How can fixed points express change?

E. Allen Emerson's ideas for the μ-calculus

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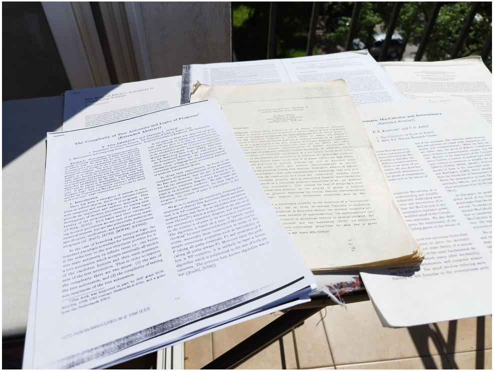
We use some sources from Wikipedia.

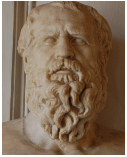
Time of logic, logic of time

1980s



Studied in our logic group in Warsaw led by **Helena Rasiowa**.

Change vs. constancy



Heraclitus, 500 BC

$\Pi\alpha'\nu\tau\alpha \quad \rho'\varepsilon\hat{\imath}$

Everything flows.

Alphonse Karr, 1849

Plus ça change, plus c'est la même chose.

The more it changes, the more it stays the same.

Change vs. constancy

$$x = f(x) = f\left(f(x)\right) = f(f(f(x))) = f\left(f(f(f(x)))\right) = \ldots$$

In various mathematical contexts, we search for

- invariants,
- equilibria,
- orbits,
- ......

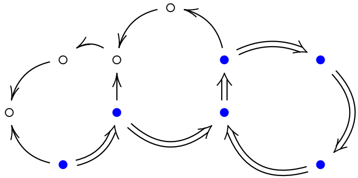
Inductive and co-inductive definitions

Knaster-Tarski theorem

$$\bigwedge\{d : f(d) \leq d\} \quad = \quad \bigvee_{\xi \in Ord} f^{\xi}(\bot) \quad = \quad \mu x.f(x)$$

$$\bigvee\{d : d \leq f(d)\} \quad = \quad \bigwedge_{\xi \in Ord} f^{\xi}(\top) \quad = \quad \nu x.f(x)$$

For example,



the origins of infinite paths $\equiv \bigcup\{x : x \subseteq \Diamond x\} = \nu x.\Diamond x$.

When induction meets co-induction

$$\begin{array}{ccccc} \mu x.\nu y.f(x,y) & & & & \\ \| & & & & \\ x & = & \nu y.f(x,y) & & \\ & & \| & & \\ & & y & = & f(x,y) \end{array}$$

Note that $a = \mu x.\nu y.f(x,y)$ satisfies $a = f(a,a)$, hence

$$\mu x.f(x,x) \leq \mu x.\nu y.f(x,y) \leq \nu y.f(y,y)$$

It makes sense!

Early history of the μ-calculi

Jaco W. de Bakker and Willem P. de Roever, A calculus for recursive program schemes, **1972**.

Lawrence Flon and Norihisa Suzuki, Consistent and Complete Proof Rules for the Total Correctness of Parallel Programs, **1978**.

David Park, On the semantics of fair parallelism, **1980**.

E. Allen Emerson and Edmund M. Clarke, Characterizing correctness properties of parallel programs using fixpoints, **1980**.

Vaughan R. Pratt, A Decidable mu-Calculus: Preliminary Report, **1981**.

Dexter Kozen, Results on the propositional μ-calculus, **1982**.

What can we express with $\nu\,\mu\,\nu\ldots$?

D. Park, On the semantics of fair parallelism, 1980.

ω-Regular properties of infinite words, like **fair merge**: both **a** and **b** occur infinitely often:

$$\nu x.\mu z.\,(\mathbf{a}z \cup \mathbf{b}\,\mu y.(\mathbf{a}x \cup \mathbf{b}y))\,.$$

E.A. Emerson and E.M. Clarke, Characterizing correctness properties. . . 1980

Computation tree properties, like: on each infinite path, **b** will ultimately always happen

$$\mu x.\nu y.\Box\,(x \vee (\mathbf{b} \wedge y))\,.$$

In general, ω iterations is not enough, and the last property is Π^1_1-complete (interpreted over $\mathbb{N}$).

Decidability issues

Can we decide the μ-calculus (at least) on the **propositional** level?

Dexter Kozen and Rohit Parikh, A Decision Procedure for the Propositional Mu-calculus, **1983**.

By a reduction to the **MSO** theory of the binary tree and the **Rabin Tree Theorem**.

Besides, over binary trees the logics are equivalent (**N. 1988**), and over all Kripke structures the μ-calculus captures precisely the bisimulation-invariant fragment (**Janin and Walukiewicz 1996**).

The Propositional Mu-Calculus is Elementary

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ABSTRACT: The propositional mu-calculus is a propositional logic of programs which incorporates a least fixpoint operator and subsumes the Propositional Dynamic Logic of Fischer and Ladner, the infinite looping construct of Streett, and the Game Logic of Parikh. We give an elementary time decision procedure, using a reduction to the emptiness problem for automata on infinite trees. A small model theorem is obtained as a corollary.

1. Introduction

First-order logic is inadequate for formalizing reasoning about programs; concepts such as termination and totality require logics strictly more powerful than first-order (Kfoury and Park, 1975). The use of a least fixpoint operator as a remedy for these deficiencies has been investigated by Park (1970, 1976), Hitchcock and Park (1973), deBakker and deRoever (1973), deRoever (1974), Emerson and Clarke (1980), and others. The resulting formal systems are often called mu-calculi and can express such important properties of sequential and parallel programs as termination, liveness, and freedom from deadlock and starvation.

Propositional versions of the mu-calculus have been proposed by Pratt (1981) and Kozen (1982). These logics use the least fixpoint operator to increase the expressive power of Propositional Dynamic Logic (*PDL*) of Fischer and Ladner (1979). Kozen's formulation captures the infinite looping construct of Streett (1982) and subsumes Parikh's Game Logic (1983a, 1983b), whereas Pratt's logic is designed to express the converse operator of *PDL*. The filtration-based decision procedure and small model theorem obtained for *PDL* extend to Pratt's mu-calculus, but the ability to express infinite looping renders the filtration technique inapplicable to Kozen's version.

Kozen (1982) and Vardi and Wolper (1984) have obtained exponential time decision procedures for fragments of Kozen's mu-calculus. Both fragments can expresses all of *PDL*, but are not strong enough to capture the infinite looping construct of Streett (1982). Kozen and Parikh (1983) have shown that the satisfiability problem for the full

Complexity issues

Robert S. Streett, E. Allen Emerson:
The Propositional Mu-Calculus is Elementary. **1984**.

- Reduction to **Rabin automata**,
- the concept of **signature**

$$\underset{\xi_5}{\mu x_5}.\quad \nu x_4.\quad \underset{\xi_3}{\mu x_3}.\quad \nu x_2.\quad \underset{\xi_1}{\mu x_1}.\quad \varphi(x_1, \ldots, x_5).$$

E. Allen Emerson and Charanjit S. Jutla:
The Complexity of Tree Automata and Logics of Programs **1988**.

- **EXPTIME**-completeness of the satisfiability problem.

The birth of parity games

ALMA observatory

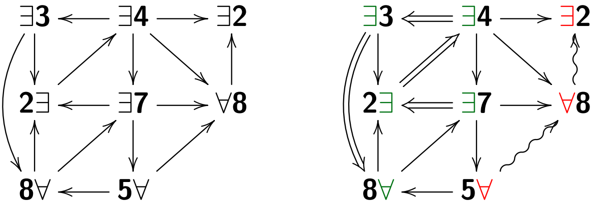
The birth of parity games

E. Allen Emerson, Charanjit S. Jutla:
Tree Automata, Mu-Calculus and Determinacy **1991**.

- a new proof of Rabin's Complementation Lemma,
- an efficient translation from μ-calculus to automata,
- **positional determinacy** of parity games (independently shown by **A. W. Mostowski** **1991**).

Established the studies of memory in infinite duration games.

Playing parity games



The μ-calculus proof of determinacy provides an explicit strategy: decrease the **signature** !

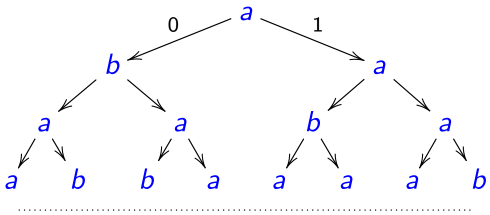
Complexity of parity games

		NP ∩ co-NP		**PPAD**	complete
		$\Cup$		$\Cup$	
parity	$\rightarrow$	**mean payoff**	$\rightarrow$	**Nash**	
$\mathbf{n}^{\mathcal{O}(\log \mathbf{n})}$		?			

C.S. Calude, S. Jain, B. Khoussainov, W. Li, F. Stephan, Deciding parity games in quasipolynomial time, **2017**

Topological complexity and measures

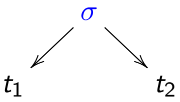
The set T_Σ of (full binary) Σ-labeled trees, i.e., $t : 2^* \to \Sigma$



has the **Cantor topology** and **coin-flipping measure**.

For $\sigma \in \Sigma$, and $L_1, L_2 \subseteq T_\Sigma$, let

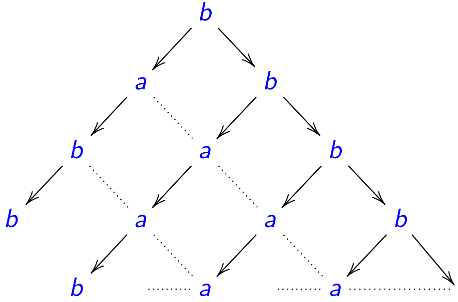
$$\sigma(L_1, L_2) \quad = \quad \{$$



$$: t_1 \in L_1, \;\; t_2 \in L_2 \;\}$$

Topological complexity and measures cont'd

$\mu x.\nu y.a(x,x) \cup b(y,y)$ $\equiv$ on each path, a occurs finitely often



This set is $\mathbf{\Pi^1_1}$-complete (hence non-Borel).

It encodes all well-founded trees $T \subseteq \omega^*$.

Recall the example in E.A. Emerson and E.M. Clarke, Characterizing correctness properties. . . 1980.

Topological complexity and measures cont'd

Are all regular sets of infinite trees **measurable** ?

Yes!

T. Gogacz, H. Michalewski, M. Mio, M. Skrzypczak,
Measure properties of regular sets of trees **2017**.

Can the measures be **effectively computed** ?

Yes! (they are algebraic numbers)

D.N., Paweł Parys and Michał Skrzypczak,
The Probabilistic Rabin Tree Theorem, **2023**.

Computing measures by fixed points

Input a tree automaton A
Output $\Pr(t \in L(A))$

We know

$$L(A) \;=\; \mu x_{d-1}.\,\nu x_{d-2} \ldots \nu x_2.\,\mu x_1.\,\nu x_0.\,F(x_0, x_1, \ldots, x_{d-1}).$$

The binary **random variables** $(t \in L(A_q))$, for all states q, are **not** independent.

The domain of **joint distributions** is **not** even a lattice.
But it nevertheless allows for an interpretation of $\nu x.\nu x.\mu \ldots .F(\vec{x})$, if suitable reconstructed.

Unary μ-calculus

The constructions $\mu x.f(x, \vec{y})$ and $\nu x.f(x, \vec{y})$ are replaced by

$$F \quad \mapsto \quad F^{\uparrow}, F^{\downarrow}$$

$F^{\uparrow} : x \mapsto$ the least fixed point **above** x
$F^{\downarrow} : x \mapsto$ the greatest fixed point **below** x,
if exist.

Then

$$\mu x_{d-1}.\,\nu x_{d-2} \ldots \nu x_2.\,\mu x_1.\,\nu x_0.\,F(x_0, x_1, \ldots, x_{d-1})$$

can be rewritten using these operations, F, and some “shuffling” of the arguments.

Then it can be interpreted in the domain of distributions, and the result is a formula of **Tarski’s theory of reals**.

Dagstuhl 1992

Seminar organised by Kevin Compton, Jean-Eric Pin, Wolfgang Thomas. From the programme:

. . . .

Nils Klarlund (Aarhus): Progress Measures for Complementation of ω-Automata

E. Allen Emerson (Austin): Complexity of Logics of Programs and Automata on Infinite Objects

Andrzej W. Mostowski (Gdańsk): Games with Forbidden Positions

Damian Niwiński (Warsaw): Problems in μ-Calculus

Paul E. Schupp (Urbana): Simulating Alternating Automata by Nondeterministic Automata

Bruno Courcelle (Bordeaux): Monadic Second-order Definability Properties of Infinite Graphs

A. L. Semenov and Andrey A. Muchnik (Moscow): Automata on Infinite Objects, Monadic Theories, and Complexity

Austin 1992

Thank you, Allen