Modal Fixpoint Logics: When Logic Meets Games, Automata and Topology

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Lecture I

Rudiments of fixpoint logics

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How to define a big object shortly ?

How to define an infinite object at all ?

Recursion

Perpetuum mobile

Complex concepts in mathematics are often defined in recursive way.

This may involve risky steps like

$$tree \quad = \quad \begin{tikzcd} & \circ \arrow[dl] \arrow[dr] & \\ tree & & tree \end{tikzcd}$$

The correctness relies on the existence of *fixed points*.

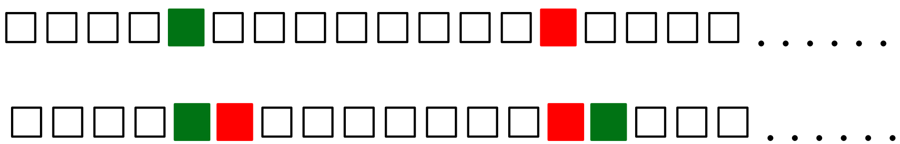
Example

Let u be a sequence of bits, such that the rewriting

$0 \to 01$

$1 \to 10$

produces the same sequence.



Does it exist ??

Example Thue-Morse sequence

$0 \to 01$

$1 \to 10$

u_0	0															
u_1	0								1							
u_2	0				1				1				0			
u_3	0		1		1		0		1		0		0		1	
u_4	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

0110100110010110100101100110100110010110011010010110100110010110...

$\lim u_n$ is a fixed point $u = u[01/0, 10/1]$.

Fixed point of a function

$$x = f(x) = f\left(f(x)\right) = f(f(f(x))) = f\left(f(f(f(x)))\right) = \ldots$$

Plus ça change, plus c'est la même chose. Alphonse Karr, 1849

Fixed point theorems

Brouwer A continuous mapping of a closed ball into itself has a fixed point.

Banach A contracting mapping of a complete metric space into itself has a (unique) fixed point.

Knaster-Tarski A monotonic mapping of a complete lattice into itself has a (least) fixed point.

……

Example von Neumann definition of $\mathbb{N}$

The least set X, such that $\emptyset \in X$ and $x \in X \implies x \cup \{x\} \in X$.

$$\underbrace{\{\emptyset\} \cup \{x \cup \{x\} : x \in X\}}_{Z} \quad \subseteq \quad X$$

$$\{\emptyset\} \cup \{z \cup \{z\} : z \in Z\} \quad \stackrel{?}{\subseteq} \quad Z$$

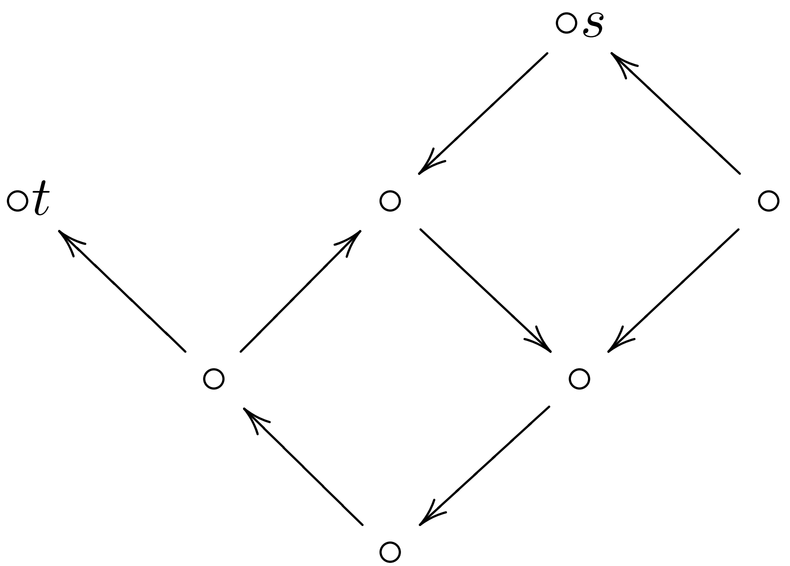
$z = x \cup \{x\} \ \wedge \ x \in X \implies z \in X \implies z \cup \{z\} \in Z.$

Yes**!** Hence,

$$\{\emptyset\} \cup \{x \cup \{x\} : x \in \mathbb{N}\} \quad = \quad \mathbb{N}$$

Example – reachability

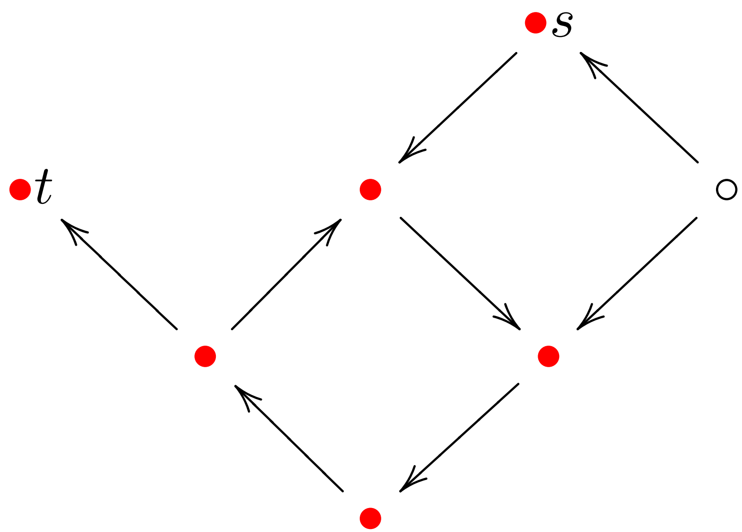
Is there a path from s to t ?



There a path from s to t iff t belongs to the **least** set of nodes X, s.t.

$$\{s\} \cup succ(X) \quad \subseteq \quad X$$

where $succ(X) = \{y : (\exists x \in X)\, x \to y\}$.



Note: this X is a **fixed point**, because $Z = \{s\} \cup succ(X)$ also satisfies $\{s\} \cup succ(Z) \subseteq Z$.

Why do we care about fixed points ?

Knowing that the least X s.t. $\{s\} \cup succ(X) \subseteq X$ satisfies

$$X = \{s\} \cup succ(X)$$

we can compute it by iteration

$$\{s\}$$

$$\{s\} \cup succ(\{s\})$$

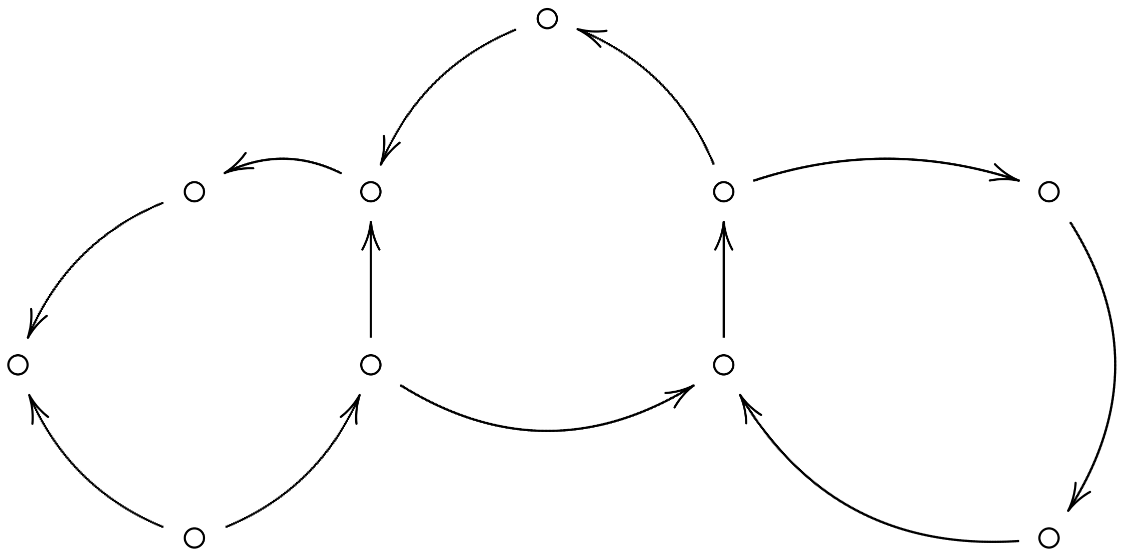
$$\{s\} \cup succ(\{s\}) \cup succ\left(succ(\{s\})\right)$$

$$\cdots\cdots\cdots\cdot$$

until it stops changes

$$X = \emptyset \cup F(\emptyset) \cup F^2(\emptyset) \cup F^3(\emptyset) \cup \ldots$$

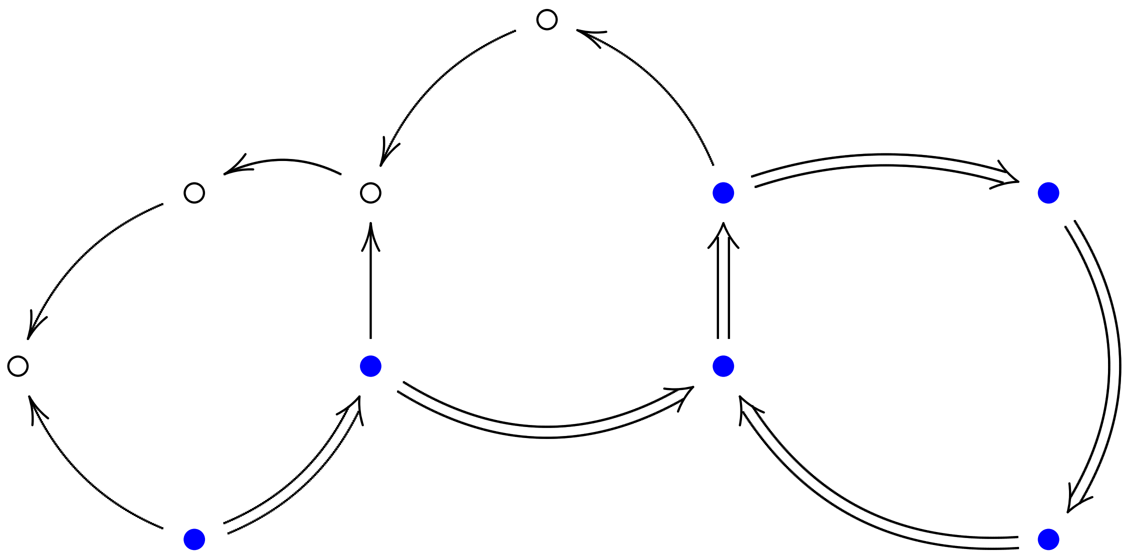
Example – infinite path



Does this graph admit an infinite path? An exhaustive search is costly. . .

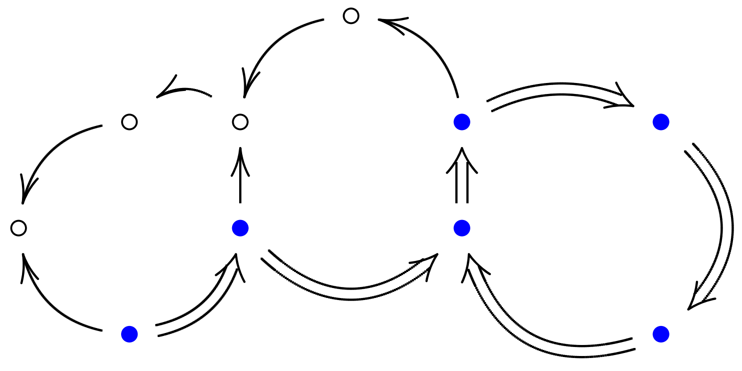
Try to characterize the **nodes**, which **originate** infinite paths.

Example – infinite path



The nodes, which originate infinite paths (**Origin-∞**) could say:

I am lucky there, because after some move I can be lucky again.



If a set Z satisfies the **"luckiness property"**

$$x \in Z \quad \Longrightarrow \quad (\exists z \in Z)\, x \to z$$

shorter notation: $$Z \quad \subseteq \quad \Diamond(Z)$$

then any $z \in Z$ originates an infinite path, i.e., $Z \subseteq$ **Origin-∞**. But

$$\textbf{Origin-}\infty \quad \subseteq \quad \Diamond(\textbf{Origin-}\infty)$$

hence, **Origin-∞** is a **maximal** set with luckiness property.

A **maximal** set satisfying the inequality $Z \subseteq \Diamond(Z)$ is a **fixed point**

$$Z \quad = \quad \Diamond(Z)$$

(otherwise $Z \subset \underline{\Diamond(Z)} \subseteq \Diamond(\underline{\Diamond(Z)})$).

Hence, it can be **computed** by iteration

$$\textbf{Origin-}\infty \quad = \quad \bigcap_{\xi} \Diamond^{\xi}(\mathbb{T})$$

On finite graphs, this yields a polynomial time algorithm.

General setting: **Knaster-Tarski Theorem**

A monote mapping $f : L \to L$ of a complete lattice L has

a least fixed point

$$\mu x.f(x) \quad = \quad \bigwedge \{d : f(d) \leq d\}$$

and a greatest fixed point

$$\nu x.f(x) \quad = \quad \bigvee \{d : d \leq f(d)\}$$

Proof for ν.

Let $a = \bigvee \underbrace{\{z : z \leq f(z)\}}_{A}$.

$a \geq A \ni z \leq f(z) \leq f(a)$. Thus $A \leq f(a)$, hence $a \leq f(a)$.

By monotonicity, $f(a) \leq f(f(a))$, hence $f(a) \in A$, hence $f(a) \leq a$.

Alternative presentation of fixed points.

$$\mu x.f(x) \quad = \quad \bigvee_{\xi \in Ord} f^{\xi}(\bot)$$

where

$$\begin{aligned} f^{\xi+1}(\bot) &= f\left(f^{\xi}(\bot)\right) \\ f^{\eta}(\bot) &= \bigvee_{\xi<\eta} f^{\xi}(\bot), \text{ for limit } \eta. \end{aligned}$$

Similarly

$$\nu x.f(x) \quad = \quad \bigwedge_{\xi \in Ord} f^{\xi}(\top)$$

A great number of concepts can be defined by μ or ν.

But the **fixpoint logics** start from an observation that

$$\mu x.\nu y.f(x, y),$$

is meaningful as well.

$$
\begin{array}{ccccc}
\mu x.\nu y.f(x,y) & & & & \\
\| & & & & \\
x & = & \nu y.f(x,y) & & \\
 & & \| & & \\
 & & y & = & f(x,y)
\end{array}
$$

Note that $a = \mu x.\nu y.f(x,y)$ satisfies $a = f(a,a)$, hence

$$\mu x.f(x,x) \leq \mu x.\nu y.f(x,y) \leq \nu y.f(y,y)$$

Example – words

Languages of finite and infinite words over alphabet Σ.

$\varepsilon \notin A \subseteq \Sigma^*$, $B \subseteq \Sigma^* \cup \Sigma^\omega$, X, Y range over $\wp(\Sigma^* \cup \Sigma^\omega)$,
$A^* = \bigcup_n A^n$ (with $A^0 = \{\varepsilon\}$), $A^\omega = \{w_0 w_1 w_2 \ldots : w_i \in A,\ i < \omega\}$.

$$
\begin{array}{llcl}
 & X & \stackrel{?}{=} & AX \cup B \\
\text{least solution} & X & = & A^*B \\
\text{greatest solution} & X & = & A^*B \cup A^\omega \\
\text{i.e.,} & \mu X.AX \cup B & = & A^*B \\
 & \nu X.AX \cup B & = & A^*B \cup A^\omega .
\end{array}
$$

$$
\begin{array}{llcl}
\text{Note} & \mu X.AX & = & \emptyset \\
 & \nu X.AX & = & A^\omega
\end{array}
$$

Further

$$
\begin{array}{lrcl}
 & \mu X.AX \cup BY & = & A^*BY \\
 & Y & \stackrel{?}{=} & A^*BY \\
\text{greatest solution} & Y & = & (A^*B)^\omega \\
\text{i.e.,} & \nu Y.\mu X.AX \cup BY & = & (A^*B)^\omega \\
\hline
\end{array}
$$

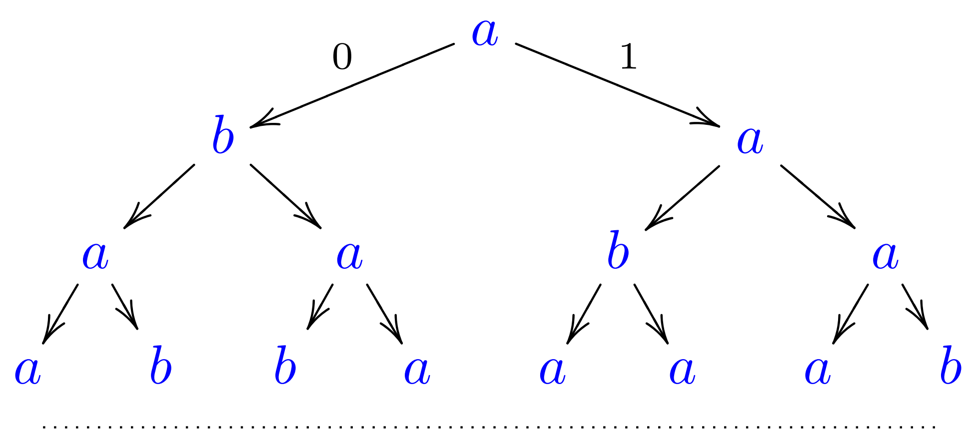
$$
\begin{array}{rcl}
\nu Y.AX \cup BY & = & B^*AX \cup B^\omega \\
X & \stackrel{?}{=} & B^*AX \cup B^\omega \\
\mu X.\nu Y.AX \cup BY & = & (B^*A)^*\,B^\omega \\
\hline
\end{array}
$$

Note

$$
\mu X.\nu Y.AX \cup BY \quad \subseteq \quad \nu Y.\mu X.AX \cup BY
$$

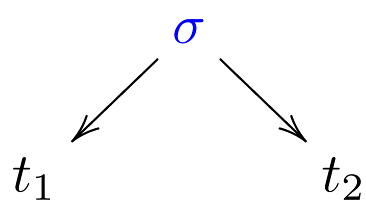
Example – trees

A (full binary) Σ-labeled tree is a mapping $t : 2^* \to \Sigma$.



Each $\sigma \in \Sigma$ induces an operation on trees

$\sigma(t_1, t_2) =$



and consequently on tree languages $L_1, L_2 \subseteq T_\Sigma$

$$\sigma(L_1, L_2) = \{\sigma(t_1, t_2) : t_1 \in L_1, \;\; t_2 \in L_2\}$$

Example – trees continued

Let $\Sigma = \{a, b\}$.

$\nu y.\mu x.a(x, x) \cup b(y, y)$ = on each path there are **infinitely many b's**

i.e., all paths are in $\nu y.\mu x.ax \cup by$,

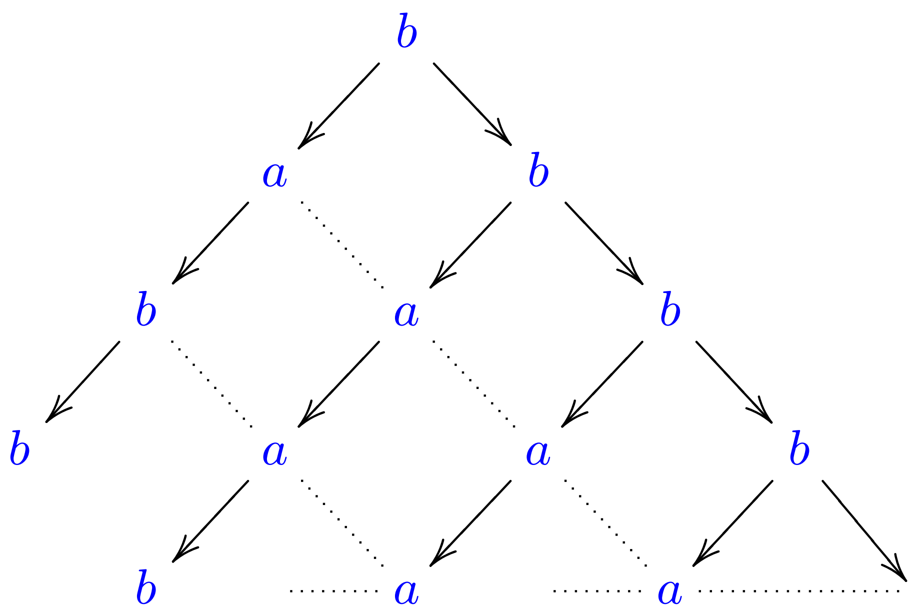
$\mu x.\nu y.a(x, x) \cup b(y, y)$ = on each path there are **only finitely many a's**

i.e., all paths are in $\mu x.\nu y.ax \cup by$.

Again $\mu x.\nu y \ldots \subseteq \nu y.\mu x \ldots$

Parenthesis.

$\mu x.\nu y.a(x,x) \cup b(y,y)$ = on each path there are only finitely many a's



This set encodes the set of well founded trees $T \subseteq \omega^*$, and can be proved $\mathbf{\Pi^1_1}$-complete, as a subset of the Cantor space $\{0,1\}^\omega$.

Example – trees continued

The pattern can be generalized.

$$
\begin{aligned}
\mu x_1.\nu x_0. &\quad a_0(x_0, x_0) \cup a_1(x_1, x_1) \\
\nu x_2.\mu x_1.\nu x_0. &\quad a_0(x_0, x_0) \cup a_1(x_1, x_1) \cup a_2(x_2, x_2) \\
\mu x_3.\nu x_2.\mu x_1.\nu x_0. &\quad a_0(x_0, x_0) \cup a_1(x_1, x_1) \cup a_2(x_2, x_2) \cup a_3(x_3, x_3) \\
\cdots\cdots &\quad \cdots\cdots
\end{aligned}
$$

On each path, if some a_i with i **odd** occurs infinitely often then there is some a_j with j **even**, which also occurs infinitely often, and $j > i$.

In short: the **highest k**, such that a_k occurs infinitely often on a path, is **even**.

Basic laws of fixed points

$$
\begin{aligned}
\mu x.\mu y.f(x, y) &= \mu x.f(x.x) \\
\nu x.\nu y.f(x, y) &= \nu x.f(x.x) \\
\mu x.\nu y.f(x, y) &\leq \nu y.\mu x.f(x, y)
\end{aligned}
$$

If $a = \theta x.\theta' y.f(x, y)$ then

$$
\begin{aligned}
a &= \theta' y.f(a, y) \\
&= \theta x.f(x, a)
\end{aligned}
$$

Example – quasi-equational proof

$$\underbrace{\mu x.\nu y.f(x,y)}_{a} \quad \leq \quad \nu y.\mu x.f(x.y)$$

$a = f(a,a)$ implies $\mu x.f(x,a) \leq a$. By monotonicity of $\nu y.f(z,y)$ (in z)

$$\nu y.f(\underline{\mu x.f(x,a)}, y) \leq \nu y.f(\underline{a}, y) = a$$

By monotonicity of f

$$f(\mu x.f(x,a), \underline{\nu y.f(\mu x.f(x,a), y)}) \leq f(\mu x.f(x,a), \underline{a})$$

By reducing both sides ($F(\theta x.F(x)) \to \theta x.F(x)$)

$$\nu y.f(\underline{\mu x.f(x,a)}, y) \leq \underline{\mu x.f(x,a)}$$

By Knaster-Tarski Theorem this implies ($\underline{a} =$) $\mu x.\nu y.f(x,y) \leq \mu x.f(x,\underline{a})$.

Again by Knaster-Tarski, $a \leq \nu y.\mu x.f(x,y)$.

Vectorial fixed points – Bekič Principle

Let $(L, \leq_L)$, $(K, \leq_K)$ be two complete lattices and

$$F : L \times K \to L \times K$$

be monotonic in two arguments. Let $F = (F_1, F_2)$. Then

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} .F(x,y) = \begin{pmatrix} \mu x.F_1\,(x, \mu y.F_2(x,y)) \\ \mu y.F_2\,(\mu x.F_1(x,y), y) \end{pmatrix}$$

Thus vectors can be eliminated at the expense of increasing the length.

Fixed point clones

A family $\mathcal{C}$ of monotonic mappings of a finite arity over a complete lattice L is a **clone** if it is closed under composition and contains all projections $\pi_k^i : L^k \to L$,

$$\pi_k^i : (a_1, \ldots, a_k) \mapsto a_i$$

It is a **μ-clone** if moreover is closed under μ, i.e.,

$$\mathcal{C} \ni f(x_1, \ldots, x_k) \Longrightarrow \mu x_i . f(x_1, \ldots, x_k) \in \mathcal{C}.$$

A **ν-clone** is defined similarly.

Comp$(\mathcal{F})$	the least clone	
$\mu(\mathcal{F})$	the least μ-clone	
$\nu(\mathcal{F})$	the least ν-clone	containing $\mathcal{F}$

Fixed point hierarchy

$$\Sigma_0^{\mu}(\mathcal{F}) = \Pi_0^{\mu}(\mathcal{F}) = \mathsf{Comp}(\mathcal{F})$$

$$\Sigma_{n+1}^{\mu}(\mathcal{F}) = \mu\,(\Pi_n^{\mu}(\mathcal{F}))$$

$$\Pi_{n+1}^{\mu}(\mathcal{F}) = \nu\,(\Sigma_n^{\mu}(\mathcal{F}))$$

$$fp(\mathcal{F}) = \bigcup_n \Sigma_n^{\mu}(\mathcal{F}) = \bigcup_n \Pi_n^{\mu}(\mathcal{F})$$

$$\begin{tikzcd}
\Sigma_0 \arrow[r, dash] \arrow[d, equal] \arrow[dr, dash] & \Sigma_1 \arrow[r, dash] \arrow[dr, dash] & \Sigma_2 \arrow[r, dash] \arrow[dr, dash] & \Sigma_3 \arrow[r, dotted, dash] \arrow[dr, dotted, dash] & \ldots \\
\Pi_0 \arrow[r, dash] \arrow[ur, dash] & \Pi_1 \arrow[r, dash] \arrow[ur, dash] & \Pi_2 \arrow[r, dash] \arrow[ur, dash] & \Pi_3 \arrow[r, dotted, dash] \arrow[ur, dotted, dash] & \ldots
\end{tikzcd}$$

The hierarchy is in general strict.

Scalar vs. vectorial fixed points

Operations in $\Sigma_n^{\mu}(\mathcal{F})$ can be characterized as components of vectorial fixed points

$$\mu \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ \dots \\ x_{1,k} \end{pmatrix} .\nu \begin{pmatrix} x_{2,1} \\ x_{2,2} \\ \dots \\ x_{2,k} \end{pmatrix} \dots . \theta \begin{pmatrix} x_{k,1} \\ x_{k,2} \\ \dots \\ x_{n,k} \end{pmatrix} .F(\vec{x}, \vec{z})$$

with the components of F in $\mathcal{F}$ (or projections).

De Morgan laws for fixed points

If a complete lattice L is a Boolean algebra (with $\overline{x} = \top - x$) then

$$\begin{array}{rcl} x = f(x) \implies \overline{x} & = & \overline{f(x)} \\ & = & \overline{f\left(\overline{\overline{x}}\right)} \end{array}$$

Thus a complement of a fixed point of f is a fixed point of the **dual function** $\widetilde{f} : x \mapsto \overline{f(\overline{x})}$.

Hence

$$\begin{array}{rcl} \overline{\mu x.f(x)} & = & \nu x.\widetilde{f}(x) \\ \overline{\nu x.f(x)} & = & \mu x.\widetilde{f}(x) \end{array}$$

Formal syntax: μ-terms

Sig is a finite set of function symbols of finite arity.

$$x$$

$$f(t_1, \ldots, t_k) \qquad \widetilde{f}(t_1, \ldots, t_k) \quad \text{for } f \in Sig \text{ of arity } k$$

$$\mu x.t \qquad \nu x.t$$

Interpretation: powerset algebras

This framework generalizes the modal μ-calculus and previous examples.

A semi-algebra $\mathbb{B} = \langle B, f^{\mathbb{B}}, g^{\mathbb{B}}, c^{\mathbb{B}}, \ldots \rangle$ over signature $Sig = \{f, g, c, \ldots\}$

$$f^{\mathbb{B}}(d_1, \ldots, d_k) \doteq b \quad \text{means} \quad (d_1, \ldots, d_k, b) \in f^{\mathbb{B}} \subseteq B^{k+1}$$

$$\text{for} \quad f \in Sig \text{ of arity } k$$

Powerset algebra

$$\wp\mathbb{B} = \Big\langle \langle \wp B, \subseteq \rangle \{ f^{\wp\mathbb{B}} : f \in Sig \} \cup \{ \widetilde{f}^{\wp\mathbb{B}} : f \in Sig \} \Big\rangle$$

$$f^{\wp\mathbb{B}}(L_1, \ldots, L_k) = \{ b : (\exists a_1 \in L_1 \ldots \exists a_k \in L_k)\ f^{\mathbb{B}}(a_1, \ldots, a_k) \doteq b \},$$

$$\widetilde{f}^{\wp\mathbb{B}}(L_1, \ldots, L_k) = \overline{f^{\wp\mathbb{B}}(\overline{L_1}, \ldots, \overline{L_k})}$$

$$= \{ b : (\forall \vec{a})\ f^{\mathbb{B}}(a_1, \ldots, a_k) \doteq b \Longrightarrow (\exists i)\ a_i \in L_i \}.$$

Recall

$f^{\wp\mathbb{B}}(L_1,\ldots,L_k) = \{b : (\exists a_1 \in L_1 \ldots \exists a_k \in L_k)\ f^{\mathbb{B}}(a_1,\ldots,a_k) \doteq b\}$,

$\widetilde{f}^{\wp\mathbb{B}}(L_1,\ldots,L_k) = \overline{f^{\wp\mathbb{B}}(\overline{L_1},\ldots,\overline{L_k})}$

The set-theoretic operations

We assume that $\mathbb{B}$ has a partial operation eq

$$eq^{\mathbb{B}}(a,b) \doteq c \quad\Longleftrightarrow\quad a = b = c$$

Then $\cap, \cup$ can be retrieved by

$$\begin{aligned}
eq^{\wp\mathbb{B}}(L_1,L_2) &= \{c : (\exists a \in L_1, \exists b \in L_2)\, a = b = c\} \\
&= L_1 \cap L_2 \\
\widetilde{eq}^{\wp\mathbb{B}}(L_1,L_2) &= L_1 \cup L_2
\end{aligned}$$

Powerset algebra of words

universe	operations	
$\Sigma^* \cup \Sigma^\omega$	σw	for $\sigma \in \Sigma$, w in universe

Powerset algebra of trees

universe	operations	
T_{Sig}	$f(t_1, \ldots, t_k)$	for $f \in Sig$, $t_1, \ldots, t_k$ in universe

Powerset algebra of a single tree $t \in T_{Sig}$

$$t : \omega^* \supseteq dom\, t \to Sig$$

universe	operations
$dom\, t$	$f(v1, \ldots, vk) \doteq v$

$$\begin{tikzcd} & f, v \arrow[dl] \arrow[dr] & \\ v_1 \arrow[rr, dotted, no head] & & v_k \end{tikzcd}$$

whenever $t(v) = f$

The modal μ-calculus of Kozen

Syntax

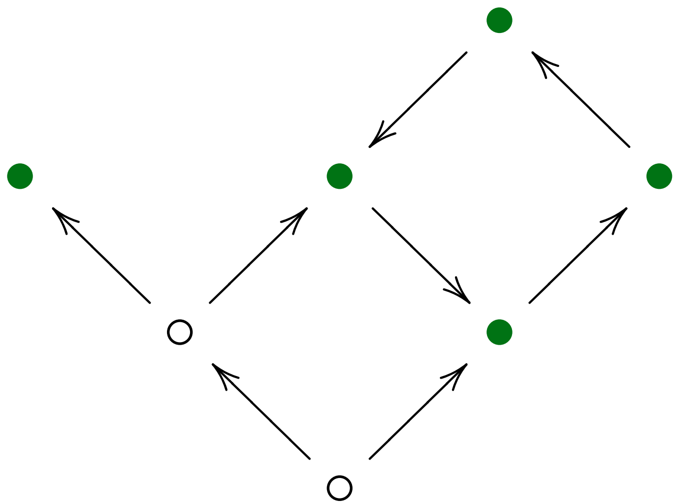
$$
\begin{array}{ll}
x & \\
p & \neg p \\
\varphi \vee \psi & \varphi \wedge \psi \\
\Diamond \varphi & \Box \varphi \\
\mu x.\varphi(x) & \nu x.\varphi(x)
\end{array}
$$

Interpretation in Kripke structures

$\mathcal{K} = \langle S, R, \rho \rangle$, with $R \subseteq S \times S$, and ρ : Prop $\to \wp S$.

$[\![\varphi]\!]_{\mathcal{K}}(v) \subseteq S$, for $v : \mathit{Var} \to \wp S$

$[\![\Diamond\varphi]\!]_{\mathcal{K}}(v) = \{s : (\exists s')\, R(s, s') \wedge s' \in [\![\varphi]\!]_{\mathcal{K}}(v)\}$
$[\![\mu x.\varphi]\!]_{\mathcal{K}}(v) = \mu X.[\![\varphi]\!]_{\mathcal{K}}(v[X/x])$.



E.g.,

$$\mu x.\nu y.\Box y \wedge (Happy \vee \Box x)$$

On each path, I will be happy from some moment on.

Kripke structure as semi-algebra

$\mathcal{K} = \langle S, R, \rho \rangle$, with $R \subseteq S \times S$, and ρ : Prop $\rightarrow \wp S$ can be identified with a semi-algebra $\mathbb{K}$.

signature	universe	operations	
Prop $\cup \{act_R\}$	S	$\rho(p) \subseteq S$,	for $p \in$ Prop;
		$act_R = R^{-1}$	i.e., $act_R(z) \doteq y$ iff $R(y, z)$
			$act_R(Z) \approx \Diamond Z$

Example

$$\begin{tikzcd}
& \mathbf{3}\, q \arrow[d, leftrightarrow] \\
\mathbf{1}\, p \arrow[ur] \arrow[r] & \mathbf{2}\, p, q
\end{tikzcd}$$

$$\begin{array}{llll}
p \doteq \mathbf{1} & p \doteq \mathbf{2} & q \doteq \mathbf{2} & q \doteq \mathbf{3} \\
act_R(\mathbf{3}) \doteq \mathbf{1} & act_R(\mathbf{3}) \doteq \mathbf{2} & & \\
act_R(\mathbf{2}) \doteq \mathbf{1} & act_R(\mathbf{2}) \doteq \mathbf{3} & &
\end{array}$$

This induces a **translation** $\alpha : \varphi \mapsto t_\varphi$ of the formulas of $L\mu$ into μ-terms.

$$
\begin{array}{lll}
\alpha : & x \mapsto x & \\
& p \mapsto p & \neg p \mapsto \widetilde{p} \\
& (\varphi \wedge \psi) \mapsto eq(\alpha(\varphi), \alpha(\psi)) & (\varphi \vee \psi) \mapsto \widetilde{eq}(\alpha(\varphi), \alpha(\psi)) \\
& \Diamond\varphi \mapsto act_R(\alpha(\varphi)) & \Box\varphi \mapsto \widetilde{act}_R(\alpha(\varphi)) \\
& \mu x.\varphi \mapsto \mu x.\alpha(\varphi) & \nu x.\varphi \mapsto \nu x.\alpha(\varphi)
\end{array}
$$

For a sentence φ,

$$
s \in [\![\varphi]\!]_{\mathcal{K}} \quad \text{iff} \quad s \in \alpha(\varphi)^{\wp\mathbb{K}}.
$$

How to understand fixed point formulas ?

$$\mu x.\nu y.\Diamond(x \wedge \Box(y \vee \mu z.\Diamond(x \wedge \Box(y \vee z))))$$

How to understand fixed point formulas ?

$$\mu x.\nu y.\Diamond(x \wedge \Box(y \vee \mu z.\Diamond(x \wedge \Box(y \vee z))))$$

A useful tool is games.

ICALP 2014. Courtesy of Henryk Michalewski

Games on graphs

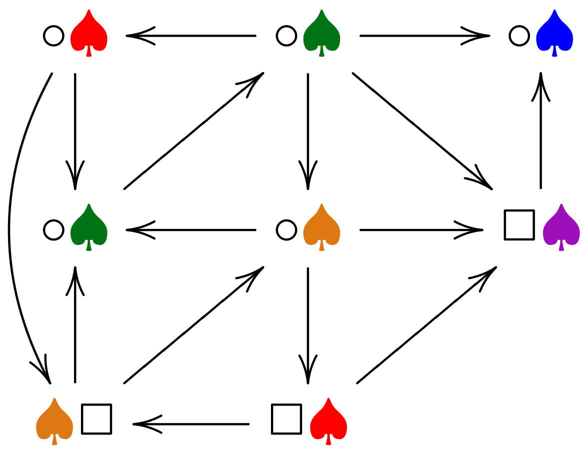
$$G \quad = \quad \langle Pos_\exists, Pos_\forall, Move, C, rank, W_\exists, W_\forall \rangle,$$

where $Pos = Pos_\exists \mathbin{\dot\cup} Pos_\forall$, $\quad Move \subseteq Pos \times Pos$,

$rank : Pos \to C$, $\quad W_\exists, W_\forall \subseteq C^\omega$, typically $W_\forall = \overline{W_\exists}$.

$\circ\, Eve$

$\square\, Adam$



Game equations

If the winning criterion $W_\exists$ is independent on finite prefixes then the set of **winning positions of Eve** satisfies

$$X \quad = \quad (E \cap \Diamond X) \cup (A \cap \Box X) \qquad =_{def} \quad Eve(X)$$

and the set of **winning positions of Adam**

$$Y \quad = \quad (A \cap \Diamond Y) \cup (E \cap \Box Y) \qquad =_{def} \quad Adam(Y)$$

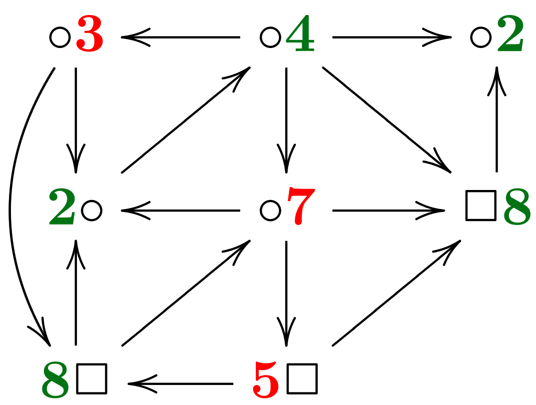
where E, A are interpreted as $Pos_\exists, Pos_\forall$, respectively.

Note $X = Eve(X)$ iff $\overline{X} = Adam\left(\overline{X}\right)$, implying
$\overline{\mu.Eve(X)} = \nu Y.Adam(Y)$.

Question. For which game is the winning set a **least** (resp. **greatest**) solution on the game equation ?

Parity games

$C \subseteq \omega$ (finite).



Eve wants to visit **even** priorities infinitely often.

Adam wants to visit **odd** priorities infinitely often.

Maximal priority wins.

$$W_\exists = \{u \in C^\omega : \limsup_{n\to\infty} u_n \text{ is } \textbf{even} \}$$
$$W_\forall = \{u \in C^\omega : \limsup_{n\to\infty} u_n \text{ is } \textbf{odd} \}.$$

Parity games are intimately linked to the μ-calculus.
Eve's winning set (for $C = \{0, 1, 2, 3\}$) is

$$
\begin{aligned}
\nu X_4.\mu X_3.\nu X_2.\mu X_1.\nu X_0. \quad & (E \cap rank_0 \cap \Diamond X_0) \cup \\
& (E \cap rank_1 \cap \Diamond X_1) \cup \\
& (E \cap rank_2 \cap \Diamond X_2) \cup \\
& (E \cap rank_3 \cap \Diamond X_3) \cup \\
\\
& (A \cap rank_0 \cap \Box X_0) \cup \\
& (A \cap rank_1 \cap \Box X_1) \cup \\
& (A \cap rank_2 \cap \Box X_2) \cup \\
& (A \cap rank_3 \cap \Box X_3)
\end{aligned}
$$

Note: its is a fixed point of $X = (E \cap \Diamond\, X) \,\cup\, (A \cap \Box\, X)$.

Game semantics for the μ-calculus

We define a parity game $\mathcal{G}(\mathbb{B}, t)$, such that, for $b \in B$

$$b \in t^{\wp\mathbb{B}} \qquad \text{iff} \qquad \text{Eve wins the game } \mathcal{G}(\mathbb{B}, t) \text{ from position } (b, t).$$

First, the variables should be indexed properly

$$\mu x\ \ .\nu y\ \ .f(x\ \ ,y\ \ ,\mu z\ \ .\nu w\ \ .f(x\ \ ,z\ \ ,w\ \ ))$$

$$\mu x_3.\nu x_2.f(x_3, x_2, \mu x_1.\nu x_0.f(x_3, x_1, x_0))$$

Better

$$\mu x_{\mathbf{1}1}.\nu x_{\mathbf{0}1}.f(x_{\mathbf{1}1}, x_{\mathbf{0}1}, \mu x_{\mathbf{1}2}.\nu_{\mathbf{0}2}.f(x_{\mathbf{1}1}, x_{\mathbf{1}2}, x_{\mathbf{0}2}))$$

ν-variables $x_{\mathbf{2m},j}$,

μ-variables $x_{\mathbf{2m+1},j}$.

If a variable $x_{\mathbf{k},\ell}$ appears in the scope of $\theta x_{\mathbf{i},j}$. then $k \geq i$.

Games for the powerset algebras

A game $\mathcal{G}(\mathbb{B}, t)$, for a semi-algebra $\mathbb{B}$ and a (closed) μ-term t.

Idea of moves (f^* stands for f or $\widetilde{f}$):

$$\begin{tikzcd}[column sep=small]
 & \mathit{Proponent} & (b, f^*(t_1, \ldots t_k), \mathit{head}) \arrow[d] & \\
 & \mathit{Opponent} & (\langle a_1 \ldots a_k \rangle, f^*(t_1, \ldots t_k), \mathit{tail}) \arrow[dl] \arrow[dr] & \\
 & (a_1, t_1, \mathit{head}) \arrow[rr, phantom, "\cdots\cdots\cdots\cdots"] & & (a_k, t_k, \mathit{head})
\end{tikzcd}$$

where $f(a_1, \ldots, a_k) \doteq b$.

Proponent is Eve for f and Adam for $\widetilde{f}$.

Positions of the game $\mathcal{G}(\mathbb{B}, t)$

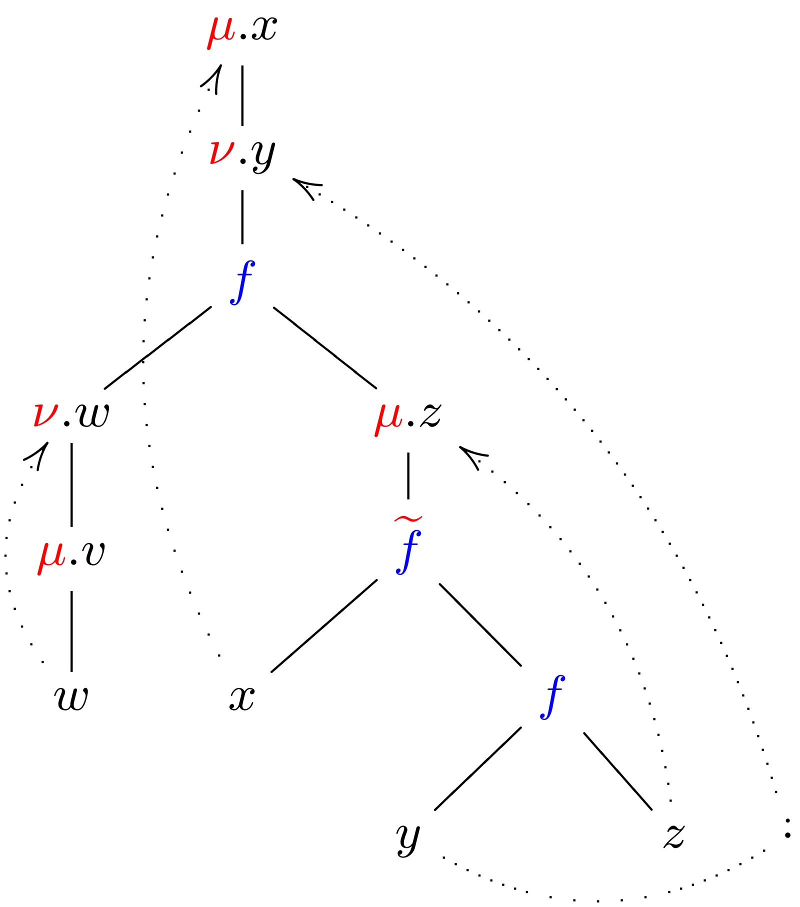
$$\begin{array}{lcl} \text{Head positions} & = & B \times Sub(t) \times \{head\} \\ \text{Tail positions} & \subseteq & B^* \times Sub(t) \times \{tail\} \end{array}$$

$$\begin{array}{rcccc} \text{of the form} & (\langle a_1, \ldots, a_k \rangle, & f^*(t_1, \ldots t_k), & & tail) \\ \text{or, more generally} & (\langle a_1, \ldots, a_k \rangle, & s & & \{tail\}) \\ & & \downarrow & & \\ & & f^*(t_1, \ldots t_k) & & \end{array}$$

whenever $s \xrightarrow{red} f^*(t_1, \ldots t_k)$.

Additionally, $(b, \perp, head)$ – Adam wins, or $(b, \top, head)$ – Eve wins.

Reduction *red* to guarded subterms $f^*(t_1, t_2)$ or $\bot, \top$.



$red(z) = red(\mu z.\widetilde{f}(x, f(y, z))) = \widetilde{f}(x, f(y, z))$

$red(w) = red(\nu w.\mu v.w) = \top$, etc.

Ownership of positions

Eve

$$
\begin{aligned}
(b, s, head) &\quad \text{if } red(s) = f(t_1, \dots, t_k), \\
(b, s, head) &\quad \text{if } red(s) = \bot, \\
(\langle a_1 \dots a_k \rangle, s, tail) &\quad \text{if } red(s) = \widetilde{f}(t_1, \dots, t_k).
\end{aligned}
$$

Adam

$$
\begin{aligned}
(b, s, head) &\quad \text{if } red(s) = \widetilde{f}(t_1, \dots, t_k), \\
(b, s, head) &\quad \text{if } red(s) = \top, \\
(\langle a_1 \dots a_k \rangle, s, tail) &\quad \text{if } red(s) = f(t_1, \dots, t_k).
\end{aligned}
$$

Size: $|Pos| = \mathcal{O}(|\mathbb{B}| \cdot |t|)$.

Moves

$$\begin{tikzcd}[column sep=small]
& (b, s, head) \arrow[d] & \\
& (\langle a_1 \ldots a_k \rangle, s, tail) \arrow[dl] \arrow[dr] & \\
(a_1, t_1, head) \arrow[rr, dotted, no head] & & (a_k, t_k, head)
\end{tikzcd}$$

whenever $red(s) = f^*(t_1, \ldots t_k)$, and $f(a_1, \ldots, a_k) \doteq b$.

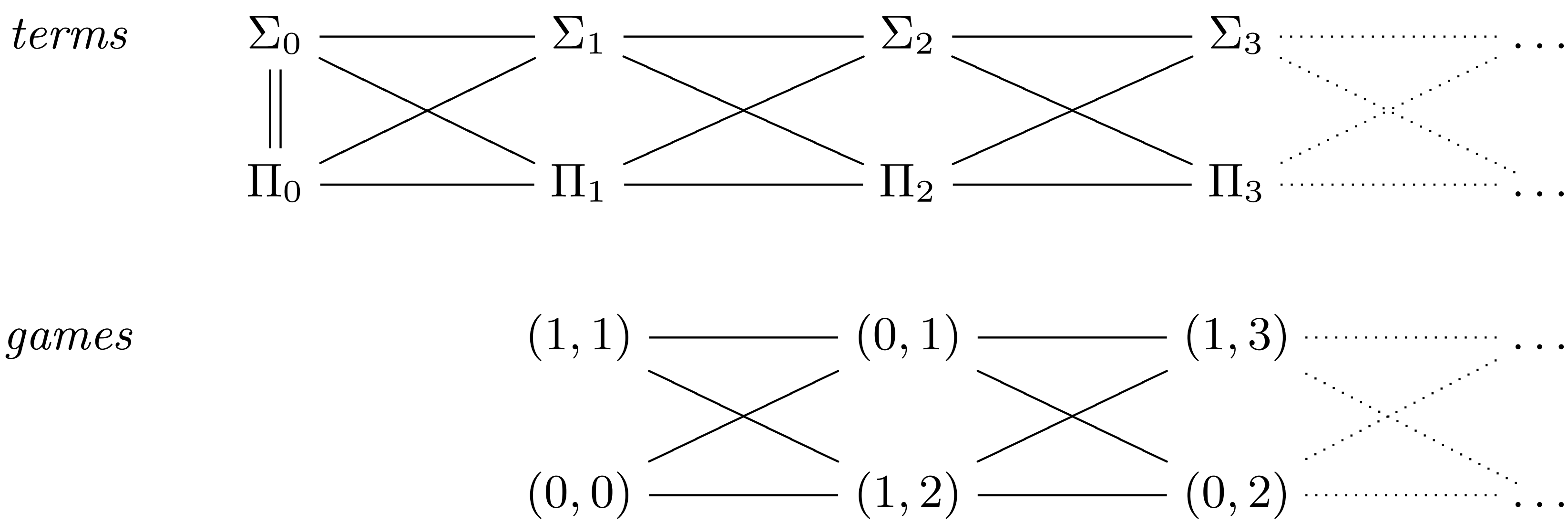
No move out from $(b, s, head)$ if $red(s) = \bot, \top$.

Ranking

$$rank\,(any, x_{\mathbf{i},j}, any) \quad = \quad \mathbf{i},$$

for all other positions, $rank = 0$.

Index of the game: $(\min rank, \max rank)$.



Parity game semantics of the μ-calculus.

Theorem. Eve wins the game $\mathcal{G}(\mathbb{B}, t)$ from a position $(b, t, head)$ iff $b \in t^{\wp\mathbb{B}}$.

We prove a more general claim for a term $t(z_1, \ldots, z_k)$, and the game $\mathcal{G}(\mathbb{B}, t, val)$, where Eve wins at the position $(b, z_i, head)$ iff $b \in val(z_i)$.

Induction on the structure of t. The case of $\mu x.t(x, \vec{z})$.

Let **A** be the set of positions from which Eve **wins the game** $\mathcal{G}(\mathbb{B}, \mu x.t, val)$.

To show $A = (\mu x.t(x, \vec{z}))^{\wp\mathbb{B}}\, val$, by Knaster-Tarski's Theorem, it is enough to prove

(i) $t^{\wp\mathbb{B}} val[\mathbf{A}/x] \subseteq \mathbf{A}$

(ii) $(\forall X)\, t^{\wp\mathbb{B}} val[X/x] \subseteq X \Longrightarrow \mathbf{A} \subseteq X$.

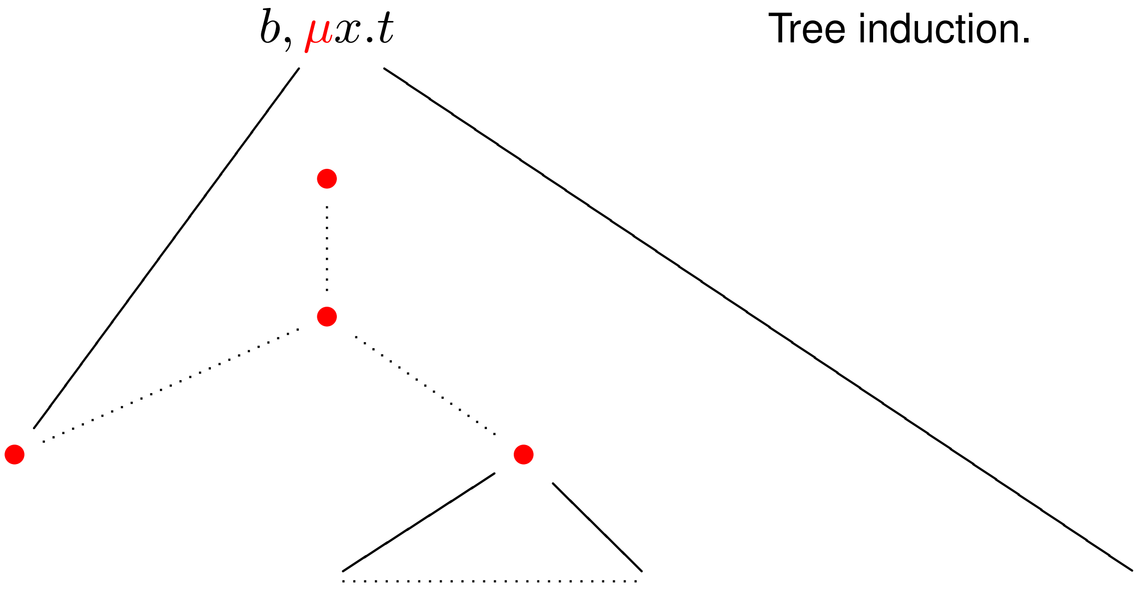
(i) $t^{\wp\mathbb{B}} val[\mathbf{A}/x] \subseteq \mathbf{A}$

(ii) $(\forall X)\, t^{\wp\mathbb{B}} val[X/x] \subseteq X \Longrightarrow \mathbf{A} \subseteq X.$

By induction hypothesis, Eve has a strategy at $t^{\wp\mathbb{B}} val[\mathbf{A}/x]$.

Ad (i). Combine the two strategies.

Ad (ii). For $b \in \mathbf{A}$, Eve has a strategy with the highest rank **odd** (well founded).

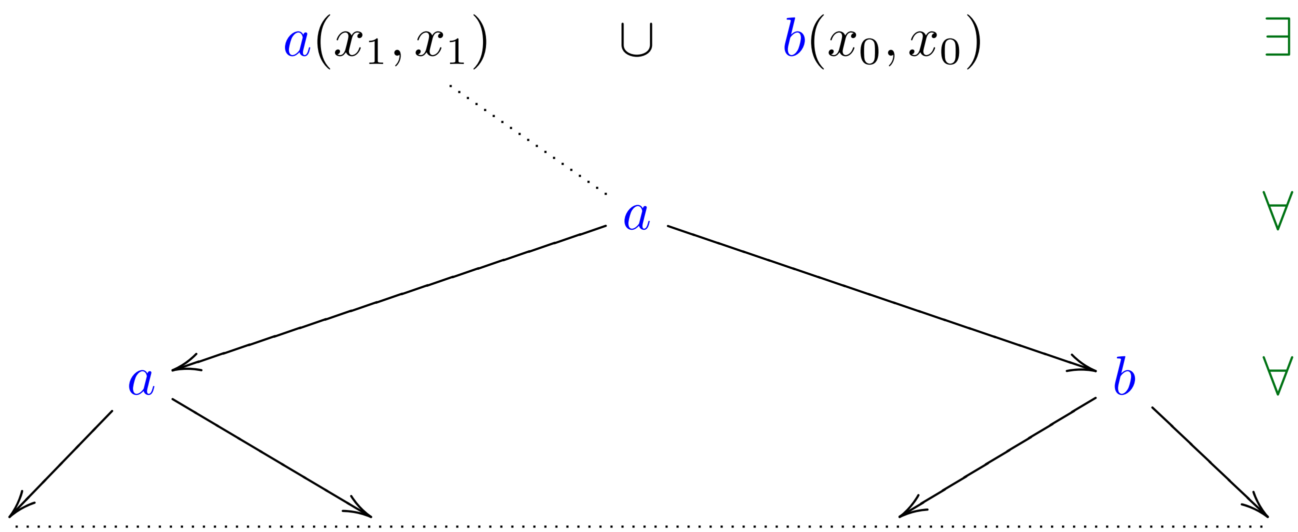


Tree induction.

Example

$\mu x_1.\nu x_0.a(x_1, x_1) \cup b(x_0, x_0) =$ the set of trees, such that on each path there are only finitely many a's.

Adam selects a path in the tree and wins if a occurs infinitely often, otherwise Eve wins.



Games for the modal μ-calculus

Eve

$$\begin{tikzcd} & \mathcal{K}, s \overset{?}{\models} \varphi \vee \psi \arrow[dl] \arrow[d] \\ \mathcal{K}, s \models \varphi & \mathcal{K}, s \models \psi \end{tikzcd}$$

$Adam$

$$\begin{tikzcd} & \mathcal{K}, s \models \varphi \wedge \psi \arrow[dl] \arrow[d] \\ \mathcal{K}, s \models \varphi & \mathcal{K}, s \models \psi \end{tikzcd}$$

Eve

$$\begin{tikzcd} \mathcal{K}, s \models \Diamond\varphi \arrow[d] \\ \mathcal{K}, s' \models \varphi \end{tikzcd}$$

with $R(s, s')$

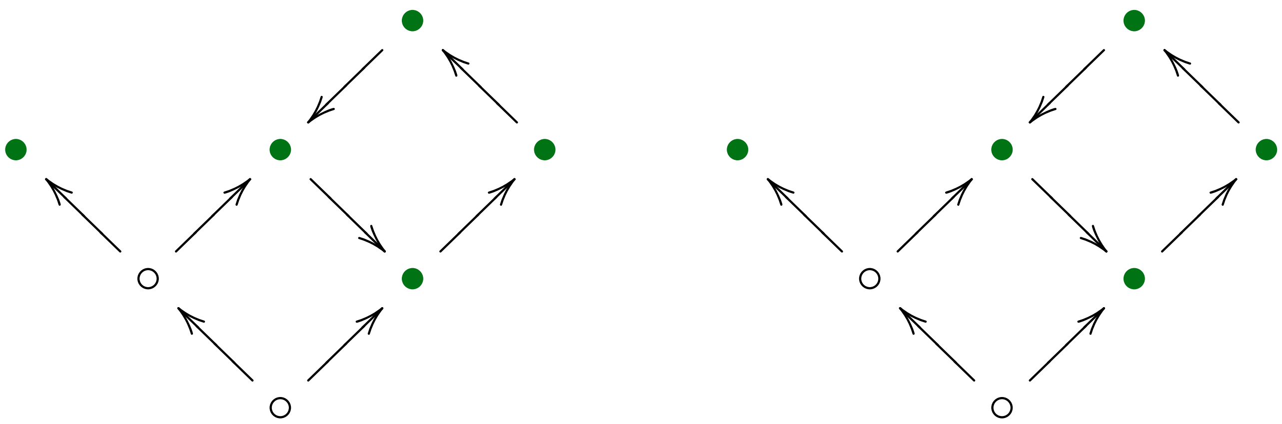
$Adam$

$$\begin{tikzcd} \mathcal{K}, s \models \Box\varphi \arrow[d] \\ \mathcal{K}, s' \models \varphi \end{tikzcd}$$

$\mathcal{K}, s \models p$ $\quad Eve \quad$ wins iff true

$$\mathcal{K}, s \models X \longrightarrow \mathcal{K}, s \models \theta X.\psi \longrightarrow \mathcal{K}, s \models \psi$$

Example



$$\mu x.\nu y.\Box y \wedge (Happy \vee \Box x)$$

Example – parity games

$$
\begin{tikzcd}
\circ\mathbf{3} \arrow[d] \arrow[dd, bend right] & \circ\mathbf{4} \arrow[l] \arrow[r] \arrow[d] \arrow[dr] & \circ\mathbf{2} \\
\mathbf{2}\circ \arrow[ur] & \circ\mathbf{7} \arrow[l] \arrow[r] \arrow[d] & \Box\mathbf{8} \arrow[u] \\
\mathbf{8}\Box \arrow[u] \arrow[ur] & \mathbf{5}\Box \arrow[l] \arrow[ur] &
\end{tikzcd}
$$

$$Win_E =$$

$$\nu X_8.\mu X_7.\ldots.\mu X_1.\nu X_0.(E \cap rank_0 \cap \Diamond X_0) \cup (E \cap rank_1 \cap \Diamond X_1) \cup \ldots$$

$$\ldots \cup (E \cap rank_7 \cap \Diamond X_7) \cup (E \cap rank_8 \cap \Diamond X_8) \cup$$

$$\cup (A \cap rank_0 \cap \Box X_0) \cup (A \cap rank_1 \cap \Box X_1) \cup \ldots \cup (A \cap rank_8 \cap \Box X_8)$$

The game induced by this formula is essentially the original game.

$$Win_E =$$

$$\nu X_8.\mu X_7.\ldots.\mu X_1.\nu X_0.(E \cap rank_0 \cap \Diamond X_0) \cup (E \cap rank_1 \cap \Diamond X_1) \cup \ldots$$
$$\ldots \cup (E \cap rank_7 \cap \Diamond X_7) \cup (E \cap rank_8 \cap \Diamond X_8) \cup$$
$$\cup (A \cap rank_0 \cap \Box X_0) \cup (A \cap rank_1 \cap \Box X_1) \cup \ldots \cup (A \cap rank_8 \cap \Box X_8)$$

By duality

$$Win_A =$$

$$\mu X_8.\nu X_7.\ldots.\nu X_1.\mu X_0.(E \cap rank_0 \cap \Diamond X_0) \cup (E \cap rank_1 \cap \Diamond X_1) \cup \ldots$$
$$\ldots \cup (E \cap rank_7 \cap \Diamond X_7) \cup (E \cap rank_8 \cap \Diamond X_8) \cup$$
$$\cup (A \cap rank_0 \cap \Box X_0) \cup (A \cap rank_1 \cap \Box X_1) \cup \ldots \cup (A \cap rank_8 \cap \Box X_8)$$

But the formulas complement each others, hence $\overline{Win_E} = Win_A$.

Thus, the game semantics result yields **determinacy of parity games**.

Note: infinite games are **not** always determined. But by Martin's Theorem, all games with **Borel** winning criteria are determined.

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Summary of the lecture

— usefulness of fixed point definitions

— basic laws of μ and ν

— logic for fixed points: μ-terms and modal μ-calculus

— parity game semantics

Plan of the course

Monday	*DN*	Basic laws and games
Tuesday	*AF*	Automata for the μ-calculus
Wednesday	*AF*	μ-calculus vs. second-order logic
Thursday	*AF*	Fixpoint hierarchies and topology
Friday	*DN*	Complexity and probabilistic extension