp, Saoudi 1986)

$$\Updownarrow$$

definable in the **alternation-free** μ-calculus (Arnold & N. 1991).

In contrast, any two disjoint sets of trees recognizable by automata of index $(1,2)$ (Büchi) **are** separable by a set from this class (Rabin 1970).

Lemma For any Borel set $B \subseteq \{0,1\}^\omega$, there is a continuous reduction

$$f : \{0,1\}^\omega \to \mathbb{T}_{\{0,1\}\times\{\exists,\forall\}}$$

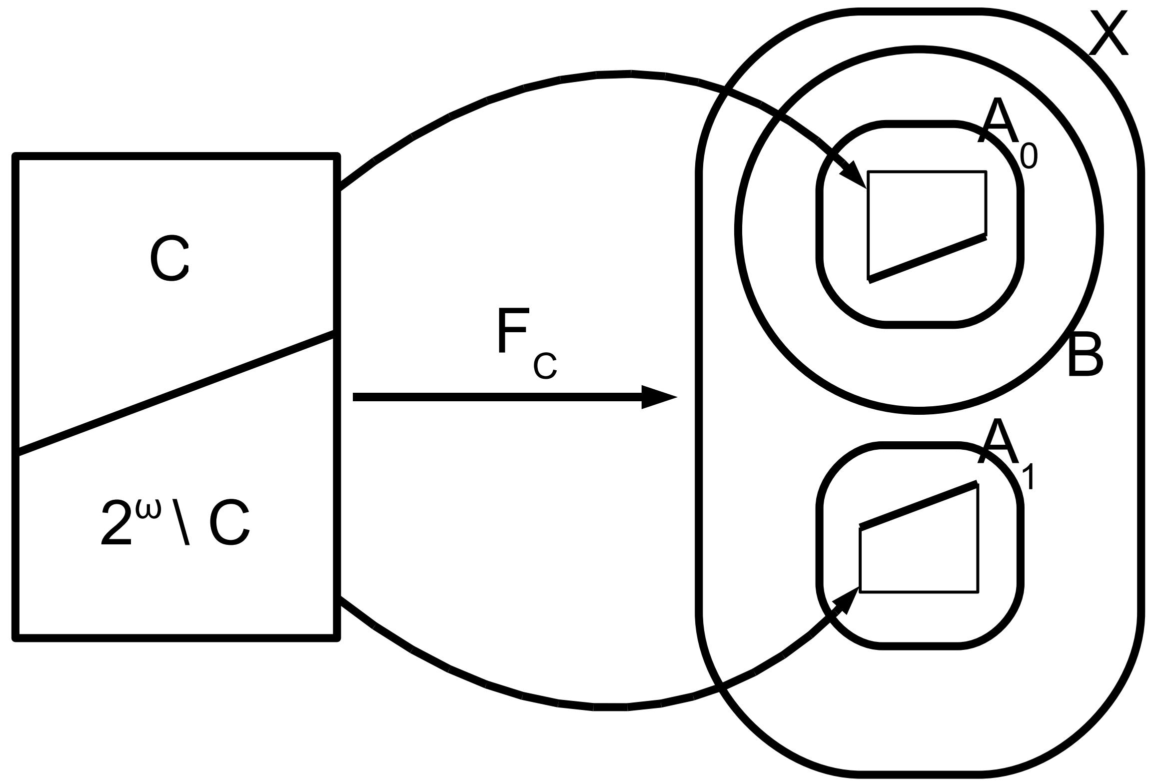
such that

$$\begin{aligned} u \in B \quad &\Rightarrow \quad f(u) \in W_{0,1} \\ u \notin B \quad &\Rightarrow \quad f(u) \in W'_{0,1} \end{aligned}$$

If there were a Borel set C s.t. $W_{0,1} \subseteq C \subseteq \overline{W'_{0,1}}$, we would have

$$\begin{aligned} u \in B \quad &\Rightarrow \quad f(u) \in C \\ u \notin B \quad &\Rightarrow \quad f(u) \in \overline{C} \end{aligned}$$

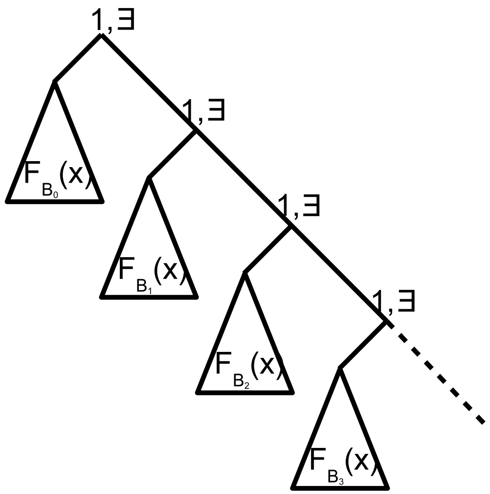
Hence any Borel set would be a continuous inverse image of C, which is impossible, since the Borel hierarchy is strict.


C
$2^\omega \setminus C$
F_C
X
A_0
B
A_1

Proof of the lemma. For a clopen set B it is enough to take

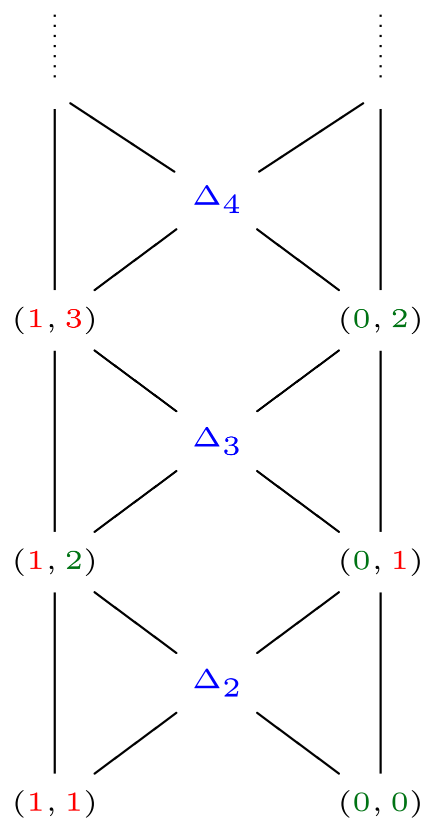
$$B \ni x \quad \mapsto \quad t_1 \in W_{0,1}$$
$$B \not\ni x \quad \mapsto \quad t_2 \in W'_{0,1}$$

Suppose $B = \bigcup B_n$ and we have suitable reductions F_{B_n} .

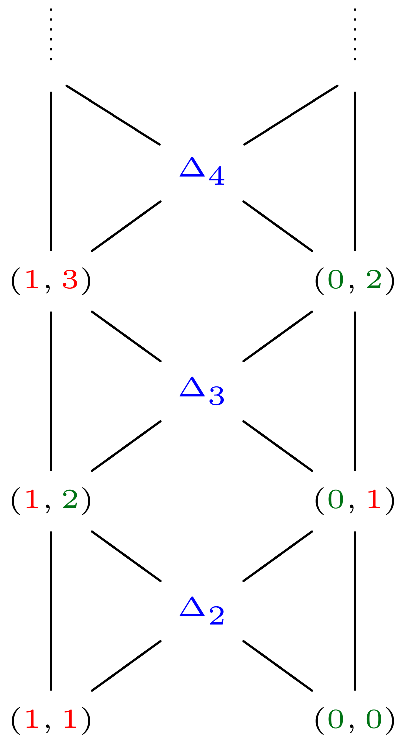


The case $B = \bigcap B_n$ follows from symmetry.

What about the next levels of the hierarchy?



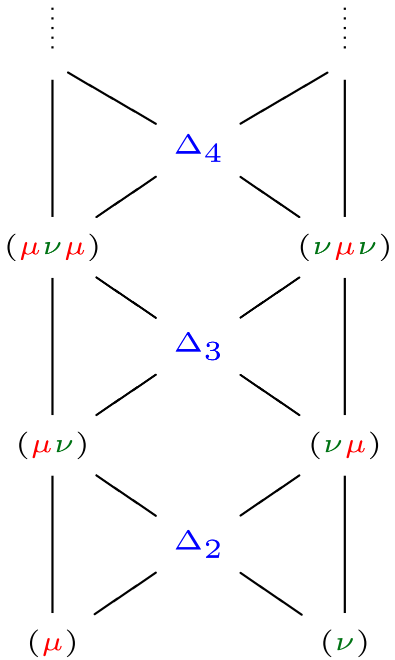
Here $\Delta_k = (1, k) \cap (0, k-1)$.



Conjecture Separation property in the class (i, n):

- holds for n **even**,
- fails for n **odd**.

In terms of the μ-calculus



Conjecture Separation property

- holds for $\nu\ldots$ classes,
- fails for $\mu\ldots$ classes.

Evidence

For n odd, one can define a copy $W'_{i,n} \subseteq \overline{W_{i,n}}$ by renaming, and this is impossible for n even.

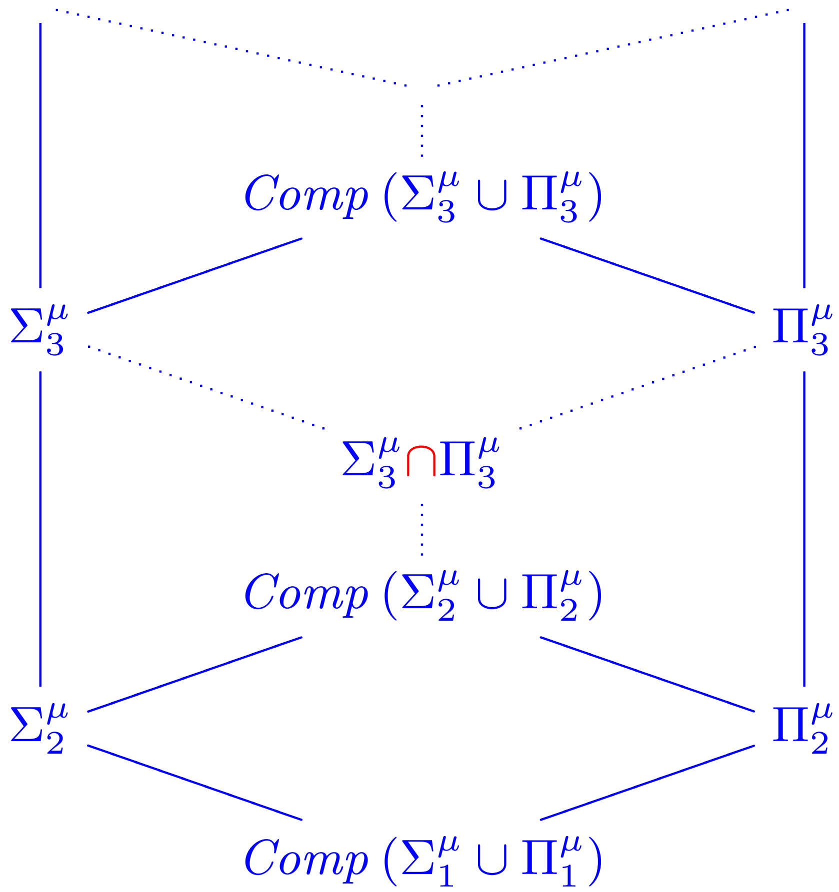
For example, $W'_{1,3}$:

$$\begin{array}{ccc} \exists & \leftrightarrow & \forall \\ 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 \end{array}$$

Conjecture: $W_{i,n}$ and $W'_{i,n}$ (for n odd) are inseparable by a Δ set.

Result: $W'_{1,3}$ is complete in the class of $\mathbf{\Sigma}^1_1$**-inductive** sets.

Related work: Santocanale and Arnold 2005 showed that the characterization of Δ_2 in terms of the alternation free μ-calculus does not extend to $n \geq 3$.



Conclusion

Topological complexity often, but not always, underlines the automata-theoretic complexity.

Topological arguments appear to work better for deterministic or alternating, rather than for non-deterministic automata (?).

The concept of inseparability sheds some light on the fine structure of finite-state recognizable sets of trees.

Parity games seem to be the core concept of the theory.