These mysterious game tree languages

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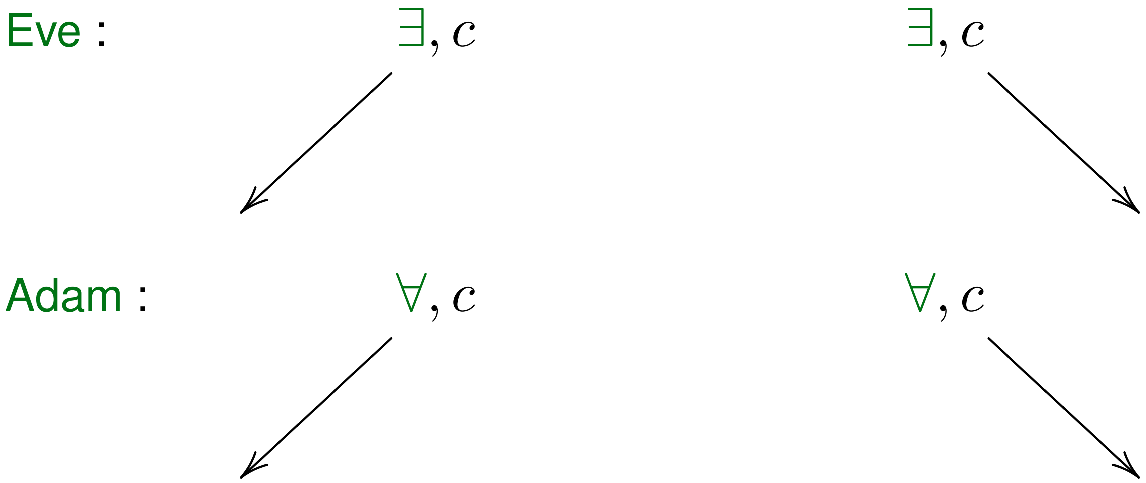
Game tree languages

A game on a tree $t : 2^* \to \{\exists, \forall\} \times C$, with condition $L \subseteq C^\omega$.



Eve wins an infinite play $(e_0, j_0),\ (e_1, j_1),\ (e_2, j_2), \ldots \quad (e_\ell \in \{\exists, \forall\})$

iff $j_0\, j_1\, j_2\, \ldots \in L$.

$$\mathrm{Win}^{\exists}(L) \quad = \quad \{t : \text{ Eve has a winning strategy } \}$$

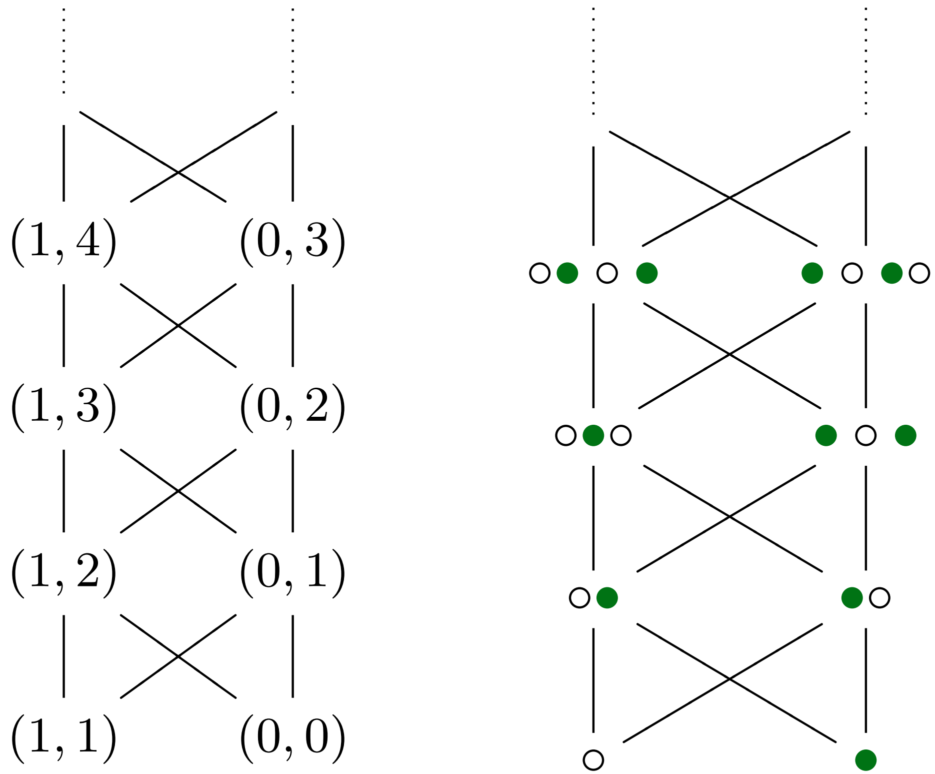
Parity game tree languages

$$\begin{aligned} A_{i,k} &= \{i, \dots, k\} \\ L_{i,k} &= \{u \in A_{i,k}^{\omega} \,:\, \limsup_{n\to\infty} u_n \text{ is even}\} \\ W_{i,k} &= \mathrm{Win}^{\exists}(L_{i,k}) \end{aligned}$$

If only Adam plays,

$$T_{i,k} = \{t \in A_{i,k}^{2^*} \,:\, (\forall \alpha \in 2^{\omega})\, t \restriction \alpha \in L_{i,k}\}$$

Hierarchy of indices



Dual indices: $\overline{(0, k)} = (1, k + 1)$.

No deterministic parity automaton of index (i,k) can recognize the set $L_{\overline{(i,k)}}$.

Consequently, the hierarchy of the Rabin-Mostowski indices is **strict**

(**Wagner 1979, Kaminski 1985**).

More generally, let $R \subseteq C^{\omega}$. A deterministic R-automaton on infinite words is

$\langle A,\ Q, q_I,\ Tr : Q \times A \to Q,\ rank : Q \to C \rangle$.

$$
\begin{array}{cccccccccc}
 & q_I & & & & & & & & \\
 & \| & & & & & & & & \\
\text{A run} & q_0 \xrightarrow{a_0} & q_1 \xrightarrow{a_1} & q_2 \xrightarrow{a_2} & q_3 \xrightarrow{a_3} & \dots & \text{is} & \textbf{accepting} \\
\text{iff} & rank(q_0) & rank(q_1) & rank(q_2) & rank(q_3) & \dots & \in & R.
\end{array}
$$

Parity automaton of index (i,k) is an $L_{i,k}$-automaton.

No deterministic R-automaton (over alphabet C) may accept $\overline{R}$.

Suppose

$$L(\mathcal{A}) \quad = \quad \overline{R}$$

Create a word

$$q_0 \xrightarrow{rank(q_0)} q_1 \xrightarrow{rank(q_1)} q_2 \xrightarrow{rank(q_2)} q_3 \xrightarrow{rank(q_3)} \quad \ldots$$

$$u = \quad rank(q_0) \quad rank(q_1) \quad rank(q_2) \quad rank(q_3) \quad \ldots$$

Then

$$u \in \overline{R} \iff u \in R,$$

a contradiction.

Remark

There is a **single** *non-regular* language R, such that **any** ω-regular language can be recognized by a deterministic R-automaton (but also some non-regular ones).

For example (M. Skrzypczak), a “universal” parity condition $R \subseteq \{0,1\}^\omega$

$$R = \{0^{m_0}\, 1\, 0^{m_1}\, 1\, 0^{m_2}\, 1\, \ldots : \limsup m_n \text{ is an } \textit{even } < \omega\}$$

From words to trees

Let $R \subseteq C^\omega$. An alternating R-automaton over binary trees $t : 2^* \to A$ is

$$Q = Q_\exists \dot\cup Q_\forall \qquad Tr \subseteq Q \times A \times \{0, 1, \varepsilon\} \times Q$$

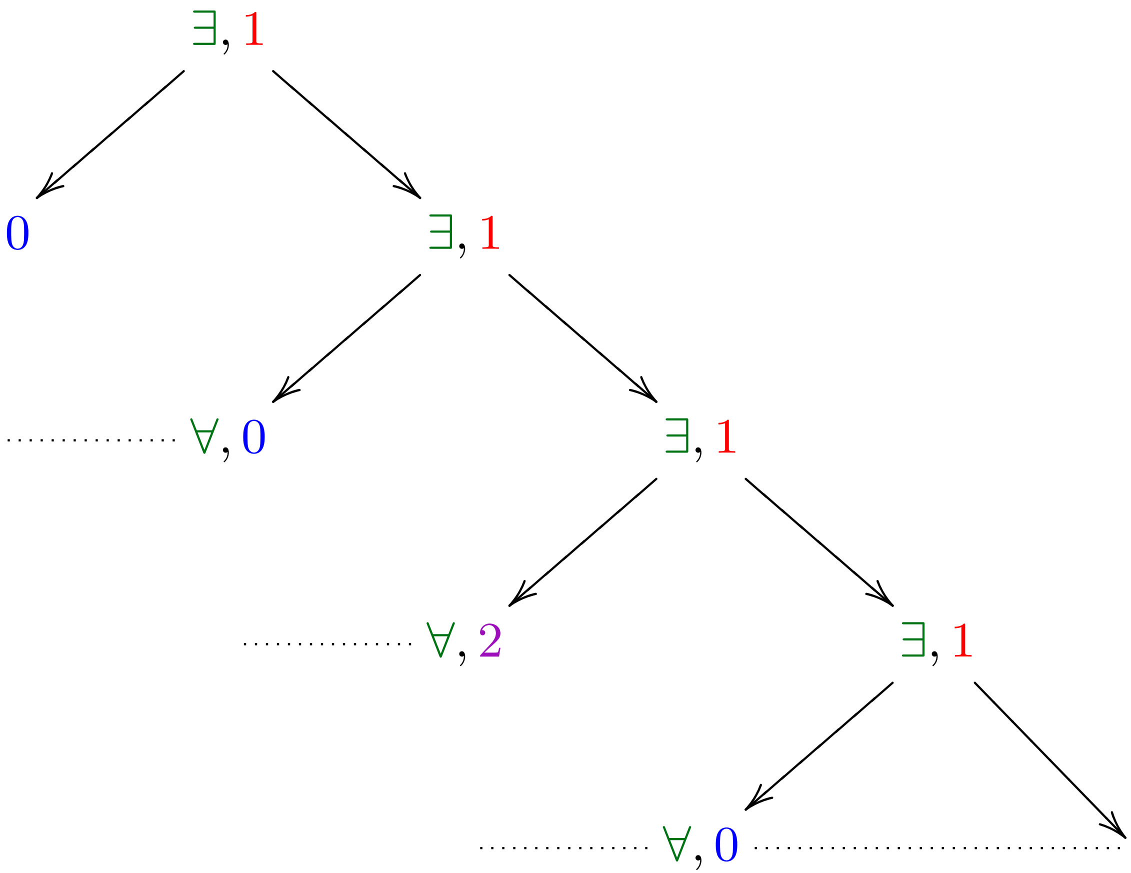
$$q_I \in Q \qquad rank : Q \to C$$

An input tree t is accepted by the automaton $\mathcal{A}$ iff Eve has a winning strategy in the game $G(\mathcal{A}, t)$

$Q_\exists \times 2^*$,	Eve's
$Q_\forall \times 2^*$,	Adam's
(q_I, ε),	initial
$\{((p, v), (q, vd)) : v \in \mathrm{dom}(t),\ (p, t(v), d, q) \in Tr\}$	moves
$rank(q, v) = \mathrm{rank}(q)$	ranking
R	winning Eve

Note. If $L \subseteq A^\omega$ is recognized by a deterministic R-automaton then $\mathrm{Win}^\exists(L)$ is recognized by an alternating R-automaton.

Example $L = R = L_{0,2}$.



No R-automaton (over alphabet $\{\exists, \forall\} \times C$) may accept $\overline{\mathrm{Win}^{\exists}(R)}$.

We use the concept of a **game tree**.

Recall that $\mathcal{A}$ accepts t iff Eve wins the game $G(\mathcal{A}, t)$ with the set of positions $2^* \times Q$ and condition R.

Unravel this game to a tree.

For a position (v, q), retain only the label $(own(q), rank(q))$, where

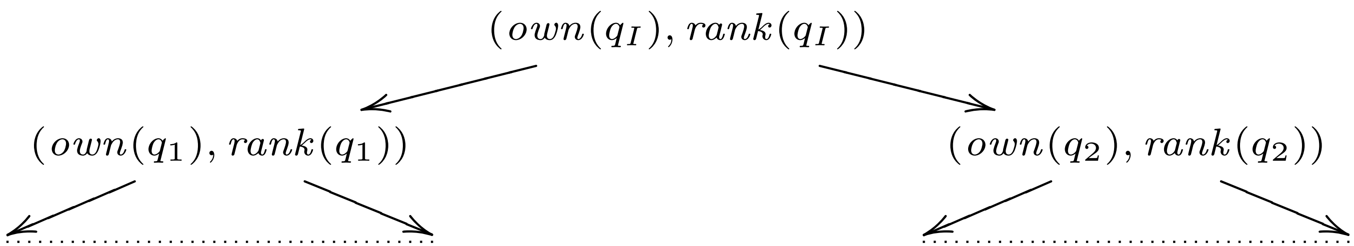
$$\begin{aligned} own(q) = \exists &\quad \text{iff} \quad q \in Q_\exists \\ own(q) = \forall &\quad \text{iff} \quad q \in Q_\forall . \end{aligned}$$

Claim. $\mathcal{A}$ accepts t iff the game tree (*mutatis mutandis*) is in $\mathrm{Win}^{\exists}(R)$.

Suppose, for an alternating R-automaton $\mathcal{A}$,

$$L(\mathcal{A}) \quad = \quad \overline{\mathrm{Win}^{\exists}(R)}.$$

Create a tree f



where

$$(q_I, (own(q_I), rank(q_I)), d_1, q_1), (q_I, (own(q_I), rank(q_I)), d_2, q_2) \in Tr.$$

Then

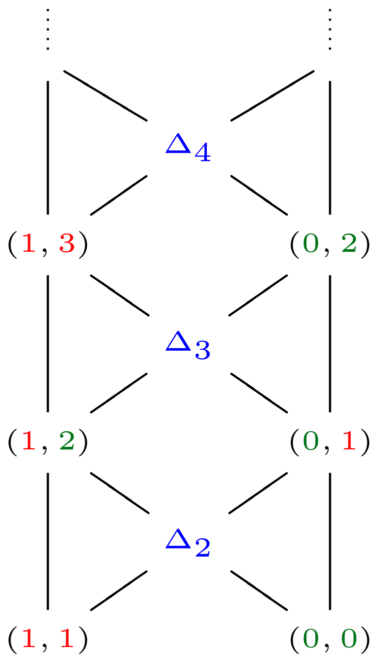
$$f \in \overline{\mathrm{Win}^{\exists}(R)} \quad \Longleftrightarrow \quad f \in \mathrm{Win}^{\exists}(R),$$

a contradiction.

Recall $W_{i,k} = \mathrm{Win}^{\exists}(L_{i,k})$.

Then $W_{0,k} \approx \overline{W_{1,k+1}}$ cannot be accepted by $(1, k+1)$-automaton

Hence the hierarchy of alternating tree automata



is **strict**, as proved by **Bradfield 1998** (credit to Walukiewicz for example), cf. also another proof by **Arnold 1999**.

Strictness of the Rabin-Mostowski index hierarchy for **non-deterministic** tree automata can be witnesses by a a family of simpler languages (**N. 1986**)

$$T_{i,k} \quad = \quad \{t : \, (\forall \alpha \in 2^{\omega}) \, t \restriction \alpha \in L_{i,k}\}$$

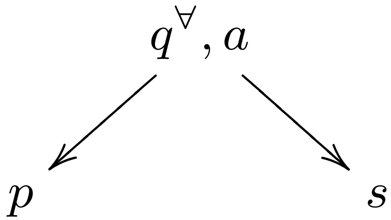
Note that these languages can be recognized by **deterministic** automata.

On the complexity of game tree languages $W_{i,k}$

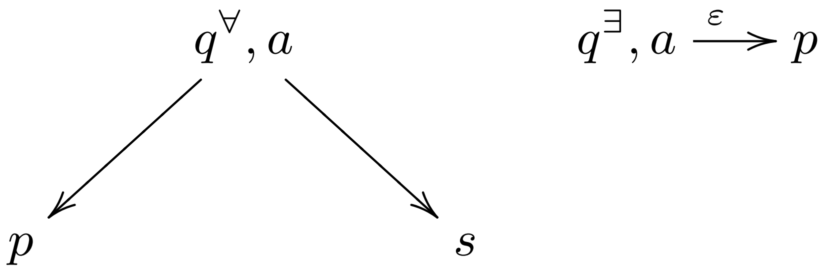
photo M.Bojańczyk

Restrictions on automata

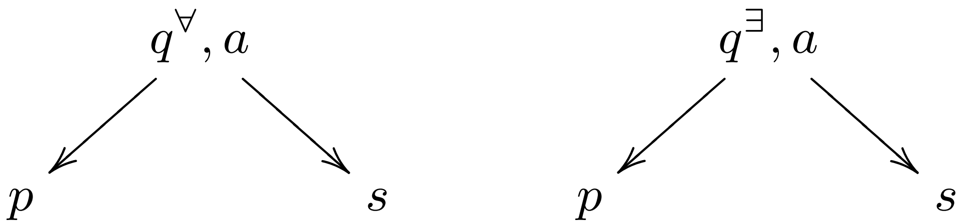
Deterministic



Nondeterministic



Game automata



Easy side of game tree languages

Tree languages of the form $\mathrm{Win}^{\exists}(L)$, for an ω-regular L, are recognizable by game automata. So are, in particular, the $W_{i,k}$.

This class, say $\mathbb{GA}$, appears more tractable than general non-deterministic (alternating) automata.

The Rabin-Mostowski index problem, open in general, is decidable for tree languages in $\mathbb{GA}$, both for non-deterministic and alternating hierarchy (**Facchini, Murlak, Skrzypczak 2013**).

Recently, **Michalewski and Mio 2015** showed an algorithm to compute probability of tree languages in $\mathbb{GA}$ in the coin-flipping measure.

In particular the measure of $W_{i,k}$ is **1** for k even, and **0**, for k odd.

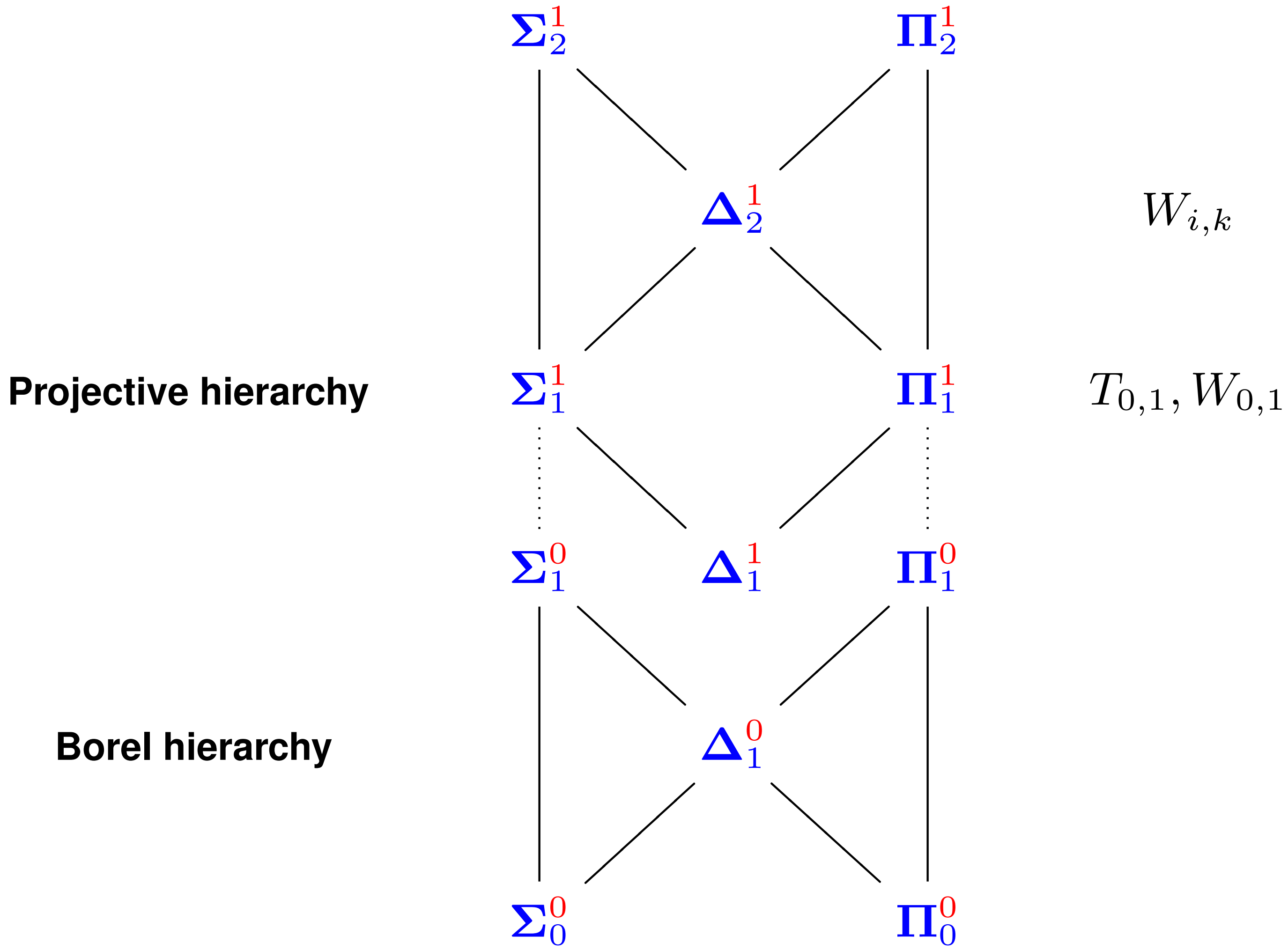
In random tree parity game, the highest priority indicates the winner.

$$
\begin{aligned}
W_{1,3} \quad = \quad & \mu x_3.\nu x_2.\mu x_1.a_{\forall,1}(x_1, x_1) \cup a_{\exists,1}(x_1, \top) \cup a_{\exists,1}(\top, x_1) \\
& \cup a_{\forall,2}(x_2, x_2) \cup a_{\exists,2}(x_2, \top) \cup a_{\exists,2}(\top, x_2) \\
& \cup a_{\forall,3}(x_3, x_3) \cup a_{\exists,3}(x_3, \top) \cup a_{\exists,3}(\top, x_3)
\end{aligned}
$$

$$
\begin{aligned}
p(W_{1,3}) \quad = \quad & \mu x_3.\nu x_2.\mu x_1.\frac{1}{6}\,(x_1 \cdot x_1 + x_1 \odot x_1 \\
& x_2 \cdot x_2 + x_2 \odot x_2 \\
& x_3 \cdot x_3 + x_3 \odot x_3)
\end{aligned}
$$

where x_i range over $[0, 1]$, and $x \odot y = 1 - (1 - x)(1 - y) = x + y - xy$.

Difficult side of game tree languages



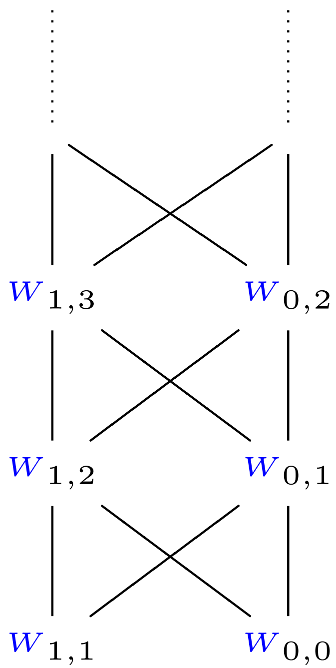
Already $T_{0,1}$ is $\mathbf{\Pi^1_1}$-complete, hence beyond the Borel hierarchy, wrt. the Cantor topology. Consequently are all $T_{i,k}$, for higher indices, as well.

A fortiori $W_{0,1}$ is $\mathbf{\Pi^1_1}$-complete, and $W_{1,2}$, $\mathbf{\Sigma^1_1}$-complete.

By Rabin's Complementation Lemma, all recognizable sets of trees are in $\mathbf{\Delta^1_2}$.

Any set of trees recognizable by an alternating automaton $\mathcal{A}$ of index (i,k) reduces to $W_{i,k}$ by a continuous "transducer" $t \mapsto G(\mathcal{A},t)$ (**Arnold 1999**).

So the $W_{i,k}$ are **Wadge complete** on the respective levels of the hierarchy.



They form a **strict hierarchy** w.r.t. the Wadge reducibility (**Arnold & N, 2008**).

Beyond $\sigma(\Sigma^1_1)$ (cf. also **Finkel & Simonnet 2009, Hummel 2012**)

$W_{1,3}$ is complete in the class of $\mathbf{\Sigma^1_1}$**-inductive** sets defined by Moschovakis (**Michalewski & N. 2012**).

The complexity of

$$F : Trees_\Sigma \times \wp(2^*) \longrightarrow \wp(2^*)$$

is the complexity of the relation

$$w \in F(t, X)$$

Fixed-point expressions are interpreted by

$$[\![\mu X.F]\!] =_{def} \{t : \varepsilon \in \mu X.F(t, X)\}$$

The $\mathbf{\Sigma^1_1}$-inductive sets of trees are those definable by $[\![\mu X.F]\!]$, with F is $\mathbf{\Sigma^1_1}$.

Proof *via* a reduction from another game, considered by **Saint Raymond 2006**.

A tree $t \subseteq \omega^*$ is *cofinal* if for every $v = (v_0, v_1, \ldots) \in \omega^\omega$ there exists a branch $(b_0, b_1, \ldots)$ in t, such that

$$(\forall i)\ b_i \geq v_i.$$

In a game $\Gamma(t)$, Player I plays natural numbers n_0, n_1, $\ldots$, and Player II answers with bits c_0, c_1, $\ldots$, observing the following.

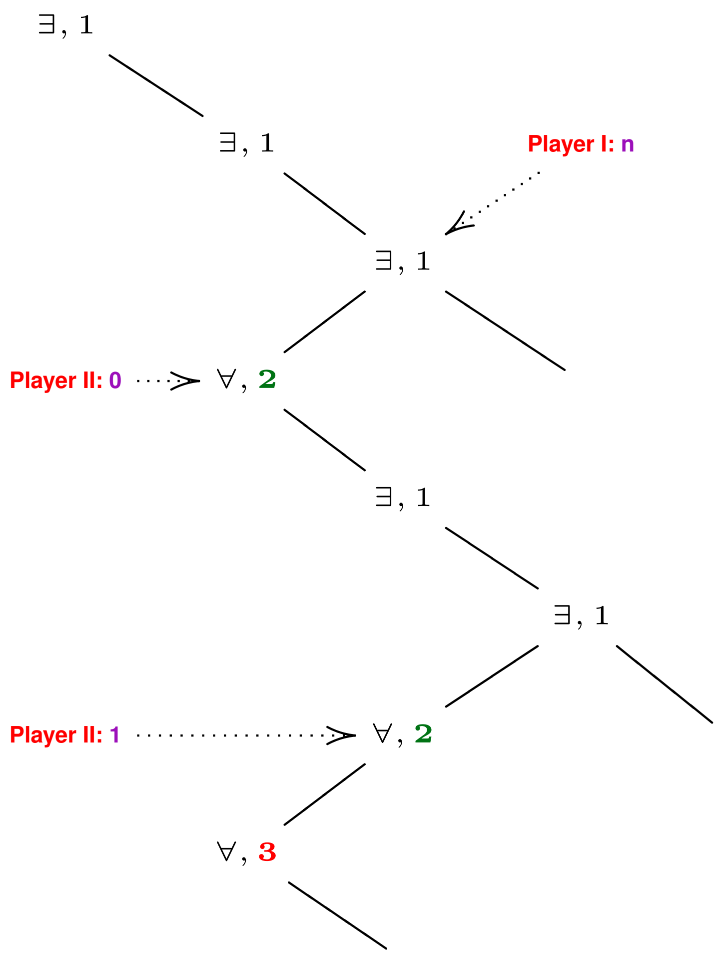
If Player II has selected $0^{m_0}10^{m_1}1\ldots10^{m_k}10^\ell$ and Player I $n_0, \ldots, n_k, \ldots$ then

1. $m_0m_1\ldots m_k \in t$,
2. $(\forall i)\ n_i \leq m_i$.

Player II wins if he plays infinitely many 1.

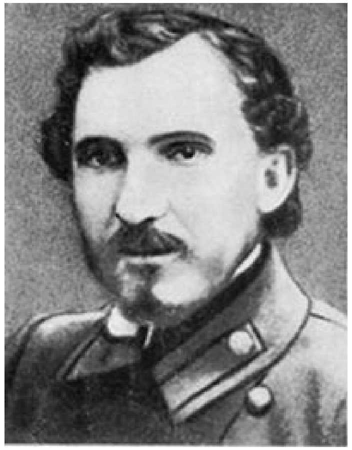
A tree $t \subseteq \omega^*$ is cofinal if and only if Player II has a winning strategy in $\Gamma(t)$ (**Saint Raymond 2006**).

Reduction transforms a tree $t \subseteq \omega^*$ onto a **game tree** in $\Gamma(t)$ of Player I, so that *Eve* wins iff Player I wins.



More generally, the sets $W_{i,k}$ are (Wadge) complete in the finite levels of the hierarchy of $\mathcal{R}$-sets introduced by A. Kolmogorov in 1928 as a foundation for measure theory (**Gogacz, Michalewski, Mio, Skrzypczak 2014**).

This hierarchy is based on generalization of **Suslin operation A**,


Mikhail Yakovlevich Suslin (1894–1919)

which was the first constructive tool to go beyond the Borel hierarchy (in **1916**).

GMMS 2014: *It is suggestive to think that the origin of the concept of parity games could be backdated to the original work of Kolmogorov.*