Modal Fixpoint Logics: When Logic Meets Games, Automata and Topology

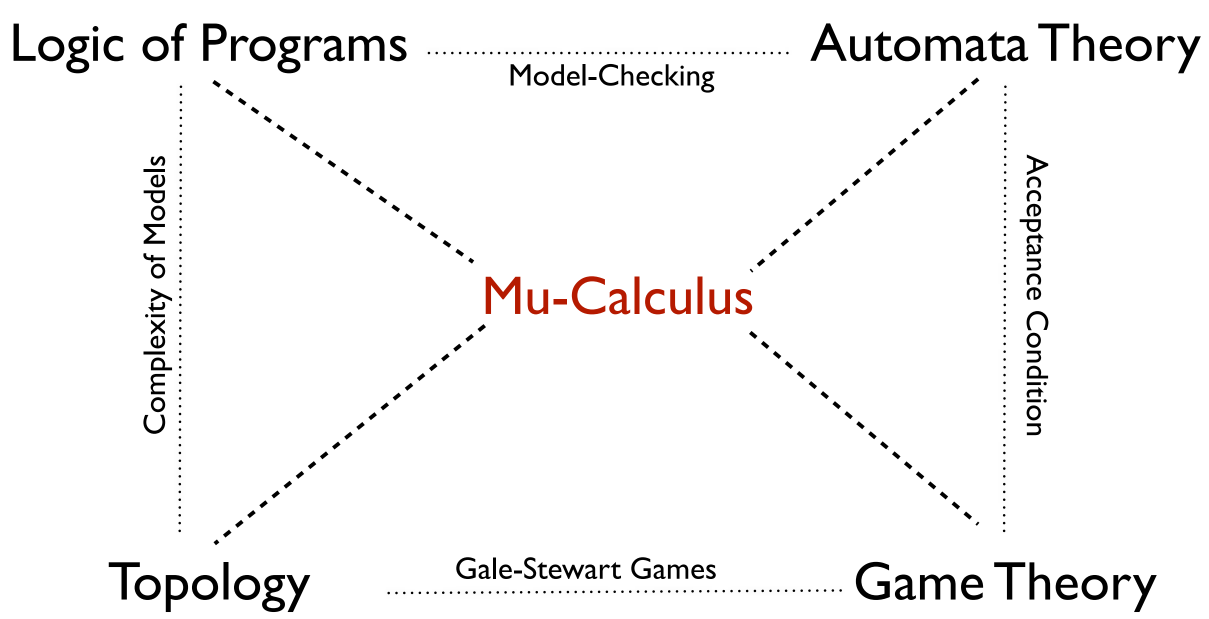
A. Facchini & D. Niwinski (U. Warsaw)

Lecture II

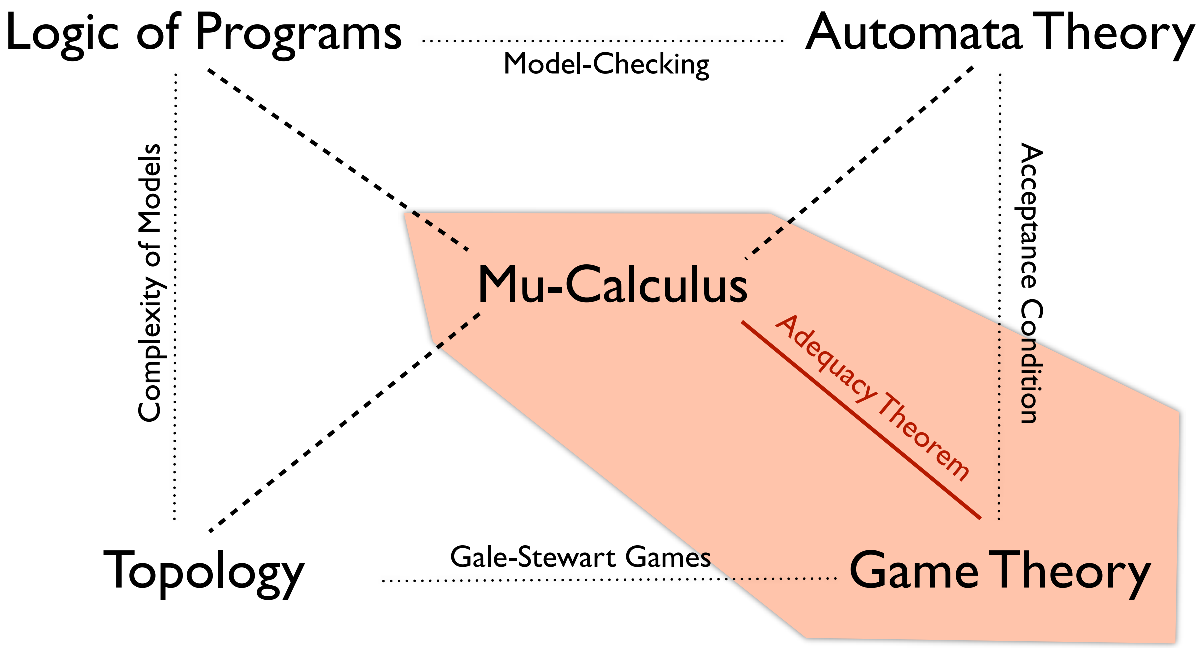
Automata for Modal Fixpoint Logics

ESSLLI 2014, Tübingen, 11-22 August 2014

The landscape of the first four days



What you have seen yesterday.....



What you have seen yesterday.....

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi$$

where $p, x \in \mathsf{Prop}$ and x occurs only positively in $\eta x.\varphi$ $(\eta = \nu, \mu)$, that is, $\neg x$ is not a subformula of φ.

What you have seen yesterday.....

Let $\mathcal{K} = (S, R, \rho)$ be a model.

- $\|p\|^{\mathcal{K}} = \rho(p)$ and $\|\neg p\|^{\mathcal{K}} = S \setminus \rho(p)$ for all $p \in \mathsf{Prop}$,
- $\|\phi \wedge \psi\|^{\mathcal{K}} = \|\phi\|^{\mathcal{K}} \cap \|\psi\|^{\mathcal{K}}$,
- $\|\phi \wedge \psi\|^{\mathcal{K}} = \|\phi\|^{\mathcal{K}} \cup \|\psi\|^{\mathcal{K}}$,
- $\|\Box\phi\|^{\mathcal{K}} = \{s \in S \mid \forall t, \text{if } (s,t) \in R \text{ then } t \in \|\phi\|^{\mathcal{K}})\}$,
- $\|\Diamond\phi\|^{\mathcal{K}}_{\rho} = \{s \in S \mid \exists t, (s,t) \in R \text{ and } t \in \|\phi\|^{\mathcal{K}})\}$.

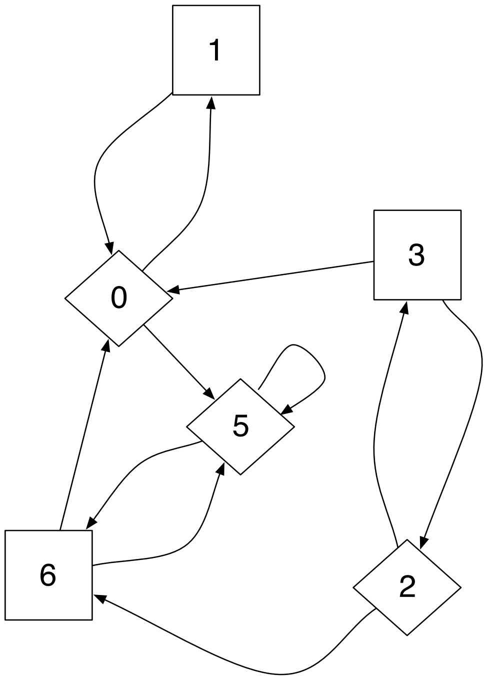
What you have seen yesterday.....

Let $\mathcal{K} = (S, R, \rho)$ be a model.

- $\ldots$
- $\|\nu x.\phi\|^{\mathcal{K}} = \bigcup\{N \subseteq S \mid N \subseteq \|\phi(x)\|^{\mathcal{K}[x \mapsto N]}\}$
- $\|\mu x.\phi\|^{\mathcal{K}} = \bigcap\{N \subseteq S \mid \|\phi(x)\|^{\mathcal{K}[x \mapsto N]} \subseteq N\}$

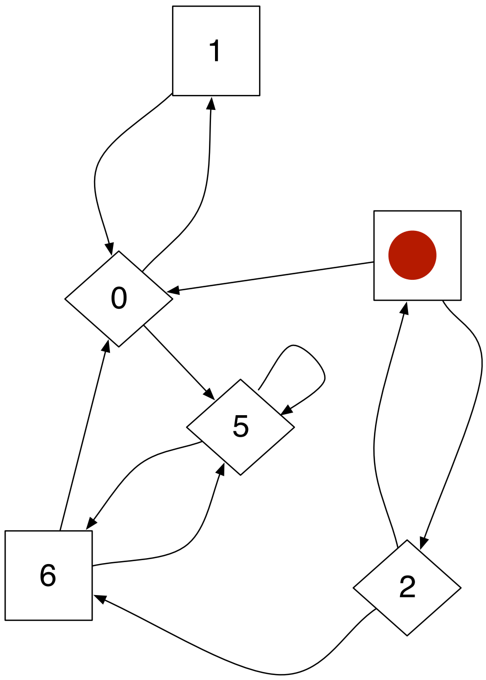
$$\|\nu x.\phi(x)\|^{\mathcal{K}} = GFP(\|\phi(x)\|^{\mathcal{K}}) \quad \text{and} \quad \|\mu x.\phi(x)\|^{\mathcal{K}} = LFP(\|\phi(x)\|^{\mathcal{K}})$$

What you have seen yesterday.....



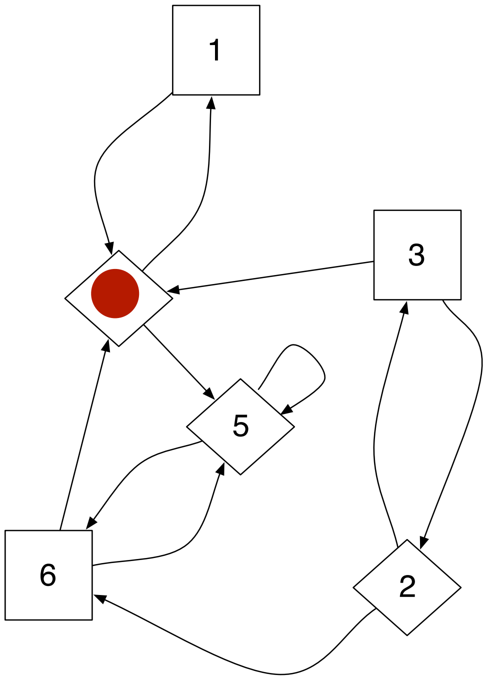
What you have seen yesterday.....

3



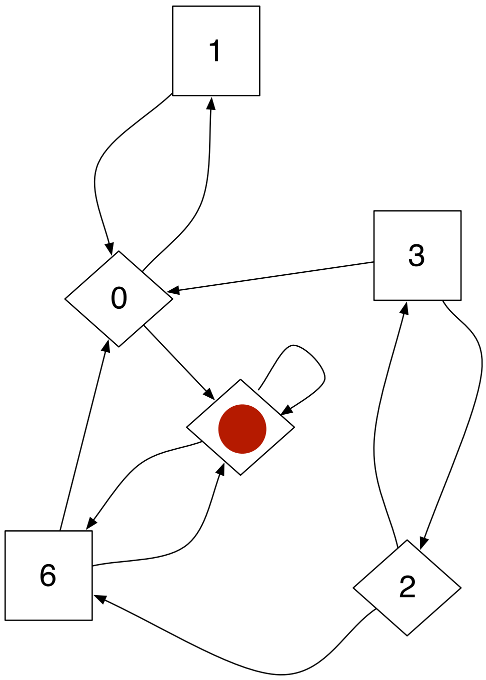
What you have seen yesterday.....

30



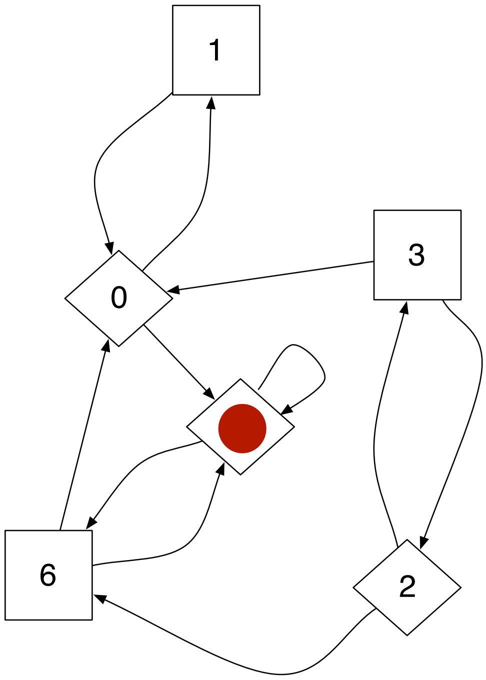
What you have seen yesterday.....

305



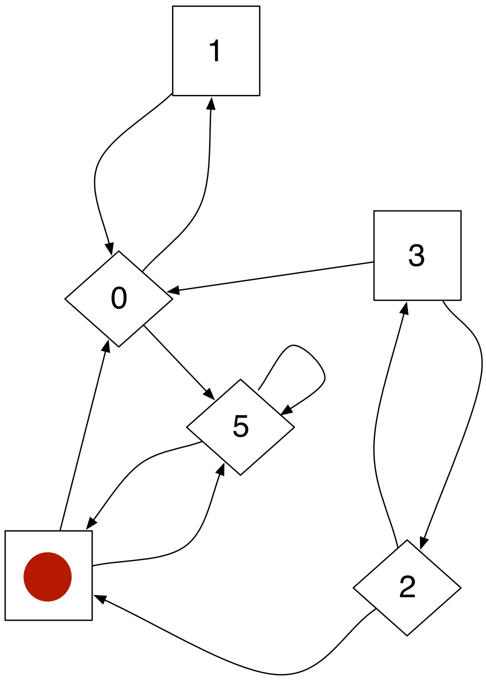
What you have seen yesterday.....

3055.....5



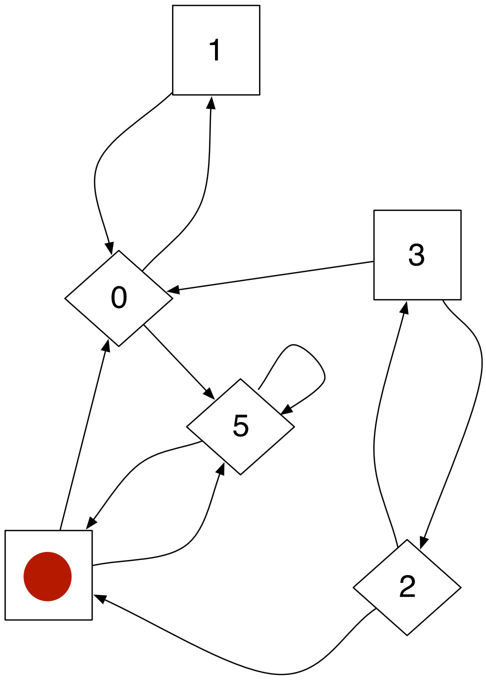
What you have seen yesterday.....

3055.....56



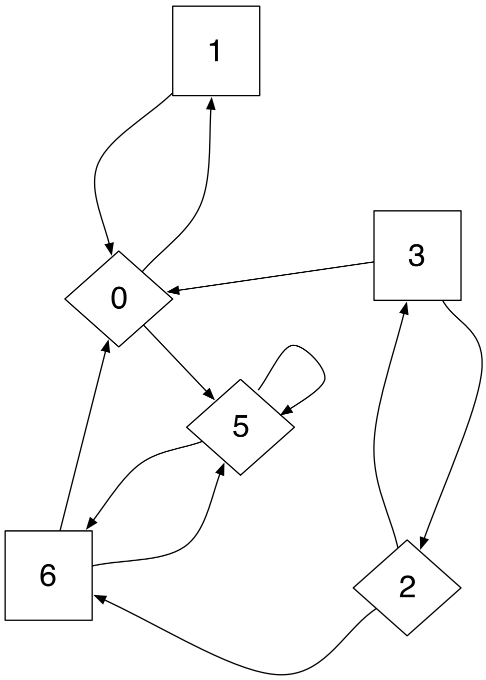
What you have seen yesterday.....

$3055.....56.... \in \{0, \ldots, 6\}^\omega$



What you have seen yesterday.....

Player $\exists$ wins iff the greatest priority occurring infinitely often is even



Theorem [Emerson & Jutla ('91), Mostowski ('91)]:
Parity games are positional determined

Theorem: Let $\mathcal{G} = (S, S_\exists, S_\forall, R, \text{rank})$ be a parity game, and let $\mathcal{K}_\mathcal{G} = (S, R, \rho)$ the associated Kripke model.
Then there is a formula $\psi_\exists$ such that

$$s \in \|\psi_\exists\|^\mathcal{K} \text{ iff } \exists \text{ has a w.s. in } \mathcal{G}@s.$$

Let $\mathcal{K} = (S, R, \rho)$ be a model, and φ be a μ-formula,

Evaluation (parity) game $\mathcal{G}(\varphi, \mathcal{K})$

odd when $\varphi_x = \mu x.\psi$, else even.

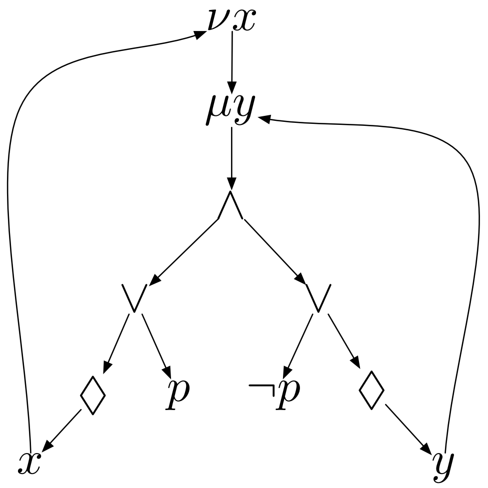
Position	Player	Admissible moves	Parity
$(\eta x.\psi, s) \in \text{sub}(\varphi) \times S$	$\exists$	$\{(\psi, s)\}$	$\text{rank}(\eta x.\psi)$
$(x, s) \in \text{sub}(\varphi) \times S$	$\exists$	$\{(\varphi_x, s)\}$	$\text{rank}(\varphi_x)$
$(\psi_1 \vee \psi_2, s)$	$\exists$	$\{(\psi_1, s), (\psi_2, s)\}$	–
$(\psi_1 \wedge \psi_2, s)$	$\forall$	$\{(\psi_1, s), (\psi_2, s)\}$	–
$(\Diamond\varphi, s)$	$\exists$	$\{(\varphi, t) \mid t \in R[s]\}$	–
$(\Box\varphi, s)$	$\forall$	$\{(\varphi, t) \mid t \in R[s]\}$	–
$(\neg p, s)$ and $p \notin \rho(s)$	$\forall$	$\emptyset$	–
$(\neg p, s)$ and $p \in \rho(s)$	$\exists$	$\emptyset$	–
(p, s) and $p \in \rho(s)$	$\forall$	$\emptyset$	–
(p, s) and $p \notin \rho(s)$	$\exists$	$\emptyset$	–

Let $\mathcal{K} = (S, R, \rho)$ be a model, and φ be a μ-formula,

Evaluation (parity) game $\mathcal{G}(\varphi, \mathcal{K})$

- $$\text{rank}(\eta x.\delta) = \begin{cases} \text{ad}(\eta x.\delta) & \text{if } \eta = \mu \text{ and ad}(\eta x.\delta) \text{ is odd, or} \\ & \eta = \nu \text{ and ad}(\eta x.\delta) \text{ is even;} \\ \text{ad}(\eta x.\delta) - 1 & \text{if } \eta = \mu \text{ and ad}(\eta x.\delta) \text{ is even, or} \\ & \eta = \nu \text{ and ad}(\eta x.\delta) \text{ is odd,} \end{cases}$$
- $\text{rank}(x) = \text{rank}(\varphi_x)$.

$$\nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$



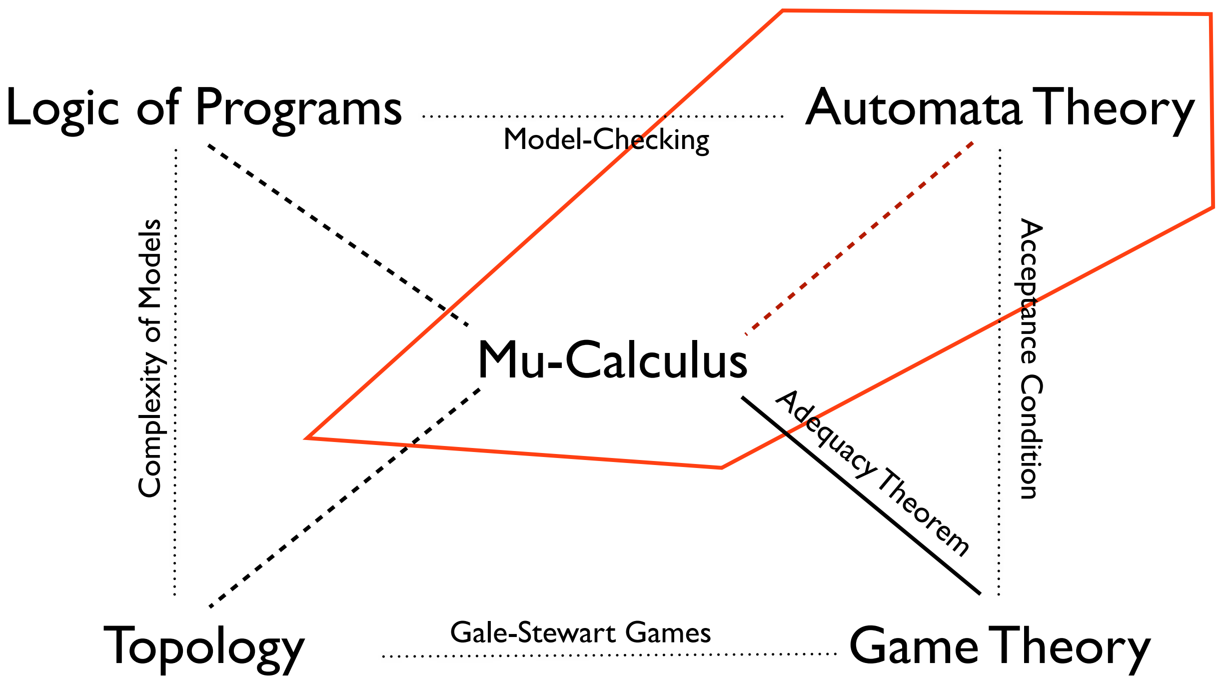
$$\text{ad}(\nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)) = 2$$
$$\text{ad}(\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)) = 1$$

What you have seen yesterday.....

Theorem [E.A. Emerson, R.S. Street (1989)]
$s \in \|\varphi\|^{\mathcal{K}}$ iff $\exists$ has a w.s. in $\mathcal{G}(\varphi, \mathcal{K})@(\varphi, s)$

$(\mathcal{K}, s) \models \varphi$

What we are going to see today...



Starting point

Given a first-order sentence, can we *decide* if the sentence is valid?

Hilbert's Entscheidungsproblem (the decision problem)

Hilbert's decision problem is unsolvable

Church-Turing theorem

Starting point

Theorem [Trakhtenbrot, Craig 1950]: First-order logic over finite graphs is undecidable.

Starting point

The decision problem became
a **classification problem**

For which sublogic L of FO is the decision problem solvable (in a efficient way) ?

The case of modal logic:

The case of modal logic:

(i) translatable into (fragment of) FO
(ii) tree model property
(iii) small model property
(iv) van Benthem-Rosser characterization theorem:

$$FO/\underline{\leftrightarrow} = ML \text{ (over } \mathcal{C}\text{)}$$

$$\mathcal{C} = \begin{cases} \text{all models} \\ \text{finite models} \end{cases}$$

The case of modal logic:

(i) translatable into (fragment of) FO
(ii) tree model property
(iii) small model property
(iv) van Benthem-Rosser characterization theorem:

$$FO/\underline{\leftrightarrow} = ML \text{ (over } \mathcal{C}\text{)}$$

$$\mathcal{C} = \begin{cases} \text{all models} \\ \text{finite models} \end{cases}$$

what about the mu-calculus?

The case of the mu-calculus:

(i) translatable into (fragment of) MSO
(ii) tree model property
(iii) small model property
(iv) Janin-Walukiewicz characterization theorem:

$$MSO/\underline{\leftrightarrow} = \mu ML \text{ (over all models)}$$

The case of the mu-calculus:

(i) translatable into (fragment of) MSO
(ii) tree model property
(iii) small model property
(iv) Janin-Walukiewicz characterization theorem.

'corollaries' of the correspondance
between parity automata and fixpoint logics

...the plan for the next two days...

Mu-Calculus vs MSO

1. Automata characterization of mu-Calculus over Kripke models (Janin & Walukiewicz, 1995)
2. Automata characterization of MSO over arbitrary trees (Walukiewicz, 1996)
3. Characterization theorem for the mu-Calculus (Janin & Walukiewicz, 1996)

‘Formula as automata’

a finite-state automaton is given by

- a finite input alphabet
- finite set of states
- an initial state
- a transition function
- an acceptance condition

'Formula as automata'

$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

- Δ tells how to move in the next position, given the properties of the actual position
- Acc tells when to accept the input

'Formula as automata'

$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

- $\Delta(1,a) = 2$
- $\Delta(1,b) = 1$
- $\Delta(2,*) = 2$
- $\mathrm{Acc} = \{2\}$

- $\Delta(1) = (a \rightarrow X2) \wedge (b \rightarrow X1)$
- $\Delta(2) = (a \rightarrow X2) \wedge (b \rightarrow X2)$
- $\mathrm{Acc} = \{2\}$

‘Formula as automata’

$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

1

‘Formula as automata’

$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

1

‘Formula as automata’

$$\mathbb{A} = (\{1, 2\}, \{a, b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

1

‘Formula as automata’

$$\mathbb{A} = (\{1, 2\}, \{a, b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

1

‘Formula as automata’

$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

2

‘Formula as automata’

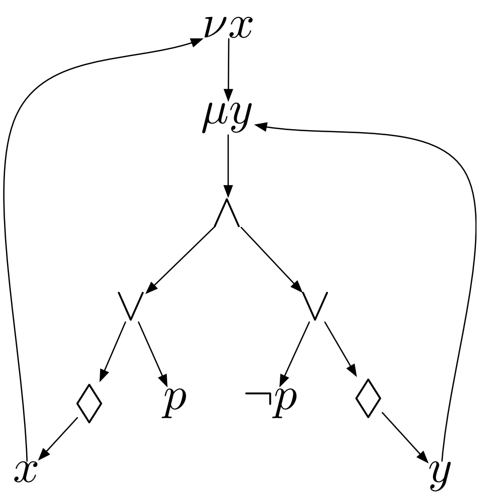
$$\mathbb{A} = (\{1,2\}, \{a,b\}, 1, \Delta, \mathrm{Acc})$$

b-b-b-a-b

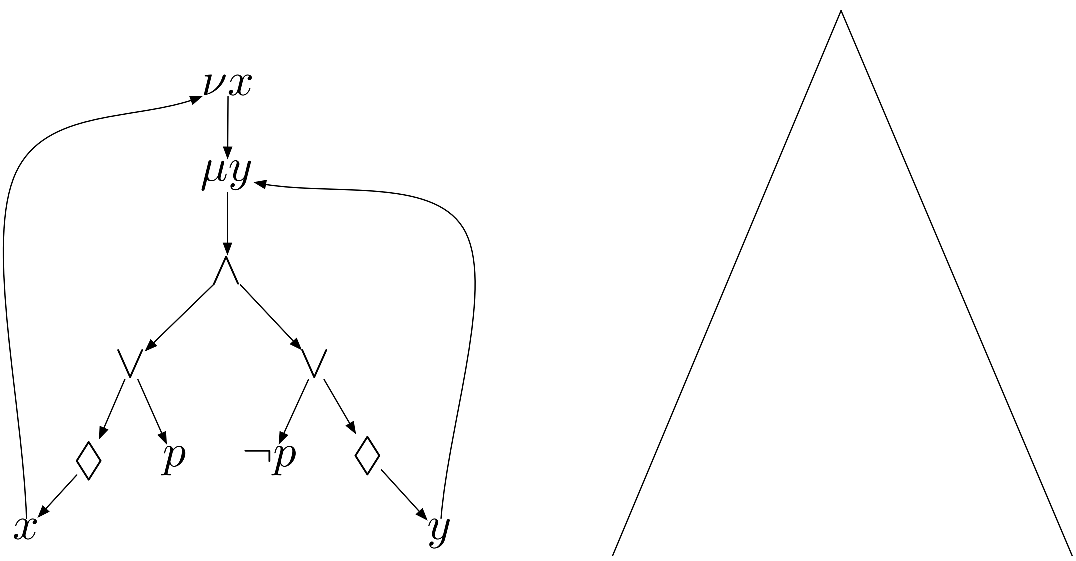
2

'Formula as automata'

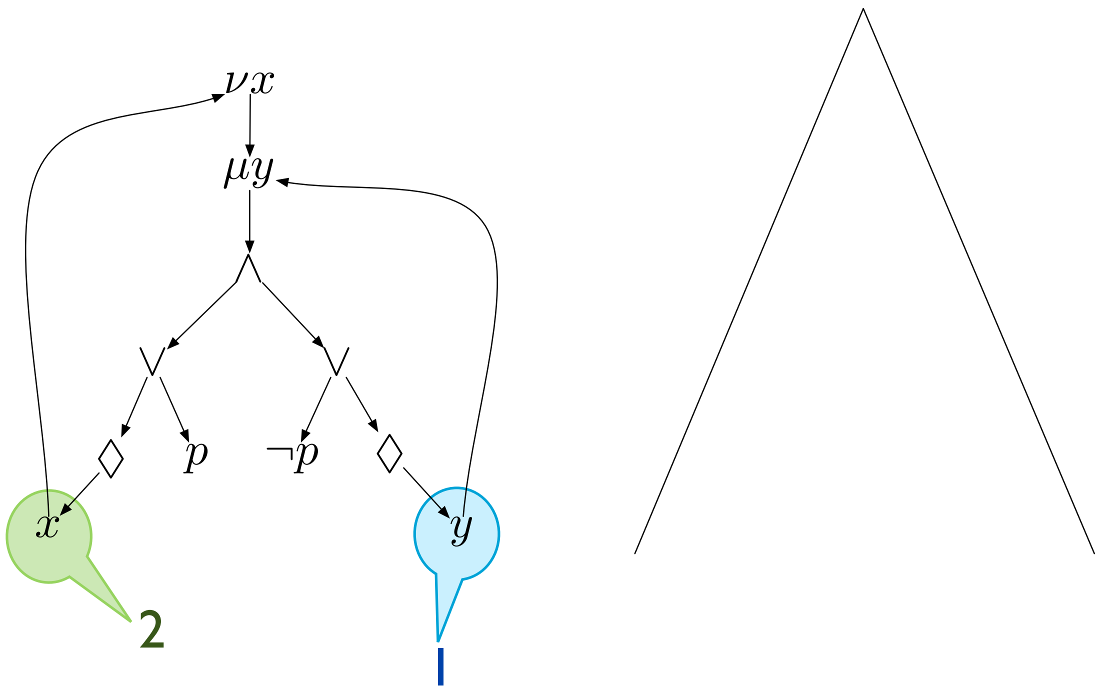
$$\nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$



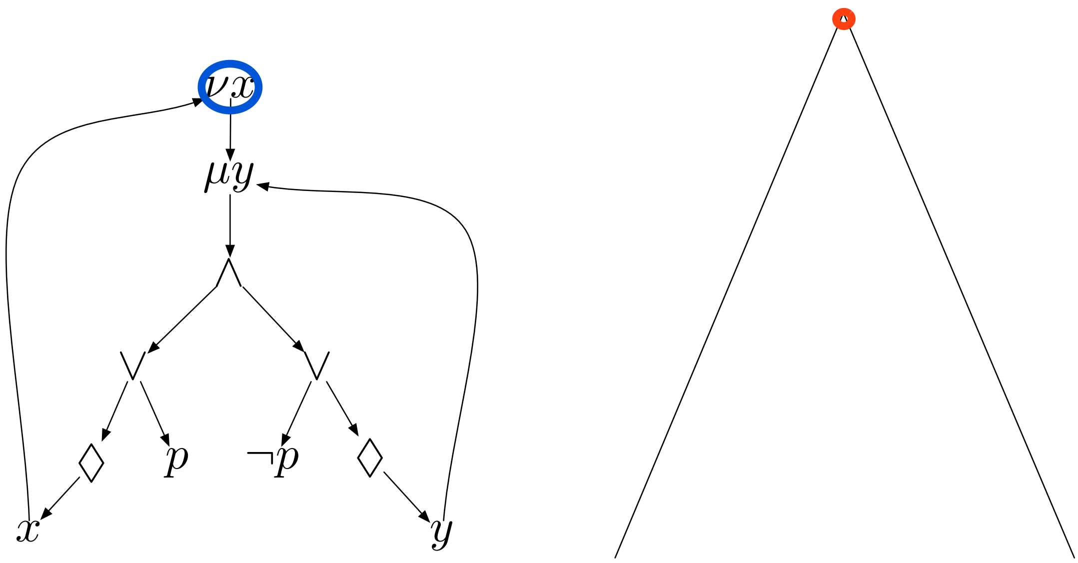
'Formula as automata'



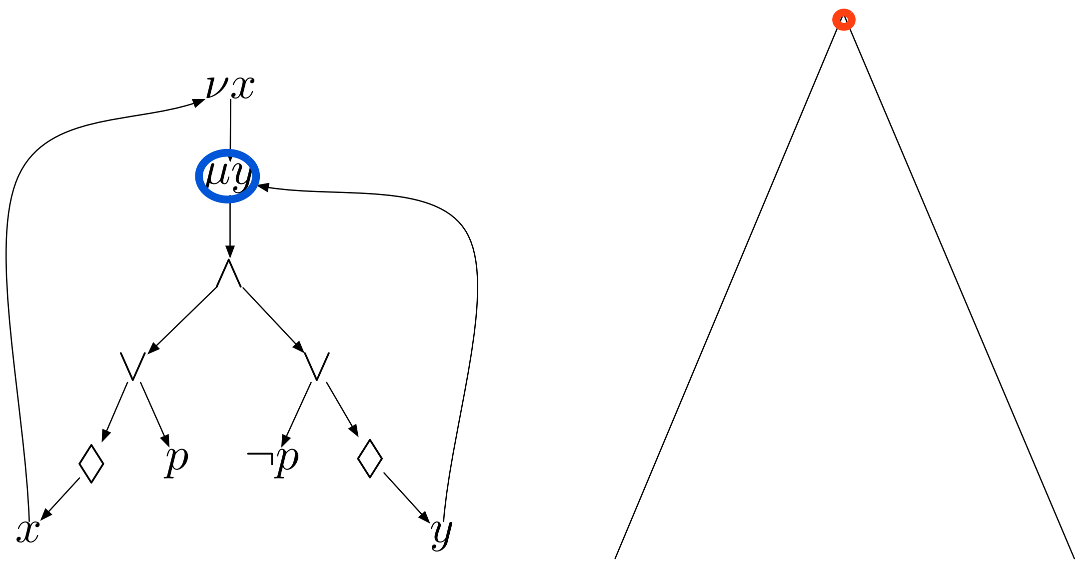
‘Formula as automata’



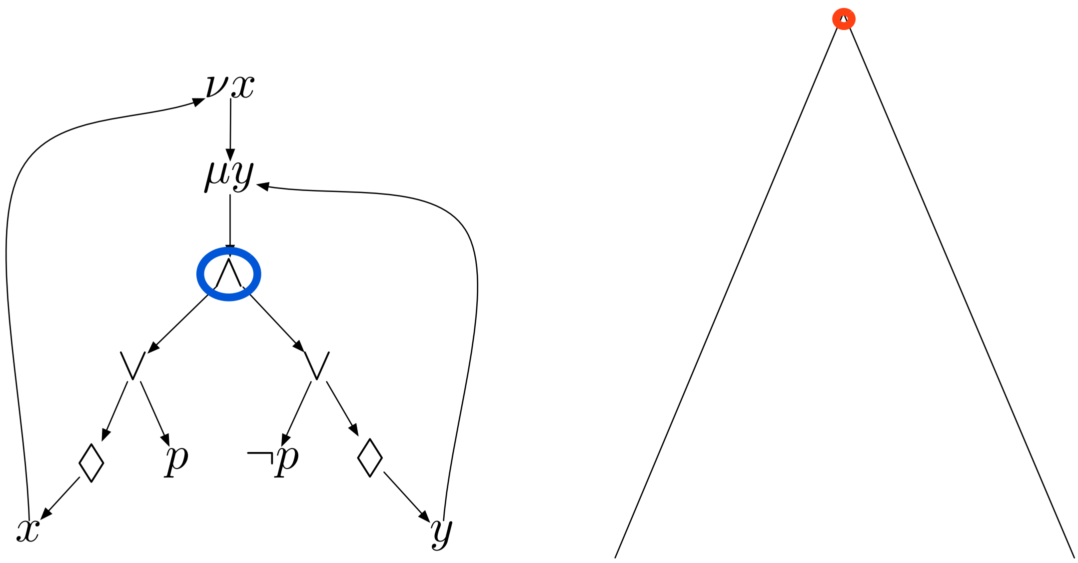
‘Formula as automata’



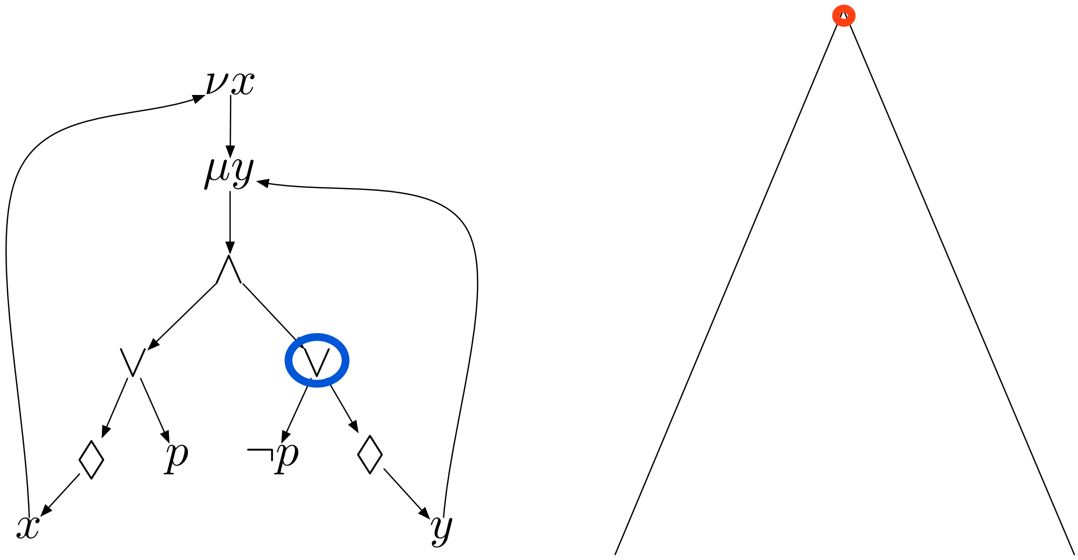
‘Formula as automata’



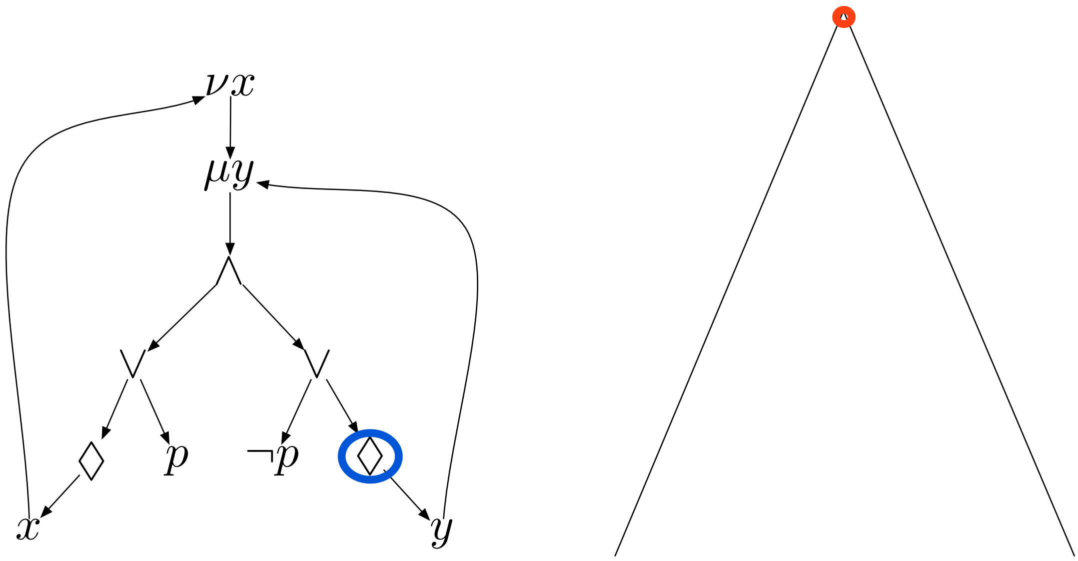
‘Formula as automata’



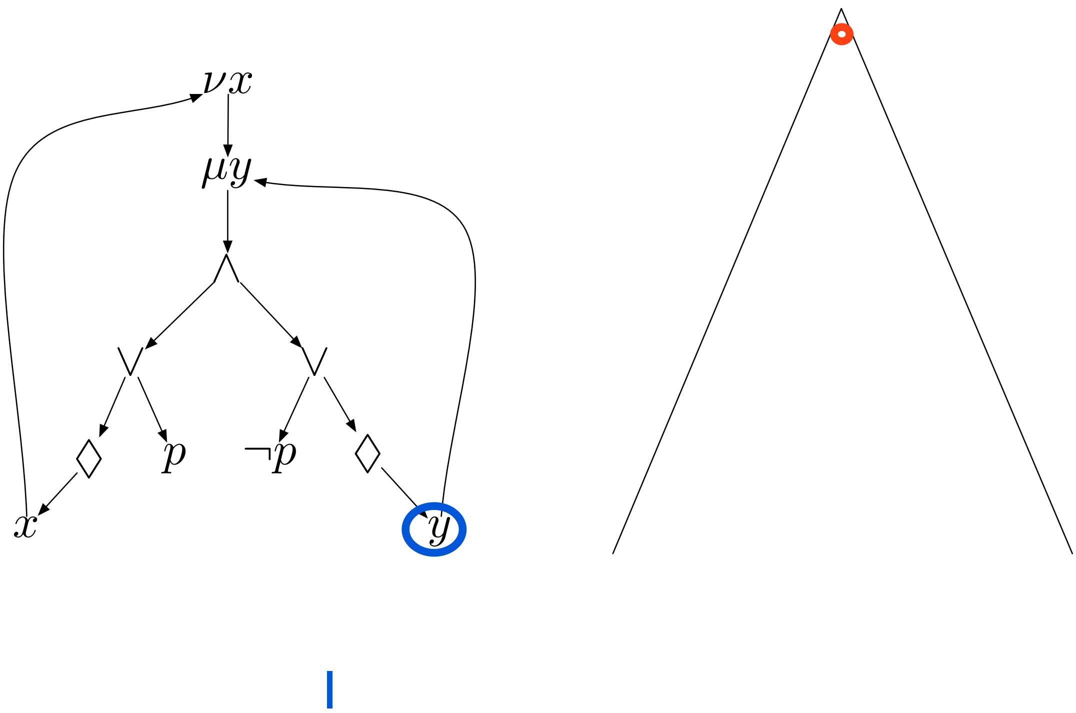
‘Formula as automata’



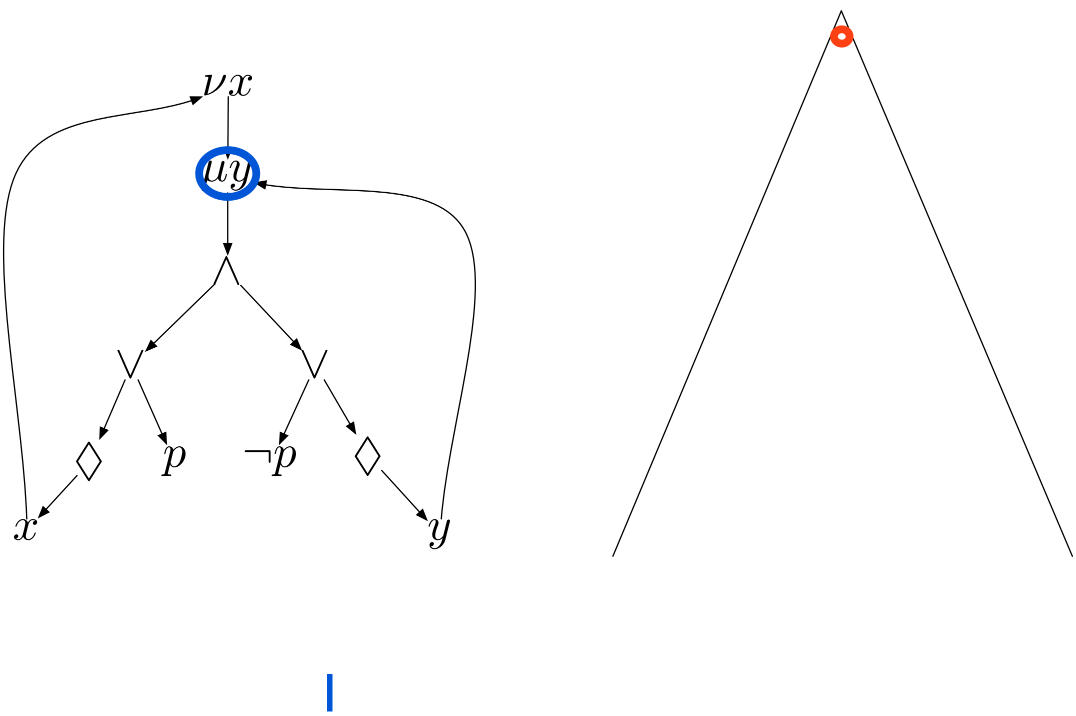
‘Formula as automata’



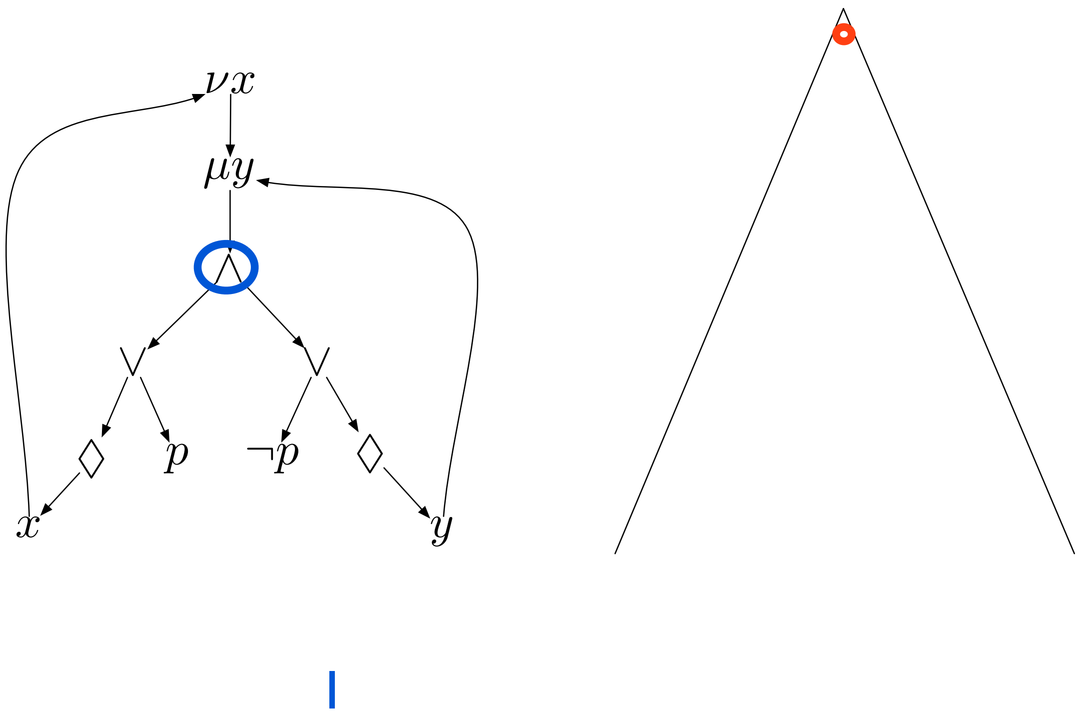
‘Formula as automata’



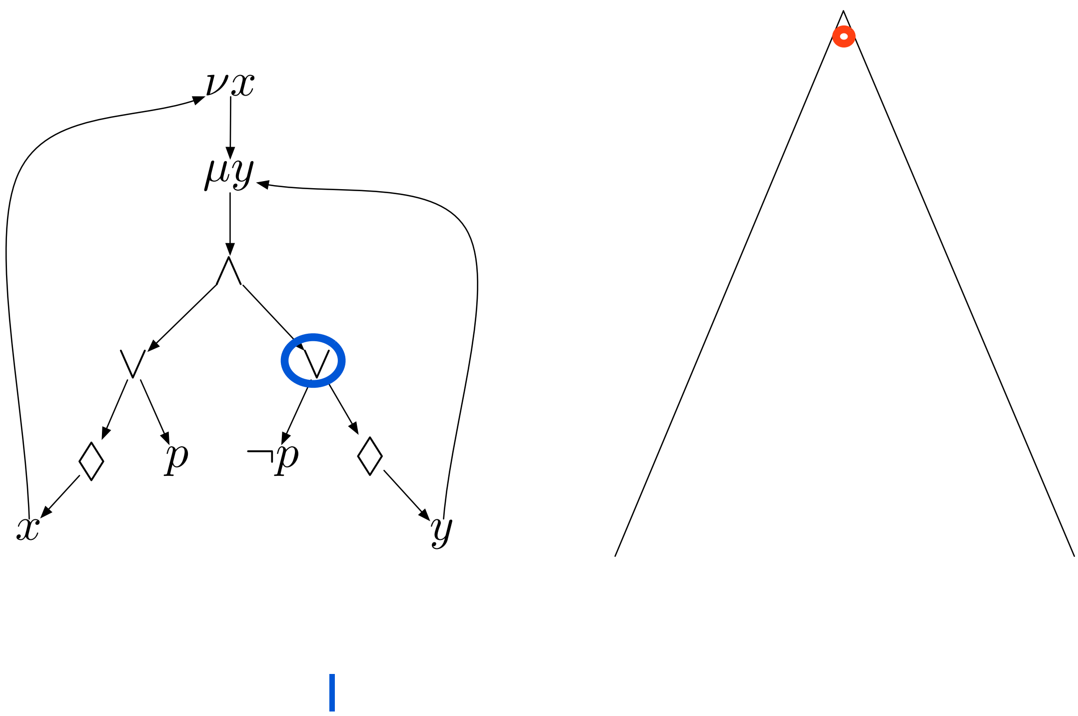
‘Formula as automata’



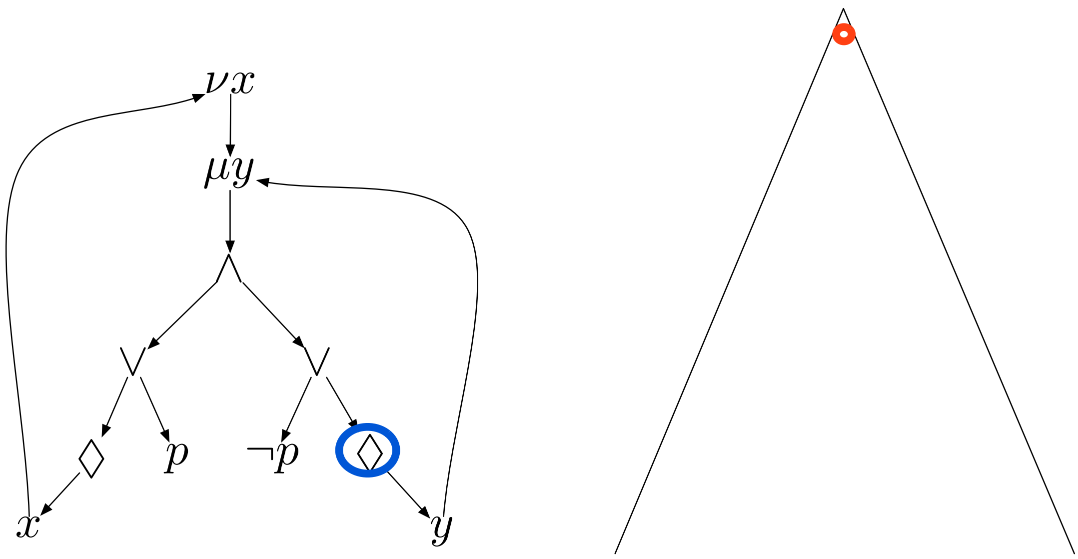
‘Formula as automata’



‘Formula as automata’

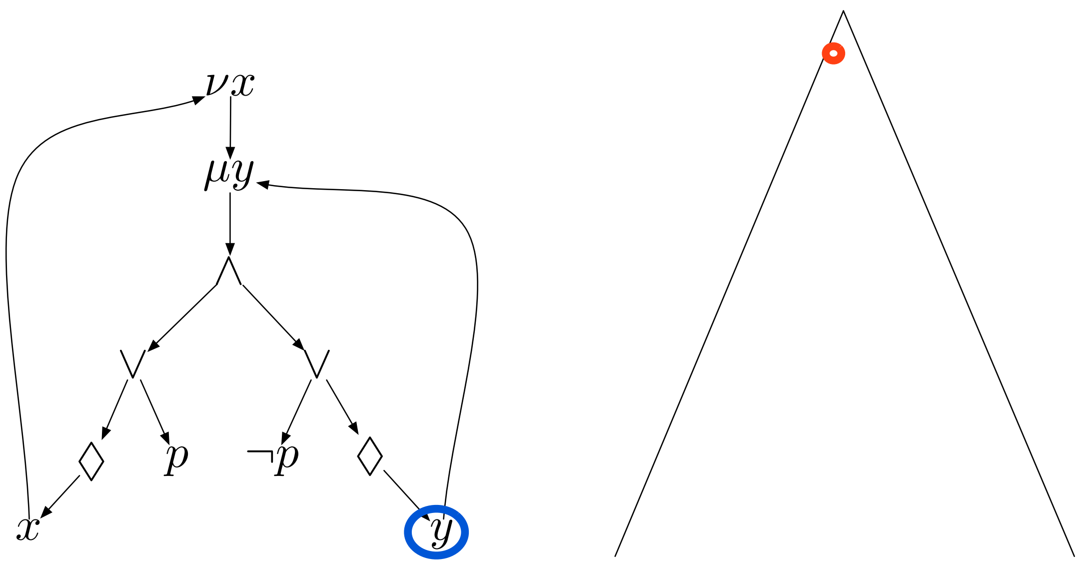


‘Formula as automata’



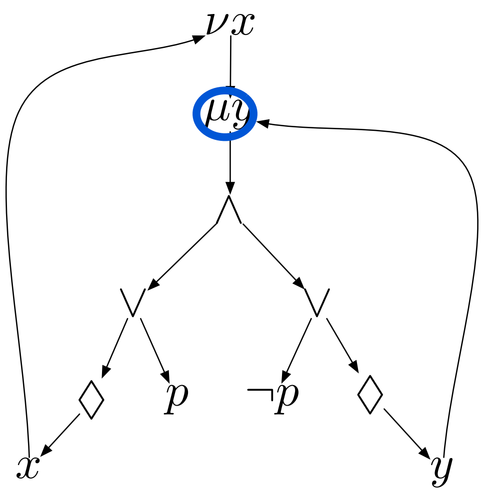
I

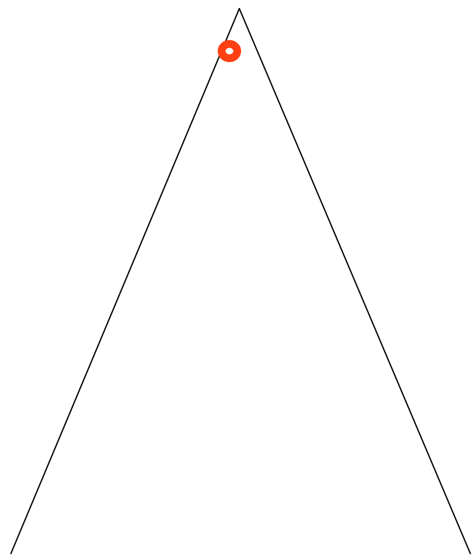
‘Formula as automata’



II

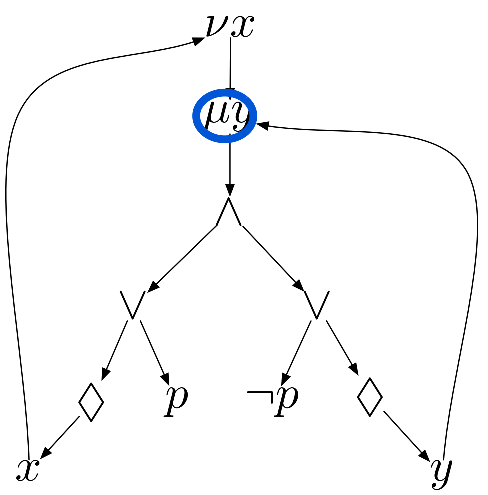
‘Formula as automata’

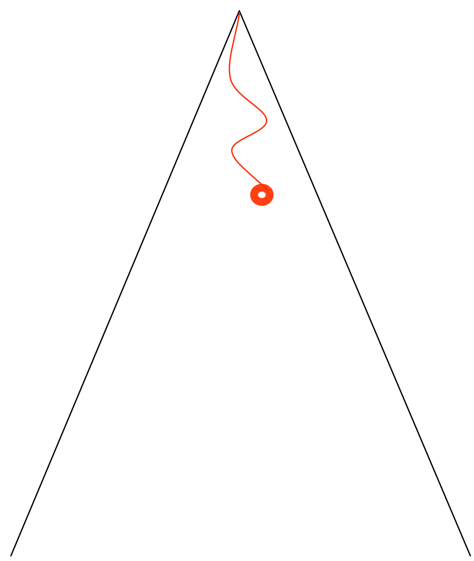




| |

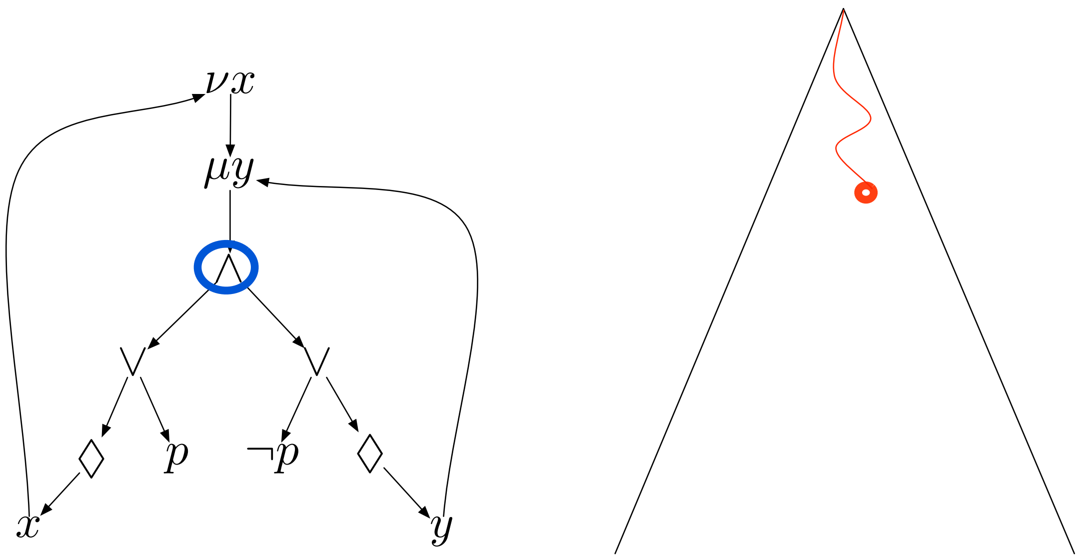
‘Formula as automata’





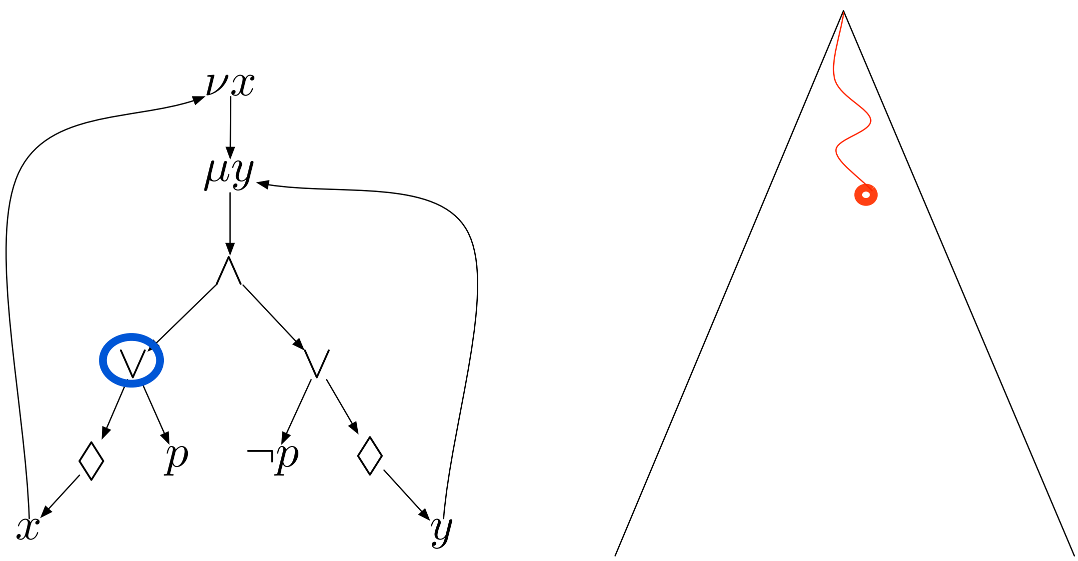
| | ..... |

‘Formula as automata’



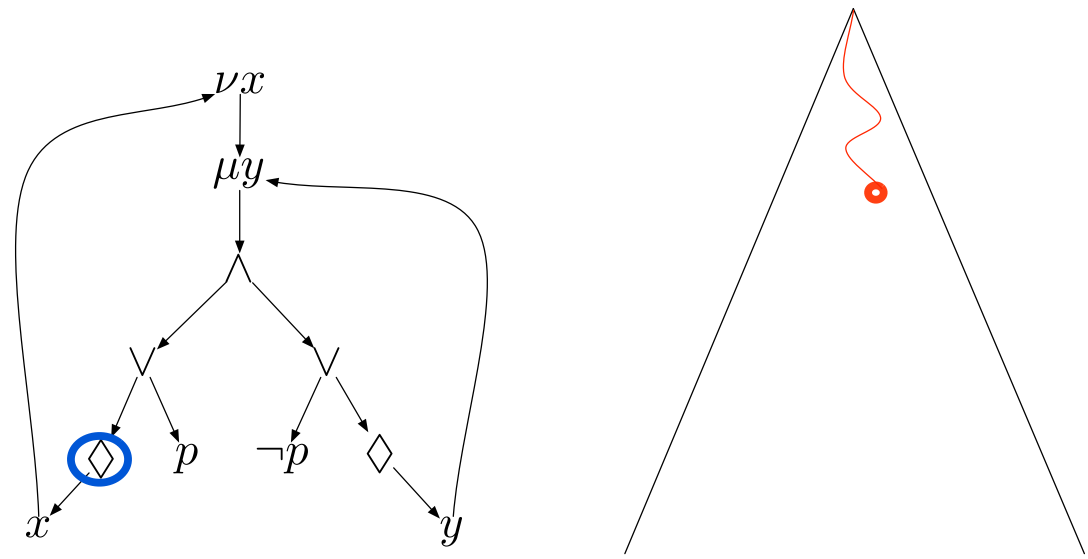
| |..... |

‘Formula as automata’



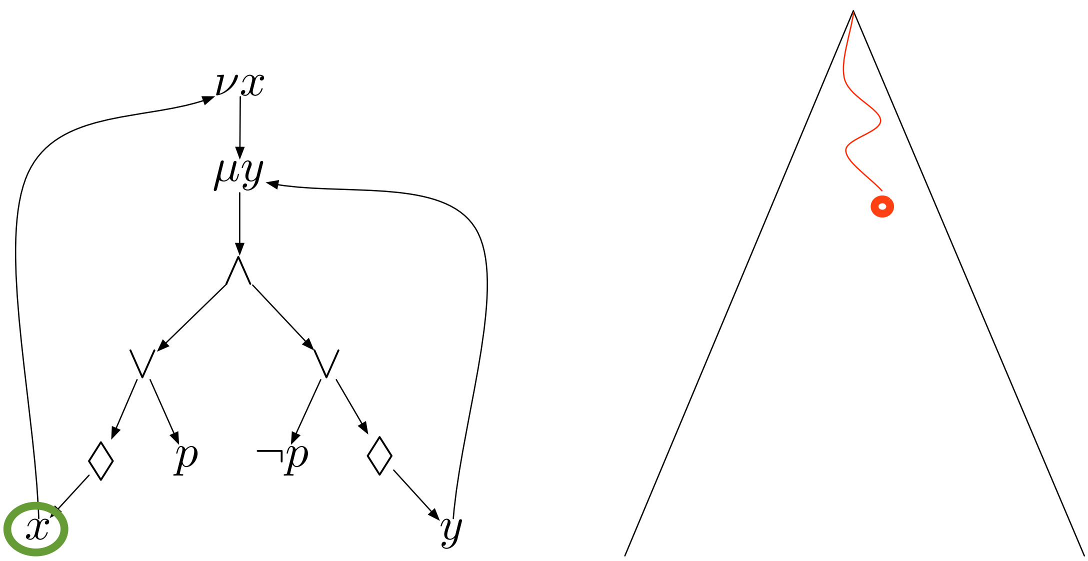
| |..... |

‘Formula as automata’



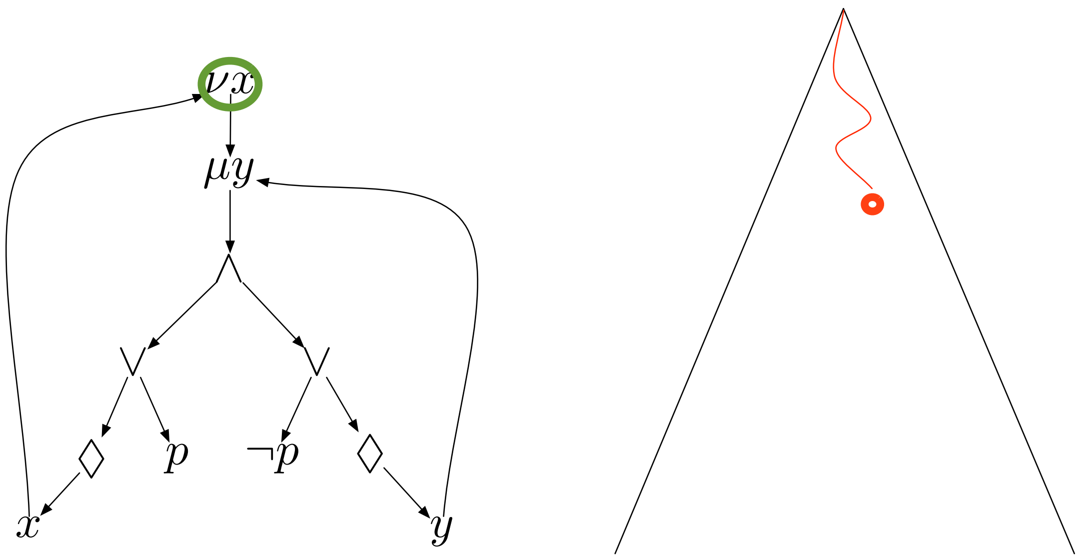
11.....1

‘Formula as automata’



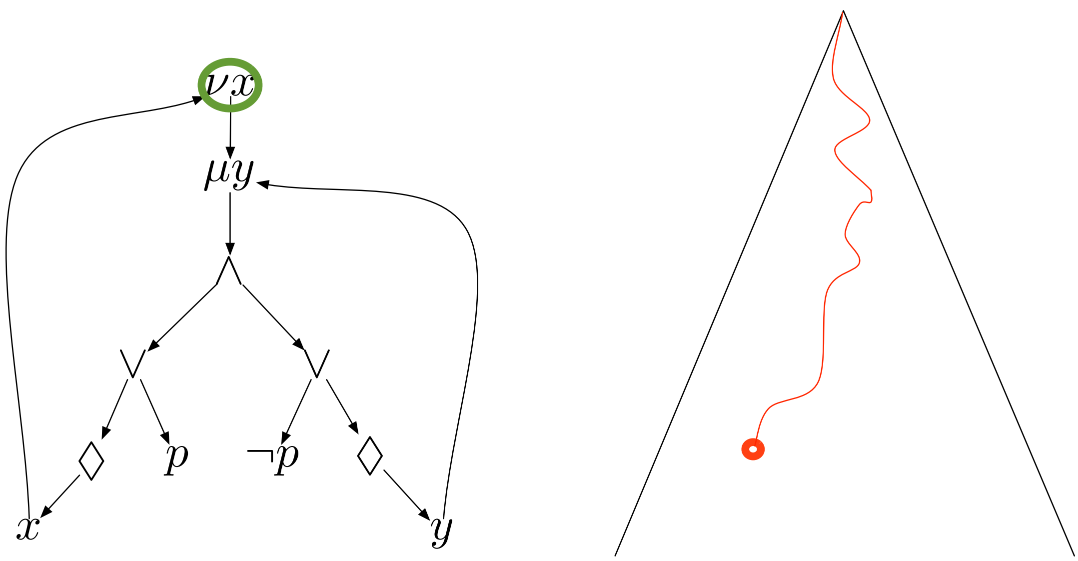
11.....12

‘Formula as automata’



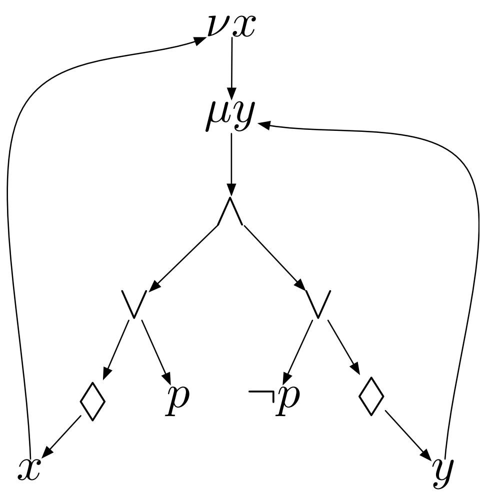
11.....12

‘Formula as automata’

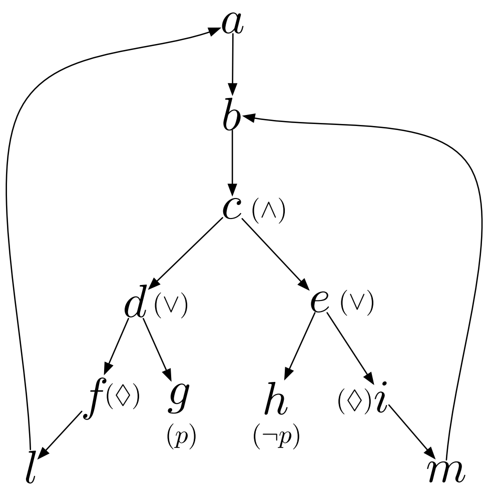


11.....12.....211.....12.....2.....

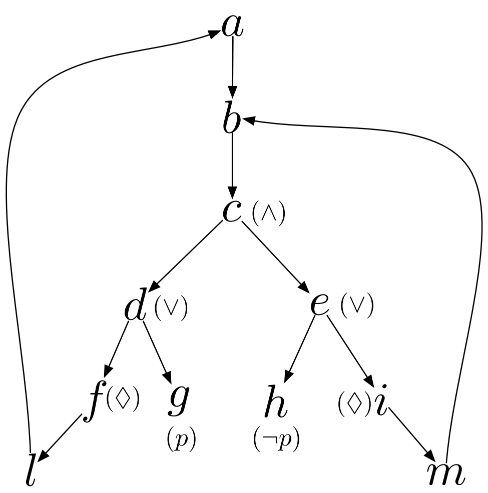
‘Formula as automata’



‘Formula as automata’



'Formula as automata'



$$(\{a, \ldots, m\}, a, \Delta, \text{rank})$$

(input alphabet subsets of propositional variables)

'Formula as automata'

$$(\{a, \ldots, m\}, a, \Delta, \text{rank}) \qquad \begin{cases} \Delta(a) = b \\ \Delta(b) = c \\ \Delta(c) = d \wedge e \\ \Delta(d) = f \vee g \\ \Delta(e) = h \vee i \\ \Delta(f) = \Diamond l \\ \Delta(g) = p \\ \Delta(h) = \neg p \\ \Delta(i) = \Diamond m \\ \Delta(l) = a \\ \Delta(m) = b \end{cases}$$

Modal automata

Given a set A of (state) variables, and a set P of propositional variables: the set $\mathrm{MLatt}(A;P)$ is defined as:

$$\phi ::= \top \mid \bot \mid a \mid p \mid \neg p \mid \Diamond a \mid \Box a \mid \bigwedge \Phi \mid \bigvee \Phi$$

with $a \in A$ and $p \in P$

Definition: A modal automaton is a tuple

$$\mathbb{A} = (A, a_I, \Delta, \mathrm{rank})$$

such that

- $a_I \in A$ (initial state)
- $\Delta : A \to \mathrm{MLatt}(A;P)$ (transition function)
- $\mathrm{rank} : A \to \mathbb{N}$ (parity/rank function)

Acceptance (parity) game $\mathcal{G}(\mathbb{A}, \mathcal{K})$

Let $\mathcal{K} = (S, R, \rho)$ be a Kripke model.

Position	Player	Admissible moves	Parity
$(a, s) \in A \times S$	$\exists$	$\{(\Delta(a), s)\}$	$\mathrm{rank}(a)$
$(\psi_1 \vee \psi_2, s)$	$\exists$	$\{(\psi_1, s), (\psi_2, s)\}$	$-$
$(\psi_1 \wedge \psi_2, s)$	$\forall$	$\{(\psi_1, s), (\psi_2, s)\}$	$-$
$(\Diamond\varphi, s)$	$\exists$	$\{(\varphi, t) \mid t \in R[s]\}$	$-$
$(\Box\varphi, s)$	$\forall$	$\{(\varphi, t) \mid t \in R[s]\}$	$-$
$(\neg p, s)$ and $p \notin \rho(s)$	$\forall$	$\emptyset$	$-$
$(\neg p, s)$ and $p \in \rho(s)$	$\exists$	$\emptyset$	$-$
(p, s) and $p \in \rho(s)$	$\forall$	$\emptyset$	$-$
(p, s) and $p \notin \rho(s)$	$\exists$	$\emptyset$	$-$
$(\top, s)$	$\forall$	$\emptyset$	$-$
$(\bot, s)$	$\exists$	$\emptyset$	$-$

Acceptance (parity) game $\mathcal{G}(\mathbb{A}, \mathcal{K})$

Definition: $\mathbb{A}$ accepts $(\mathcal{K}, s_I)$ iff $\exists$ has a winning strategy in $\mathcal{G}(\mathbb{A}, \mathcal{K})@(a_I, s_I)$

$$(\mathcal{K}, s_I) \in L(\mathbb{A})$$

Modal automata

$$\varphi = \nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$

$$\mathbb{A} = (\{a, \dots, m\}, a, \Delta, \text{rank})$$

$$\begin{cases} \Delta(a) = b \\ \Delta(b) = c \\ \Delta(c) = d \wedge e \\ \Delta(d) = f \vee g \\ \Delta(e) = h \vee i \\ \Delta(f) = \Diamond l \\ \Delta(g) = p \\ \Delta(h) = \neg p \\ \Delta(i) = \Diamond m \\ \Delta(l) = a \\ \Delta(m) = b \end{cases}$$

Modal automata

$$\varphi = \nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$

$$\mathbb{A} = (\{a, b\}, a, \Delta, \text{rank})$$

$$\Delta(a) = \Delta(b) = (\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\text{rank}(a) = 2$$
$$\text{rank}(b) = 1$$

$$(\mathcal{K}, s_I) \models \varphi$$

iff

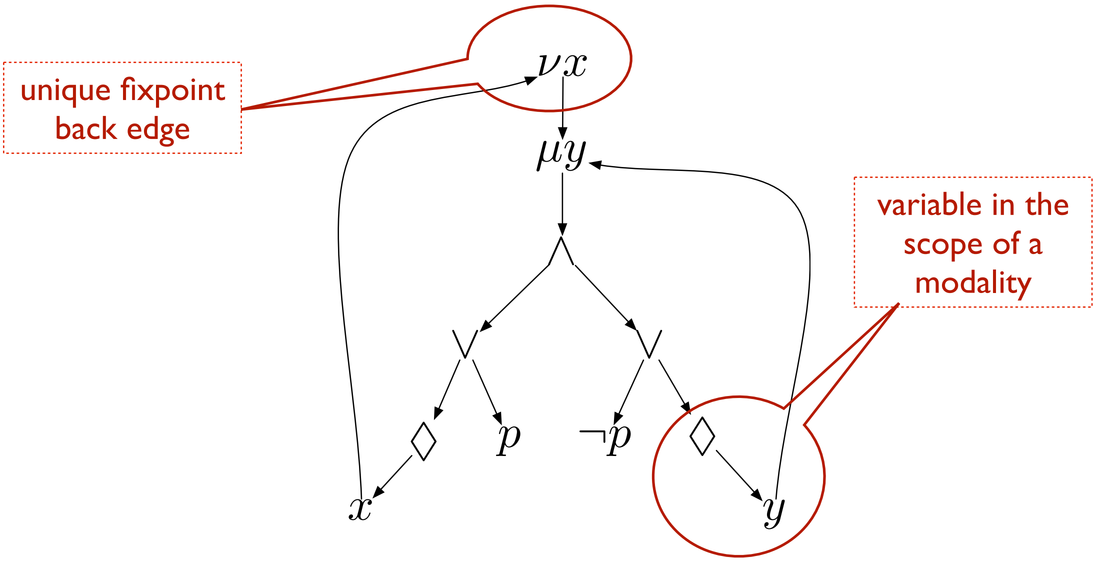
$$(\mathcal{K}, s_I) \in L(\mathbb{A})$$

Theorem:

1. For every μ-formula ϕ there is an equivalent modal automaton $\mathbb{A}_\phi$,
2. for every modal automaton $\mathbb{A}$ there is an equivalent μ-formula $\phi_\mathbb{A}$.

Modal automata

Proof: For item 1, let φ be a well-named and guarded μ-formula.



Modal automata

Proof: For item 1, let φ be a well-named and guarded μ-formula. Let $\mathbb{A}_\varphi$ given by

- $A_\varphi = \{\hat{\psi} \mid \psi \leq \varphi\}$,
- $a_I := \hat{\varphi}$,
- $\Delta(\hat{\psi}) = \begin{cases} \hat{\delta} \circ \hat{\theta} & \text{for } \psi = \delta \circ \theta \\ \circ\hat{\delta} & \text{for } \psi = \circ\delta, \circ = \Diamond, \Box \\ \psi & \text{for } \psi = p, \neg p, \bot, \top \\ \hat{\varphi}_x & \text{for } \psi = x \\ \hat{\theta} & \text{for } \psi = \eta x.\theta \end{cases}$

Proof (cont): and by

- $\text{rank}(\hat{\eta x.\delta}) = \begin{cases} \text{ad}(\eta x.\delta) & \text{if } \eta = \mu \text{ and ad}(\eta x.\delta) \text{ is odd, or} \\ & \eta = \nu \text{ and ad}(\eta x.\delta) \text{ is even;} \\ \text{ad}(\eta x.\delta) - 1 & \text{if } \eta = \mu \text{ and ad}(\eta x.\delta) \text{ is even, or} \\ & \eta = \nu \text{ and ad}(\eta x.\delta) \text{ is odd,} \end{cases}$
- $\text{rank}(\hat{x}) = \text{rank}(\hat{\varphi_x})$,
- $\text{rank}(\hat{\psi}) = \min(\{\text{rank}(\hat{\eta x.\delta}) \mid \eta x.\delta \leq \varphi\}$, for $\psi \neq x$ and $\psi \neq \eta x.\delta$.

Then $(\mathcal{K}, s_I) \models \varphi$ iff $(\mathcal{K}, s_I) \in L(\mathbb{A}_\varphi)$.

Proof (cont): For item 2, we reason as follows. Let $\mathbb{A} = (A, a_I, \Delta, \text{rank})$ over $P' = P \cup X$, and $\Delta : A \to \text{MLatt}(A \cup X; P)$.

Proof (cont): For item 2, we reason as follows.
Let $\mathbb{A} = (A, a_I, \Delta, \mathrm{rank})$ over $P' = P \cup X$, and
$\Delta : A \to \mathrm{MLatt}(A \cup X; P)$.

$p \wedge q \wedge (\Diamond a \vee \Diamond x)$

Modal automata

Proof (cont): For item 2, we reason as follows.
Let $\mathbb{A} = (A, a_I, \Delta, \mathrm{rank})$ over $P' = P \cup X$, and
$\Delta : A \to \mathrm{MLatt}(A \cup X; P)$.

(P,X)-automata

(P,∅)-automata = modal automata

Proof (cont): For item 2, we reason as follows. Let $\mathbb{A} = (A, a_I, \Delta, \text{rank})$ over $P' = P \cup X$, and $\Delta : A \to \text{MLatt}(A \cup X; P)$.

Claim: For every (P,X)-automata $\mathbb{A}$, there is an equivalent μ-formula $\varphi_{\mathbb{A}}$, where each $x \in X$ occurs positively in $\varphi_{\mathbb{A}}$.

Proof of claim: By induction on

$$\text{index(rank)} = \begin{cases} -1 & \text{if no cycles in } \mathbb{A}, \\ \max\{\text{rank}(a) \mid a \text{ is in a cycle }\} & \text{else.} \end{cases}$$

Proof of claim: By induction on the index.

If index = -1, just write down the corresponding modal formula.

$A = \{a_I, a, b\}$

$$\Delta(a_I) = (p \vee q) \wedge \Diamond a \wedge \Box b$$
$$\Delta(a) = \neg p \wedge \Box x$$
$$\Delta(b) = \bot$$

$$\varphi_{\mathbb{A}} = (p \vee q) \wedge \Diamond(\neg p \wedge \Box x) \wedge \Box\bot$$

Proof of claim: By induction on the index.

If index(rank) ≥ 0, let

$$M = \{a \in A \mid \text{rank}(a) = \text{index}(\text{rank}) \text{ and } a \text{ lies in some scc}\}$$

Wlog $a_I \notin M$.

Proof of claim: By induction on the index.

If index(rank) ≥ 0, let

$$M = \{a \in A \mid \text{rank}(a) = \text{index(rank) and } a \text{ lies in some scc}\}$$

$$\mathbb{A}_M = (A \setminus M, a_I, \Delta|_{A\setminus M}, \text{rank}|_{A\setminus M})$$

This is a $(P, X \cup M)$-automaton of lower rank.

Proof of claim: By induction on the index.

If index(rank) ≥ 0, let

$$M = \{a_0, \ldots, a_k\}$$

$$\begin{aligned}\mathbb{A}_i =& ((A \setminus M) \cup \{a_i^\star\}, a_i^\star,\\ & \Delta|_{A\setminus M} \cup \{(a_i^\star, \Delta(a_i))\}, \text{rank}|_{A\setminus M} \cup (a_i^\star, 0))\end{aligned}$$

All $(P, X \cup M)$-automata of lower rank.

Proof of claim (cont.):

$$\begin{array}{ccccc} \mathbb{A}_M, & \mathbb{A}_0, & \ldots, & \mathbb{A}_k \\ \| & \| & & \| \\ \varphi_M, & \varphi_0, & \ldots, & \varphi_k \end{array}$$

Proof of claim (cont.):

Let $\overline{\varphi} = (\varphi_0, \ldots, \varphi_k)$.

$$\|\overline{\varphi}\|_{\mathcal{K}} : \wp(S)^{k+1} \to \wp(S)^{k+1}$$

$$\|\overline{\varphi}\|_{\mathcal{K}}(X_0, \ldots, X_k) := (\|\varphi_0\|_{\mathcal{K}[\overline{a} \mapsto \overline{X}]}, \ldots, \|\varphi_k\|_{\mathcal{K}[\overline{a} \mapsto \overline{X}]})$$

is monotone.

Proof of claim (cont.):

Let $\overline{\varphi} = (\varphi_0, \ldots, \varphi_k)$.

$$\|\overline{\varphi}\|_{\mathcal{K}} : \wp(S)^{k+1} \to \wp(S)^{k+1}$$

From the first lesson, we know that there are $\varphi_0^\mu, \ldots, \varphi_k^\mu$ and $\varphi_0^\nu, \ldots, \varphi_k^\nu$ s.t.

$$\begin{cases} (\|\varphi_0^\mu\|_{\mathcal{K}}, \ldots, \|\varphi_k^\mu\|_{\mathcal{K}}) & \text{is the lfp of } \|\overline{\varphi}\|_{\mathcal{K}} \\ (\|\varphi_0^\nu\|_{\mathcal{K}}, \ldots, \|\varphi_k^\nu\|_{\mathcal{K}}) & \text{is the gfp of } \|\overline{\varphi}\|_{\mathcal{K}} \end{cases}$$

Proof of claim (cont.):

Let $\varphi_{\mathbb{A}} = \varphi_M[a_0/\varphi_0^{\eta_0}, \ldots, a_k/\varphi_k^{\eta_k}]$, where

$$\eta_\ell = \begin{cases} \mu & \text{if rank}(a_\ell) = \text{index(rank) odd} \\ \nu & \text{else.} \end{cases}$$

Proof of claim (cont.):

Let $\varphi_{\mathbb{A}} = \varphi_M[a_0/\varphi_0^{\eta_0}, \ldots, a_k/\varphi_k^{\eta_k}]$, where

$$\eta_\ell = \begin{cases} \mu & \text{if } \mathrm{rank}(a_\ell) = \mathrm{index(rank)} \text{ odd} \\ \nu & \text{else.} \end{cases}$$

One can then check that

$$(\mathcal{K}, s) \models \varphi_{\mathbb{A}} \text{ iff } (\mathcal{K}, s) \in L(\mathbb{A})$$

■

$$\mathbb{A} = (\{a, b\}, a, \Delta, \mathrm{rank})$$

$$\Delta(a) = \Delta(b) = (\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\mathrm{rank}(a) = 2$$
$$\mathrm{rank}(b) = 1$$

Modal automata

$$\mathbb{A} = (\{a_I, a, b\}, a_I, \Delta, \text{rank})$$

$$\Delta(c) = (\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\text{rank}(a_I) = \text{rank}(a) = 2$$
$$\text{rank}(b) = 1$$

irrelevant priority

Modal automata

$M = \{a\}$

$$\mathbb{A}_a = (\{a_I, b\}, a_I, \Delta|_{\{a_I, b\}}, \text{rank}|_{\{a_I, b\}})$$

$$\Delta(c) = (\Diamond x_a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\text{rank}(a_I) = 2$$
$$\text{rank}(b) = 1$$

irrelevant priority

Modal automata

$M' = \{a, b\}$

$$(\mathbb{A}_a)_b = (\{a_I\}, a_I, \Delta|_{\{a_I\}}, \text{rank}|_{\{a_I\}})$$

$$\Delta(a_I) = (\Diamond x_a \vee p) \wedge (\Diamond x_b \vee \neg p)$$

$$\varphi_{(\mathbb{A}_a)_b} = (\Diamond x_a \vee p) \wedge (\Diamond x_b \vee \neg p)$$

Modal automata

$$\mathbb{A}_a = (\{a_I, b\}, a_I, \Delta|_{\{a_I, b\}}, \text{rank}|_{\{a_I, b\}})$$

$$\Delta(c) = (\Diamond x_a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\text{rank}(a_I) = 2$$
$$\text{rank}(b) = 1$$

$$\varphi_{\mathbb{A}_a} = \mu b.(\Diamond x_a \vee p) \wedge (\Diamond b \vee \neg p)$$

Modal automata

$$\mathbb{A} = (\{a, b\}, a, \Delta, \mathrm{rank})$$

$$\Delta(a) = \Delta(b) = (\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\mathrm{rank}(a) = 2$$
$$\mathrm{rank}(b) = 1$$

$$\varphi_{\mathbb{A}} = \nu a.\mu b.(\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

Guarded modal automata

Given a set A of (state) variables, and a set P of propositional variables: the set $\mathrm{MLatt}_g(A; P)$ is defined as:

$$\phi ::= \top \mid \bot \mid p \mid \neg p \mid \Diamond a \mid \Box a \mid \bigwedge \Phi \mid \bigvee \Phi$$

with $a \in A$ and $p \in P$

Guarded modal automata

Theorem: For every modal automaton there is an equivalent guarded one.

Proof hint: 'Syntactical massage'.

A general approach

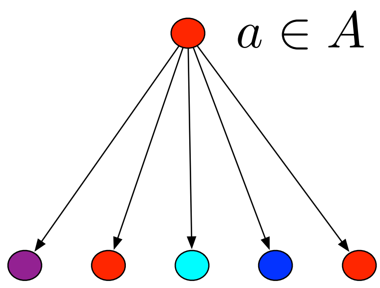
Parity automata: $\mathrm{Aut}(\mathcal{L})$

$$(A, \Sigma, a_I, \Delta, \mathrm{rank} : Q \to \mathbb{N})$$

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$

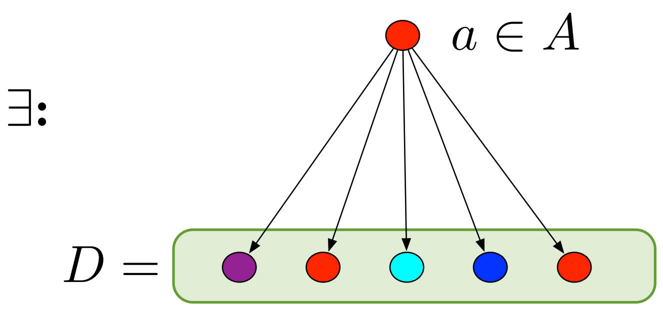
A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$



A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$



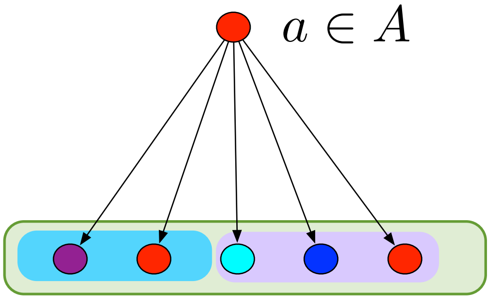
A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$

$a \in A$

$\exists$:

$(D, V) =$



$$V : A \to \wp D$$

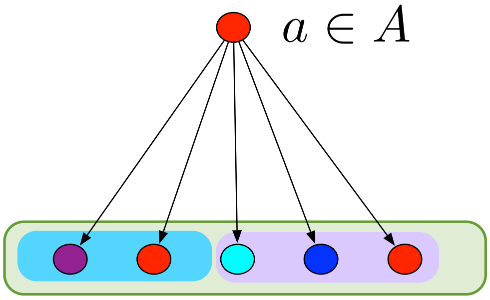
A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$

$a \in A$

$\exists$:

$(D, V) =$

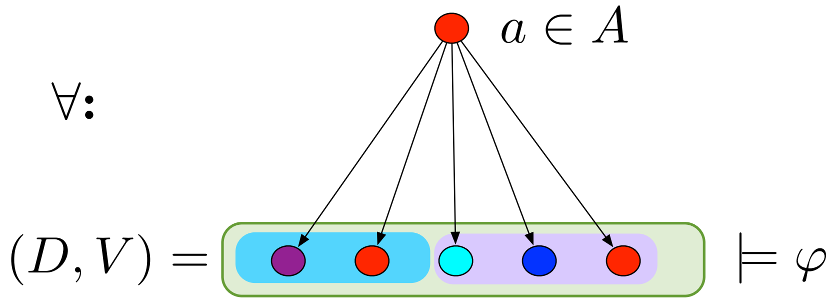


$\models \varphi$

$$V : A \to \wp D$$

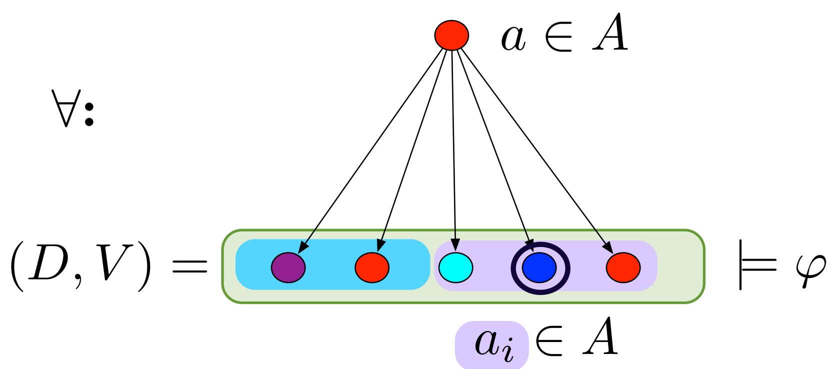
A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$



A general approach

$$\Delta : (a, c) \mapsto \varphi \in \mathcal{L}(A)$$



Fact: Every $\phi \in \mathrm{MLatt}_g(A;P)$ if equivalent to disjunction of formulas of the form

$$\bigwedge_{p\in Q} p \wedge \bigwedge_{p\notin Q} \neg p \wedge \psi$$

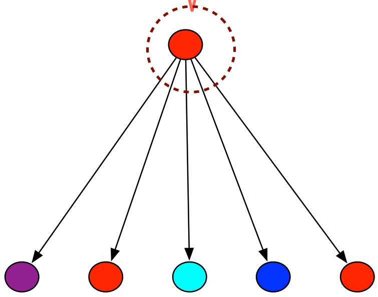
for $Q \subseteq P$ and $\psi \in \mathrm{MLatt}_g(A;\emptyset)$

A general approach

$$\bigwedge_{p\in Q} p \wedge \bigwedge_{p\notin Q} \neg p \wedge \psi$$

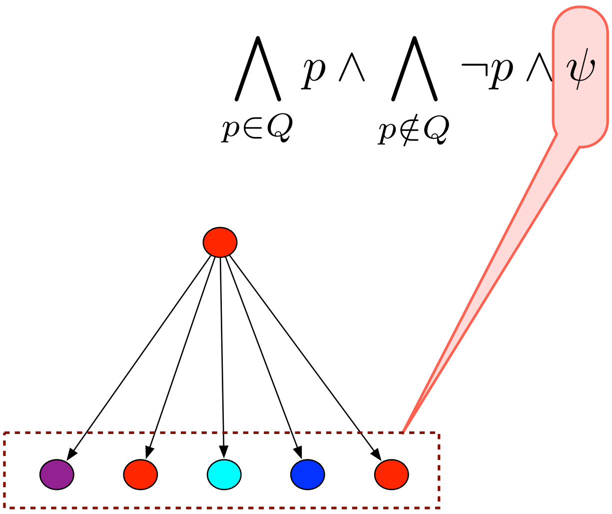
A general approach

$$\bigwedge_{p\in Q} p \wedge \bigwedge_{p\notin Q} \neg p \wedge \psi$$



A general approach

$$\bigwedge_{p\in Q} p \wedge \bigwedge_{p\notin Q} \neg p \wedge \psi$$



A general approach

$$\Delta : (a, Q) \mapsto \psi \in \mathrm{Mlatt}_g(A; \emptyset)$$

$$\bigwedge_{p \in Q} p \wedge \bigwedge_{p \notin Q} \neg p \wedge \psi$$

A general approach

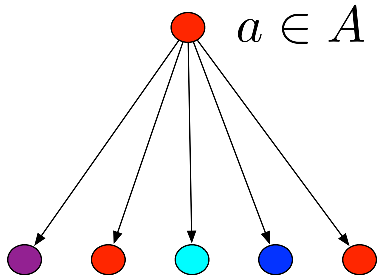
$$\Delta : (a, Q) \mapsto \psi \in \mathrm{Mlatt}_g(A; \emptyset)$$

$$\begin{cases} \Diamond a \mapsto \exists x.a(x) \\ \Box a \mapsto \forall x.a(x) \end{cases}$$

$$\Delta : (a, Q) \mapsto \psi \in \mathrm{FO}^+(A)$$

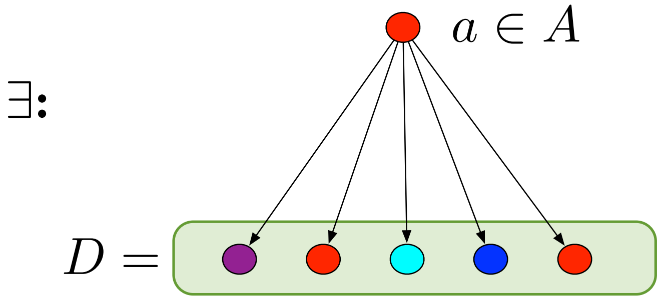
A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$



A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$



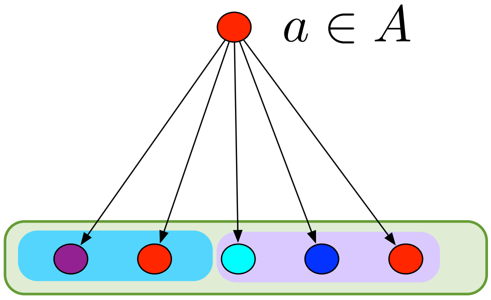
A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$

$a \in A$

∃:

$(D, V) =$



$$V : A \to \wp D$$

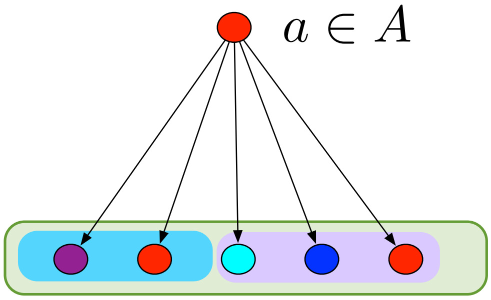
A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$

$a \in A$

∃:

$(D, V) =$

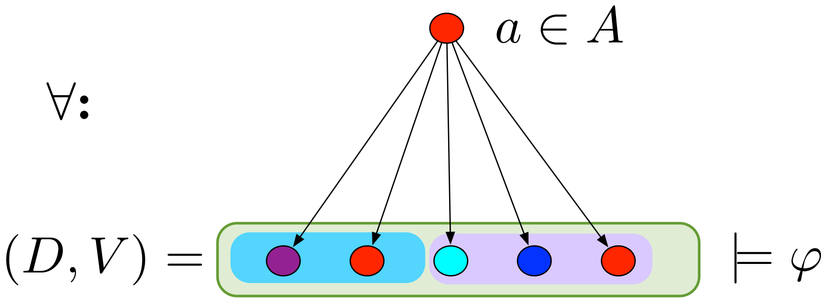


$\models \varphi$

$$V : A \to \wp D$$

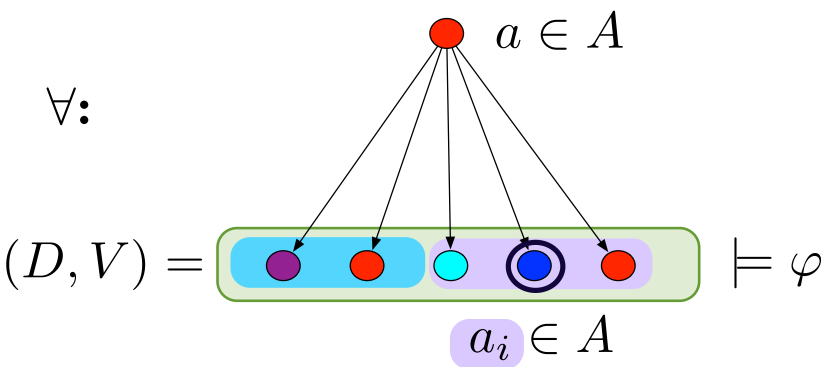
A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$



A general approach

$$\Delta : (a, Q) \mapsto \varphi \in \mathrm{FO}^+(A)$$



One-step logic

Given a set A of (state) variables, the set of formula $\mathrm{FO}(A)$ is defined as:

$$\phi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x.\phi \mid \forall x.\phi$$

with $a \in A$.

One-step logic

Given a set A of (state) variables, the set of formula $\mathrm{FO}^+(A)$ is defined as:

$$\phi ::= \top \mid \bot \mid a(x) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x.\phi \mid \forall x.\phi$$

with $a \in A$.

One-step logic

Given a set A of (state) variables, the set of formula $\mathrm{FOE}(A)$ is defined as:

$$\phi ::= \top \mid \bot \mid x = y \mid x \neq y \mid a(x) \mid \neg a(x) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x.\phi \mid \forall x.\phi$$

with $a \in A$.

One-step logic

Given a set A of (state) variables, the set of formula $\mathrm{FOE}^{+}(A)$ is defined as:

$$\phi ::= \top \mid \bot \mid x = y \mid x \neq y \mid a(x) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x.\phi \mid \forall x.\phi$$

with $a \in A$.

One-step logic

Models of one-step formulas are pairs

$$(D, V)$$

- D is a non-empty set
- $V : A \to \wp D$

Mu automata

Definition: A μ-automaton is a tuple

$$\mathbb{A} = (A, \wp P, a_I, \Delta, \Omega)$$

such that

- $a_I \in A$ (initial state)
- $\Delta : A \times \wp P \to \mathrm{FO}^+(A)$ (transition fct)
- $\mathrm{rank} : A \to \mathbb{N}$ (parity fct)

Acceptance (parity) game $\mathcal{G}(\mathbb{A}, \mathcal{K})$

Let $\mathcal{K} = (S, R, \rho)$ be a Kripke model.

Position	Player	Admissible moves	Parity
$(a, s) \in A \times S$	$\exists$	$\{V : A \to \wp(R[s]) \mid$ $(R[s], V) \models \Delta(a, \rho(s))\}$	$\mathrm{rank}(a)$
$V : A \to \wp S$	$\forall$	$\{(b, t) \mid t \in V(b)\}$	$\max(\mathrm{rank}[A])$

Acceptance (parity) game $\mathcal{G}(\mathbb{A}, \mathcal{K})$

Definition: $\mathbb{A}$ accepts $(\mathcal{K}, s_I)$ iff $\exists$ has a winning strategy in $\mathcal{G}(\mathbb{A}, \mathcal{K})@(a_I, s_I)$

$$(\mathcal{K}, s_I) \in L(\mathbb{A})$$

Mu automata

$$\varphi = \nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$

$$\mathbb{A} = (\{a, b\}, a, \Delta, \mathrm{rank})$$

$$\Delta(a) = \Delta(b) = (\Diamond a \vee p) \wedge (\Diamond b \vee \neg p)$$

$$\mathrm{rank}(a) = 2$$
$$\mathrm{rank}(b) = 1$$

Mu automata

$$\varphi = \nu x.\mu y.(\Diamond x \vee p) \wedge (\Diamond y \vee \neg p)$$

$$\mathbb{A} = (\{a, b\}, \wp P, a, \Delta, \mathrm{rank})$$

$$\Delta(a, Q) = \Delta(b, Q) = \begin{cases} \exists x.a(x) & \text{if } p \notin Q \\ \exists x.b(x) & \text{if } p \in Q \end{cases}$$

$$\mathrm{rank}(a) = 2$$
$$\mathrm{rank}(b) = 1$$

Theorem:

1. For every modal automaton there is an equivalent μ-automaton ,
2. for every μ-automaton there is an equivalent modal automaton.

Proof: Point 1 is immediate from what precede. Point 2 is a corollary of the simulation theorem.

A **type** is a subset of P.

Let Q be a type.

- $\tau_Q(x) := \begin{cases} \top & \text{if } Q = \emptyset \\ \bigwedge_{p \in Q} p(x) \wedge \bigwedge_{p \notin Q} \neg p(x) & \text{else;} \end{cases}$
- $\tau_Q^+(x) := \begin{cases} \top & \text{if } Q = \emptyset \\ \bigwedge_{p \in Q} p(x) & \text{else.} \end{cases}$

Definition: A formula $\phi \in \mathrm{FO}^+(A)$ is in special basic normal form if it is of the form

$$\exists x_0 \ldots \exists x_k \bigwedge_{i \leq k} \tau^+_{Q_i}(x_i) \wedge \forall y. \bigvee_{i \leq k} \tau^+_{Q_i}(x)$$

where each type Q_i is either empty or a singleton. We say that $\phi \in \mathrm{SBF}^+(A)$.

Definition: A μ-automaton $\mathbb{A}$ is non-deterministic if

$$\Delta : A \times \wp P \to \mathrm{SLatt}(SBF^+(A))$$

Simulation Theorem: Every μ-automaton is equivalent to a non-deterministic one.

Proof: ... (tomorrow, for MSO-automata.)

Theorem: Given a μ-automaton $\mathbb{A}$ it is decidable whether $L(\mathbb{A}) = \emptyset$.

Proof: Let $\mathbb{A}$ be a μ-automaton. By the Simulation Theorem, there is a non-deterministic μ-automaton $\mathbb{B}$ such that

$$L(\mathbb{A}) = L(\mathbb{B})$$

It is thus enough to check that the emptiness problem is decidable for $\mathbb{B}$.

Proof (cont.): Transitions of $\mathbb{B}$ are disjunctions of formulas of the form

$$\exists x_0 \ldots \exists x_k \bigwedge_{i \leq k} \tau^+_{Q_i}(x_i) \wedge \forall y. \bigvee_{i \leq k} \tau^+_{Q_i}(x)$$

where each type Q_i is either empty or a singleton.

Proof (cont.): We define the following emptiness game over $\mathbb{B}$, denoted by $\mathcal{E}(\mathbb{B})$

Position	Player	Admissible moves	Parity
$a \in B$	$\exists$	$\{(\phi, Q) \mid Q \in \wp P \wedge \exists i \leq k$ $\Delta(a, Q) = \bigvee_{\ell \leq k} \psi_\ell \wedge \psi_i = \phi\}$	$\mathrm{rank}(a)$
$(\exists \overline{x} \bigwedge_{i \leq k} \tau^+_{Q_i}(x_i)$ $\wedge \forall y. \bigvee_{i \leq k} \tau^+_{Q_i}(x), Q)$	$\forall$	$\bigcup_{i \leq k} Q_i$	$-$

Claim: $L(\mathbb{B}) \neq \emptyset$ iff $\exists$ has a winning strategy in $\mathcal{E}(\mathbb{B})@b_I$.

Proof of claim: From left to right, let $\mathcal{K} \in L(\mathbb{B})$. Thus $\exists$ has a w.s. σ in $\mathcal{G}(\mathbb{B}, \mathcal{K})@(b_I, s_I)$. Such σ induces a w.s. for $\exists$ in $\mathcal{E}(\mathbb{B})@b_I$.

Proof of claim (cont.): From right to left, let σ be a w.s. for $\exists$ in $\mathcal{E}(\mathbb{B})@b_I$. By positional determinacy of parity games, we can assume σ positional. Consider T_σ, the tree representing σ. Since σ is positional, we can define a model $\mathcal{K}_\sigma$ as follows:

- $S_\sigma = B \cap T_\sigma$ and $s_I = b_I$,
- $(b, b') \in R_\sigma$ iff $b' \in \bigcup_i Q_i$ and $(\exists \overline{x} \bigwedge_{i \leq k} \tau^+_{Q_i}(x_i), Q) = \sigma(b)$,
- $\rho_\sigma(b) = Q$, where $\sigma(b) = (\phi, Q)$.

Proof of claim (cont.): Notice that $|\mathcal{K}_\sigma| \leq |B|$. Clearly σ induces a w.s. for $\exists$ in $\mathcal{G}(\mathbb{B}, \mathcal{K}_\sigma)@(b_I, s_I)$.

The Simulation Theorem

Corollary (Small Model Property): Let ϕ be a μ-formula. Then if ϕ is satisfiable, it has a model of size exponential in the size of the formula.

On the usefulness of mu-automata

Mu automata - and the corresponding simulation theorem - are crucially used in proving some other important results in the theory of the modal mu-calculus

Kozen's axiom system

(Prop) propositional tautologies,

(Sub) if $\vdash \varphi$ then $\vdash \varphi[p/\psi]$,

(K) $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$,

(Nec) if $\vdash \varphi$ then $\vdash \Box\varphi$,

(FA) $\vdash \varphi[x/\mu x.\varphi] \rightarrow \mu x.\varphi$,

(FR) if $\vdash \varphi[x/\psi] \rightarrow \psi$ then $\vdash \mu x.\varphi \rightarrow \psi$,

with $x \notin \text{bound}(\varphi)$ and $\text{free}(\psi) \cap \text{bound}(\varphi) = \emptyset$.

Theorem [Walukiewicz (1995)]: Kozen's axiomatisation is (weakly) sound and complete (i.e. $\text{Ax} \vdash \varphi$ iff $\models \varphi$).

W.'s proof makes crucial use of mu-automata (and of the simulation theorem). At the moment is the unique proof we know for this result.

On the usefulness of mu-automata

μ-automata can also be used in order to prove that:

- the μ-calculus enjoys uniform interpolation and Łoś-Tarski theorem [D'Agostino, Hollenberg (2000)],
- it can be decided whether φ is continuous in p [Fontaine (2008)],
- the μ-calculus is the bisimulation invariant fragment of MSO [Janin, Walukiewicz (1996)]
- ...

What we have seen today...