On topological completeness of regular tree languages

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Abstract

We identify the class of $\mathbf{\Sigma}^1_1$–inductive sets studied by Moschovakis as a set theoretical generalization of the class $(1,3)$ of the Rabin-Mostowski index hierarchy of alternating automata on infinite trees. That is, we show that every tree language recognized by an alternating automaton of index $(1,3)$ is $\mathbf{\Sigma}^1_1$–inductive, and exhibit an automaton whose language is complete in this class w.r.t. continuous reductions.

Classification Automata on infinite trees, Logic in computer science, Infinite games.

1 Introduction

A common feature of computational complexity theory, recursion theory, automata theory, or descriptive set theory, is that they organize their realms into various hierarchies according to their sense of complexity. The complexity levels are usually understood through some concrete examples, genuine to a complexity level. For instance, the complexity class *NL* is understood through the problem of maze, and the topological class $\mathbf{\Pi}^1_1$ through the set of well-founded (infinite) trees. Of a special interest are examples which separate complexity levels, or are conjectured to do so. This often involves some concept of *completeness*, which is also a common feature of the above theories, although the actual reductions vary from polynomial-time (or log-space) reductions in complexity theory to continuous reductions in descriptive set theory.

The introduction of the μ-calculus by Kozen [12] (anticipated by the work of Emerson and Clarke, Pratt, Park, and others, see, e.g., [3] for references) gave rise to investigation of the hierarchy induced by the alternation of the least (μ) and greatest (ν) fixed point operators. Bradfield [5] proved that this hierarchy is strict, giving also [6] a natural family of examples based on parity games [8]. This model-theoretic result yielded the strictness of another hierarchy, classifying sets of (infinite) trees recognizable by finite automata, first considered by Rabin [21]. More specifically, Bradfield proved the strictness of the hierarchy induced by the Rabin-Mostowski index of *alternating* automata[1], corresponding level by level to the hierarchy of the μ-calculus. The witness family consists of the so-called game tree languages $W_{\iota,k}$, $\iota \in \{0,1\}$, $\iota \leq k$, obtained as the tree encoding of parity games (the author [6] credits I.Walukiewicz for this example).

Unfortunately, although the questions about finite automata are usually decidable, no method is known up to date to decide the exact level of a tree language in the hierarchy, if an automaton

[1]For non-deterministic automata, the result was proved earlier [17].

is given[2]. This challenge seems to be related to the search for a suitable concept of reduction (and completeness) for tree automata.

On the other hand, infinite trees can be naturally viewed as elements of a Cantor (topological) space, where the concept of continuous reduction is available and several hierarchies are well understood. In order to take advantage of this page of mathematics, we need first to accurately place tree automata into the realm of descriptive set theory.

Finite-state recognizable sets of infinite words were classified already by Landweber [13, Corollary 3.6] as Boolean combinations of F_σ sets (see also [26]). Finite automata on trees are more interesting from this perspective. They recognize some Borel sets on any finite level [23], as well as some non-Borel sets [19], although by definition cannot go beyond $\mathbf{\Delta}^1_2$. It was observed by Arnold [1] that the game tree languages $W_{\iota,k}$ are complete on the subsequent levels of the alternating hierarchy w.r.t. the Wadge (i.e., continuous) reductions; on the other hand they form themselves a Wadge hierarchy [4].

The low classes of the index hierarchy are comparable to the analytic ($\mathbf{\Sigma}^1_1$) and co-analytic classes in projective hierarchies. Rabin [22] proved that (in our current terminology) the level $(1, 2)$ of the index hierarchy, corresponding to the $\nu\mu$ level in the μ-calculus, is definable in the existential fragment of $S2S$, and consequently is included in the class $\mathbf{\Sigma}^1_1$. By symmetry, the level $(0, 1)$ is included in $\mathbf{\Pi}^1_1$. On the other hand, there are recognizable sets of trees of levels $(0, 1)$ and $(1, 2)$ complete in the classes $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$, respectively, w.r.t. the continuous reductions [19]; in particular $W_{0,1}$ and $W_{1,2}$ have this property. It is natural to ask if the subsequent levels of the hierarchy do also enjoy some meaningful topological extensions.

In this paper, we show that the class of $\mathbf{\Sigma}^1_1$–inductive sets forms such an extension for the level $(1, 3)$ of the index hierarchy, corresponding to the level $\mu\nu\mu$ in the μ-calculus. The concept was analyzed by Moschovakis in [15] (see also [14], which contains historical information about inductive sets). The $\mathbf{\Sigma}^1_1$–inductive sets are those that can be obtained as the least fixed points of $\mathbf{\Sigma}^1_1$–definable operators. We verify that the game tree language $W_{1,3}$ is $\mathbf{\Sigma}^1_1$–inductive, and then show that it is actually complete among all $\mathbf{\Sigma}^1_1$–inductive sets, by a reduction from the set of *quasi bounded* trees invented by Saint Raymond [24]. By the property of $W_{1,3}$, it implies that every tree language recognized by an alternating automaton of index $(1, 3)$ is $\mathbf{\Sigma}^1_1$–inductive. We terminate by providing a game characterization of the class of $\mathbf{\Sigma}^1_1$–inductive sets, in which we explore the aforementioned completeness of $W_{1,3}$.

A similar characterization of all the levels of the μ-calculus hierarchy in terms of game quantifiers has been established by Bradfield [7] for fixed-point definable sets of natural numbers. It possibly can be adapted to sets of trees, if an appropriate extension of the concept of $\mathbf{\Sigma}^1_1$–inductive sets is made.

2 Index hierarchy

Notation. Throughout the paper, ω stands for the set of natural numbers, which we identify with its ordinal type. We also identify a natural number $n < \omega$ with the set $\{0, 1, \ldots, n-1\}$.

The concept of alternating automaton (see [26]) is best presentable via games. A *parity game* is a perfect information game of possibly infinite duration played by two players, say Eve and Adam.

[2] An algorithm is known only in the case if a given automaton is deterministic [18].

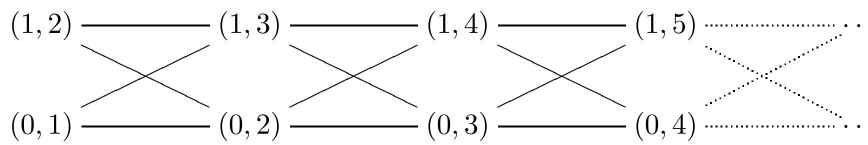

Figure 1: The Mostowski–Rabin index hierarchy.

We present it as a tuple $(V_\exists, V_\forall, \mathit{Move}, p_0, \mathit{rank})$, where $V_\exists$ and $V_\forall$ are (disjoint) sets of positions of Eve and Adam, respectively, $\mathit{Move} \subseteq V \times V$ is the relation of possible moves, with $V = V_\exists \cup V_\forall$, $p_0 \in V$ is a designated initial position, and $\mathit{rank} : V \to \omega$ is the ranking function which admits only a finite number of values.

The players start a play in the position p_0 and then move the token according to relation Move (always to a successor of the current position), thus forming a path in the directed graph (V, Move). The move is selected by Eve or Adam, depending on who is the owner of the current position. If a player cannot move, she/he looses. Otherwise, the result of the play is an infinite path in the graph, $v_0, v_1, v_2, \ldots$ Eve wins the play if $\limsup_{n\to\infty} \mathit{rank}(v_n)$, is even, otherwise Adam wins. It is known that parity games are *positionally determined*: one of the players has a winning strategy which moreover can be made *positional*, i.e., represented by a (partial) function $\sigma : V \to V$ [8, 16]. We say that Eve *wins* the game if she has a winning strategy, the similar for Adam.

A full binary *tree* over a finite alphabet Σ is a mapping $t : 2^* \to \Sigma$. (Recall that $2 = \{0, 1\}$.)

An *alternating parity tree automaton* running on such trees can be presented by

$$\mathcal{A} = \langle \Sigma, Q_\exists, Q_\forall, q_0, \delta, \mathrm{rank} \rangle$$

where the set of states Q is partitioned into existential states $Q_\exists$ and universal states $Q_\forall$, $\delta \subseteq Q \times \Sigma \times \{0, 1, \varepsilon\} \times Q$ is a transition relation, and $\mathit{rank} : Q \to \omega$ a *rank* function. An input tree t is accepted by $\mathcal{A}$ iff Eve has a winning strategy in the parity game $\langle Q_\exists \times 2^*, Q_\forall \times 2^*, (q_0, \varepsilon), \mathrm{Mov}, \Omega \rangle$, where $\mathrm{Mov} = \{((p, v), (q, vd))\colon\ v \in \mathrm{dom}(t),\ (p, t(v), d, q) \in \delta\}$ and $\Omega(q, v) = \mathrm{rank}(q)$.

We assume without loss of generality that $\min \mathit{rank}(Q)$ is 0 or 1. The *Mostowski-Rabin index* is the pair $(\min \mathit{rank}(Q), \max \mathit{rank}(Q))$. It is useful to have a partial ordering on indices; it is represented on Figure 1. The idea is that we let $(\iota, \kappa) \sqsubseteq (\iota', \kappa')$ if either $\{\iota, \ldots, \kappa\} \subseteq \{\iota', \ldots, \kappa'\}$, or $\iota = 0$, $\iota' = 1$, and $\{\iota + 2, \ldots, \kappa + 2\} \subseteq \{\iota', \ldots, \kappa'\}$. We consider the indices $(1, \kappa)$ and $(0, \kappa - 1)$ as *dual*, and let $\overline{(\iota, \kappa)}$ denote the index dual to (ι, κ).

We recall an example of a witness family used by Bradfield [6] to show that the hierarchy induced by the indices of alternating parity tree automata is strict. The family consists of languages $W_{\iota,k}$, $\iota \in \{0, 1\}$, $\iota \le k$ which are themselves based on parity games. The alphabet of $W_{\iota,k}$ is $\{\exists, \forall\} \times \{\iota, \iota+1, \ldots, k\}$. We let $T_{\iota,k}$ denote the set of all binary trees over this alphabet. With each tree t in $T_{\iota,k}$, we associate a parity game $G(t)$, with

- $V_\exists = \{v \in 2^* : t(v) \downarrow_1 = \exists\}$,
- $V_\forall = \{v \in 2^* : t(v) \downarrow_1 = \forall\}$,
- $\mathit{Move} = \{(w, wi) : w \in 2^*,\ i \in \{0, 1\}\}$,

- $p_0 = \varepsilon$ (the root of the tree),
- $rank(v) = t(v) \downarrow_2$, for $v \in 2^*$.

(In the above, $\alpha \downarrow_i$, $i = 1, 2$, means the projection on the ith component.) The set $W_{\iota,k}$ consists of those trees for which Eve wins the game $G(t)$.

3 Basic topological concepts

All topological spaces under consideration are completely metrizable and separable. Let $\mathcal{T}_\Sigma$ denote the set of all k–ary trees over a finite alphabet Σ. This set can be equipped with a metric

$$d(t_1, t_2) = \begin{cases} 0 & \text{if } t_1 = t_2 \\ 2^{-n} \text{ with } n = \min\{|w| : t_1(w) \neq t_2(w)\} & \text{otherwise.} \end{cases}$$

It is well known and easy to see that the topological space induced by this metric is homeomorphic to the Cantor discontinuum $\{0,1\}^\omega$. We call a *Cantor space* any space homeomorphic with $\{0,1\}^\omega$.

The space ω^ω consists of all sequences of natural numbers. The distance between two sequences u, v is defined

$$d(u, v) = \begin{cases} 0 & \text{if } u = v \\ 2^{-n} \text{ with } n = \min\{m : u(n) \neq w(n)\} & \text{otherwise.} \end{cases}$$

For a topological space $\mathcal{H}$, the family of F_σ sets consists of all countable unions of closed sets in $\mathcal{H}$. *Borel sets* over $\mathcal{H}$ constitute the least family containing open sets and closed under complement and countable union. The *Borel relations* are defined similarly, starting with open relations (i.e., open subsets of $\mathcal{H}^n$, for some n, considered with product topology). The *analytic* (or $\mathbf{\Sigma}^1_1$) sets are those representable by

$$L = \{t : (\exists t')\, R(t, t')\}$$

where $R \subseteq \mathcal{H} \times \mathcal{H}$ is a Borel relation. The *co-analytic* (or $\mathbf{\Pi}^1_1$) sets are the complements of analytic sets. A continuous mapping $f : \mathcal{H} \to \mathcal{H}$ *reduces* a set $A \subseteq \mathcal{H}$ to $B \subseteq \mathcal{H}$ if $f^{-1}(B) = A$. As in complexity theory, a set $L \in \mathcal{K}$ is *complete* in class $\mathcal{K}$ if all sets in this class reduce to it.

4 $\mathbf{\Sigma}^1_1$–inductiveness — classical definition

The original definition of $\mathbf{\Sigma}^1_1$–inductiveness (see [15]) refers to the least fixed points of the $\mathbf{\Sigma}^1_1$-definable operators and however formally identical to the Definition formulated below, instead of using the constructiveness in the sense of this paper, that is via arbitrary countable unions, intersections, projections and game–quantifiers, the book [15] assumes, that all the above mentioned notions are limited by the assumption of recursiveness. In Set Theory this recursive approach is called *Effective Descriptive Set Theory*, and leads to so called *lightface* classes of sets, but since recursiveness does not seem to offer at the moment any benefits to Automata Theory, we use larger and more robust classes from *Classical Descriptive Set Theory*, so called *boldface* classes. The theory of lightface and boldface classes essentially coincide when it comes to really important results, but in this article, in order to avoid any confusion, we will introduce all the boldface concepts from scratch and will not use any lightface results from [15]. In our work, we have been much inspired by [24], and our approach to $\mathbf{\Sigma}_1^1$–inductive sets follows the exposition in [24]. Also, in Section 5 we will use an interesting combinatorial example from [24] of a set complete in the class of $\mathbf{\Sigma}_1^1$–inductive.

In the definitions below C is a Cantor space and the set I is an arbitrary countable set of indices with one special element $i_0 \in I$. We will assume that I is equipped with a discrete topology. We will be mostly interested in $I = 2^\star$, $i_0 = \varepsilon$ and $C = T_\Sigma$, but for certain applications in the following sections we will need the more general approach. Let

$$F : \wp(I) \times C \to \wp(I)$$

be a mapping monotone on the first argument w.r.t. the inclusion ordering. Keeping our main example in mind, if t is a tree and $I = 2^\star$, we view $F(.,t)$ as a mapping on the sets of nodes of t. We define the sets $F^\xi(t)$ by induction on ordinal ξ.

$$\begin{aligned} F^0(t) &= \emptyset \\ F^{\xi+1}(t) &= F\left(F^\xi(t), t\right) \\ F^\lambda(t) &= \bigcup_{\xi<\lambda} F^\xi(t), \text{ for limit } \lambda. \end{aligned}$$

Since F is monotone and I is countable, there is a countable ordinal ζ, such that $F^{\zeta+1}(t) = F^\zeta(t)$, and consequently $F^\zeta(t) = F^\xi(t)$, for all $\xi > \zeta$. We denote this set by $F^\infty(t)$. Finally, we let

$$\mathrm{Ind}(F) = \{t : i_0 \in F^\infty(t)\} \tag{1}$$

and for $t \in \mathrm{Ind}(F)$ we define $\mathrm{rk}(t)$ as the minimal ordinal ζ such that $F^{\zeta+1}(t) = F^\zeta(t)$ and for $t \notin \mathrm{Ind}(F)$ we define $\mathrm{rk}(t) = \omega_1$.

The complexity of a mapping F is defined in terms of the relation $w \in F(Y,t)$. More specifically, we represent a set Y by its characteristic function $\chi_Y : I \to \{0,1\}$, which in turn can be viewed as an element of a Cantor space $\{0,1\}^I$.

Definition 1. A set of trees $A \subseteq C$ is $\mathbf{\Sigma}_1^1$*–inductive* if it can be presented as $A = \mathrm{Ind}(F)$, for some mapping $F : \wp(I) \times C \to \wp(I)$ (monotone on the first argument), such that the relation

$$\{(w, \chi_Y, t) : w \in F(Y,t)\}$$

is $\mathbf{\Sigma}_1^1$.

To show that the set $W_{1,3}$ is $\mathbf{\Sigma}_1^1$–inductive in the above sense, we have to present it as $\mathrm{Ind}(F)$, for a suitable operator F, where $C = T_{1,3}$. For a tree $t \in T_{1,3}$, a set of nodes $Y \subseteq 2^*$, and a node $w \in 2^*$, we consider a game $G(t,Y,w)$ similar to the game $G(t)$ defined on page 3 but with the following modifications. The initial position is w (rather than ε). Whenever the token arrives in a node in the set Y, the play stops. Eve wins a play $\pi = (v_0, v_1, v_2, \ldots)$ (with $v_0 = w$) if

- the label 3 can occur only at the initial position, i.e., $t(v_0)\downarrow_2 \in \{1,2,3\}$, but $t(v_i)\downarrow_2 \in \{1,2\}$, for $i \geq 1$,
- either π is finite and ends in Y,

- or π is infinite and $\limsup_{n\to\infty} t(v_n)\downarrow_2 = 2$.

We let

$$F(Y,t) \quad = \quad \{w : \text{ Eve has a winning strategy in the game } G(t,Y,w)\}$$

Since the winning condition in $G(t,Y,w)$ is similar as in $W_{1,2}$ (i.e., of Büchi type), it is straightforward to verify that F satisfies the requirements of Definition 1. And it is not very difficult to verify that

$$\mathrm{Ind}(F) \quad = \quad W_{1,3}.$$

We will need later the following standard

Lemma 1. *If D, C are Cantor spaces, $\phi : D \to C$ is a continuous mapping and $A \subseteq C$ is a $\mathbf{\Sigma}^1_1$–inductive set, then the preimage $B = \phi^{-1}[A]$ is also a $\mathbf{\Sigma}^1_1$–inductive set.*

Proof. Let

$$F : \wp(I) \times C \to \wp(I)$$

be such that $A = \mathrm{Ind}(F)$, where

$$R = \{(w, \chi_Y, t) : w \in F(Y,t)\}$$

analytic. Define

$$G : \wp(I) \times C \to \wp(I)$$

by the formula $G(Y,t) = F(Y, \phi(t))$. Then clearly $B = \mathrm{Ind}(G)$. We have to verify that G is $\mathbf{\Sigma}^1_1$–definable.

Define

$$\psi : I \times \wp(I) \times D \to I \times \wp(I) \times C$$

by the formula

$$\psi(w, \chi_Y, t) = (w, \chi_Y, \phi(t)).$$

Then

$$\{(w, \chi_Y, t) : w \in G(Y,t)\} = \{(w, \chi_Y, t) : w \in F(Y, \phi(t))\} = \psi^{-1}[R].$$

Since ψ is a continuous mapping, the preimage of an analytic set R remains analytic, hence B is $\mathbf{\Sigma}^1_1$–inductive. $\square$

From the Lemma and from a result of Arnold [1] that any tree language of level (ι, k) is continuously reducible to $W_{\iota,k}$, it follows

Corollary 1. *Tree languages recognized by alternating automata of index $(1,3)$ are $\mathbf{\Sigma}^1_1$–inductive.*

5 Completeness via the quasi–bounded trees

We will prove that $W_{1,3}$ is complete in the class of $\mathbf{\Sigma}_1^1$–inductive sets reducing to it a set of so-called (unlabeled) *quasi-bounded trees*, whose completeness has been established by Saint Raymond [24] directly from the definition. We call a subset of ω^* an *unlabeled tree* if it closed with respect to prefixes. Characteristic function of an unlabeled tree t belongs to the Cantor space 2^{ω^*} and the topology on the space of unlabeled trees is inherited from the space 2^{ω^*}.

A tree $t \subset \omega^*$ is *cofinal* if for every $v = (v_0, v_1, \ldots) \in \omega^\omega$ there exists a branch $b = (b_0, b_1, \ldots) \in [t]$ such that $b \geq v$, that is for every $n \in \omega$ holds the inequality $b_n \geq v_n$. We define

$$QB = \{t \subset \omega^* : t \text{ is a not a cofinal tree}\}.$$

Let ψ be a mapping from 2^* to ω^* such that for a given sequence $s \in 2^*$ if $s = 0^{n_0}10^{n_1}1 \ldots 10^{n_k}10^l$ ($l \geq 0$) then

$$\psi(s) = (n_0, n_1, \ldots, n_k).$$

For a given tree $t \subset \omega^*$ we define a game $\Gamma(t)$ such that Player I plays natural numbers n_0, n_1, $\ldots$, one number in every round, and Player II answers with c_0, c_1, $\ldots$, where $c_n \in \{0, 1\}$. Moreover, Player II is obliged at every step to preserve the following two conditions

1. if $\psi(c_0, \ldots, c_k)$ has length $l \geq 1$, then
$$\psi(c_0, \ldots, c_k) \geq (n_0, \ldots, n_{l-1}).$$

2. $\psi(c_0, \ldots, c_k) \in t$.

Player II wins if he managed to play infinitely many 1. In [24] one can find the following characterization

Theorem 1 (J. Saint Raymond)**.** *A tree $t \subset \omega^{<\omega}$ is not cofinal if and only if Player I has a winning strategy in $\Gamma(t)$.*

Using a method of proof from [24, Theorem 3], we will prove the following

Theorem 2. *There exists ϕ which continuously reduces QB to $W_{1,3}$.*

Proof. For a given tree $t \subset \omega^{<\omega}$ we have to construct in a continuous way a tree $\phi(t) \in \mathrm{Tr}_{1,3}$ such that $\exists$ has a winning strategy in $\phi(t)$ if and only if t is not cofinal.

We will subsequently add new vertices to $\phi(t)$, starting from a partial tree s_0 consisting of $s_0(2, 2, \ldots, 2) = (\exists, 1)$, $s_0(2, 2, \ldots, 2, 1) = (\forall, 2)$. The essence of the inductive definition of $\phi(t)$ is depicted in Figure 2. We start from a tree s_0, then add it again in certain vertices marked in the Figure. There are some restrictions described below on placement of the vertices $(\forall, 3)$. Assume, that

- we already defined a partial tree $s_1 \subset \phi(t)$ and
- for v such that $s_1(v) = (\forall, 2)$, we already defined sequences $n_0^v, \ldots, n_{k+1}^v$, $c_0^v, \ldots, c_k^v$ such that conditions (1), (2) from the definition of $\Gamma(t)$ are fulfilled.

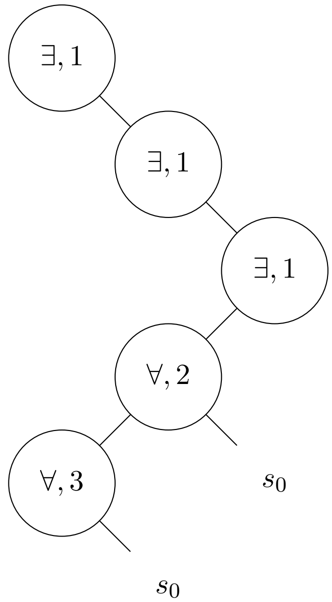

Figure 2: An approximation of the tree $\phi(t)$.

We will extend it to a partial tree $s_2 \subset \phi(t)$. For a given leaf $v \in s_1$ such that $s(v) = (\forall, 2)$, we add s_0 to the right, that is $s_2(v2w) = s_0(w)$. In every new leaf $v' \in s_1$ such that $s_2(v') = (\forall, 2)$, we define $n_i^{v'} = n_i^v$ $(i \leq k+1)$, $c_i^{v'} = c_i^v$ $(i \leq k)$ and $n_{k+2}^{v'}$ is the number of 2 from v to v' (formally speaking $n_{k+2}^{v'}$ is defined as $|v'| - |v| - 1$) and $c_{k+1}^{v'} = 0$.

To the left we add

1. $s_2(v1) = (\forall, 3)$, but only under the condition, that sequences $(n_0^v, \dots, n_{k+1}^v)$, $(c_0^v, \dots, c_k^v, 1)$ fulfill conditions (1), (2) from the definition of $\Gamma(t)$. We add $s_2(v12r) = s_0(r)$ and for a new leaf v' such that $s_2(v') = (\forall, 2)$ we define sequences as above with the exception, that $c_{k+1}^{v'} = 1$,

2. otherwise we add $s_2(v1r) = (\exists, 2)$ for every r; this choice ensures that Player $\forall$ will not be tempted to enter this subtree.

Claim 1. *If Player $\exists$ has a winning strategy in $\phi(t)$, then $t \in QB$.*

Proof. Assume $\exists$ has a winning strategy. Player $\exists$ plays n_0 and later in every place such that $\forall$ has a choice of going left or right, $\exists$ answers with a natural number. It means, that strategy for $\exists$ defines a mapping σ from $2^{<\omega}$ into $\omega^{<\omega}$. This extends to a continuous $\overline{\sigma}$ from 2^ω into ω^ω and bound of the image is a quasi–bound of the whole tree t. $\square$

Claim 2. *If Player $\forall$ has a winning strategy, then t is a cofinal tree.*

Proof. Take any $w \in \omega^\omega$ and play it as $\exists$. Answers of $\forall$ will lead us to a sequence in t dominating w. $\square$

This finishes the proof of the Theorem. □

6 Topological games

Now we will consider $W_{1,3}$ from a different angle, namely from the point of view of topological games and game quantifiers. We first recall the concepts of the Gale-Stewart game and the game quantifier (see, e.g., [11]). Let Y be an arbitrary set and let $A \subseteq Y^\omega$. The game $\Gamma(A)$ is played by two players, I and II, who consecutively select elements of ω.

$$\begin{array}{lcccccccc} \text{I} & y_0 & & y_2 & & y_4 & & y_6 & \cdots \\ \text{II} & & y_1 & & y_3 & & y_5 & & \cdots \end{array}$$

The result of a play is thus an infinite sequence $y_0 y_1 y_2 \ldots$. Player I wins the play if this sequence belongs to A; otherwise II is the winner. We define $\Gamma'(A)$ as the same game with the same winning conditions, except that the first move belongs to Player II.

$$\begin{array}{lcccccccc} \text{I} & & y_1 & & y_3 & & y_5 & \cdots \\ \text{II} & y_0 & & y_2 & & y_4 & & \cdots \end{array}$$

For a set C and $A \subseteq C \times Y^\omega$, we let

$$\Game(A) = \{x : \text{ I has a winning strategy in the game } \Gamma(A_x)\}$$

(where $A_x = \{w : (x, w) \in A\}$).

Definition 2. A subset B of a Cantor space C is game–definable, if it can be presented as $\Game(A)$, where

$$A \subseteq C \times \omega^\omega$$

is an F_σ set in the product space of $C \times \omega^\omega$.

This concept has been analyzed by Y. N. Moschovakis [15] for the lightface sets and the equivalence of this notion and lightface notion of $\mathbf{\Sigma}^1_1$–inductiveness is the content of Exercise 7.C.10 in [15] (a conjunction of a Theorem of Wolfe and a Theorem of Solovay). The equivalence of these two notions in the boldface sense follows from [24]. We will not use directly any of this results, however certainly they inspire the following proofs and in Section 5 we already used completeness of QB in the class of $\mathbf{\Sigma}^1_1$–inductive sets proved in [24], which is part of the proof that the two notions coincide.

Proposition 1. *$W_{1,3}$ is game–definable.*

Proof. We cannot directly use the parity game from definition of $W_{1,3}$, as the parity condition of index $(1, 3)$ is not F_σ. Instead, for a tree $t \in T_{1,3}$, we consider the following modification of the game $G(t)$. The players move as previously except in the case when *rank* of the actual position is 1. In this case, Eve must exhibit a (finite) strategy to reach a node with a rank *greater* than 1 in finite time. (If it is not possible, Eve cannot win in $G(t)$.) Next, Adam chooses one of the nodes reachable by Eve's strategy, and the game continues.

To make it precise, we first define a *local strategy* (for Eve) at a node v of t. It is a *finite* subset S of the descendants of v, such that $v \in S$ and, whenever $w \in S$, then

- if $t(w) = (\forall, 1)$ then $w0, w1 \in S$;
- if $t(w) = (\exists, 1)$ then w has exactly one successor in S;
- if $t(w) \downarrow_2 \geq 2$ then w is a *leaf*, i.e., has no successor in S.

Now, for a tree $t \in T_{1,3}$, we define a parity game $H(t)$, as follows. The positions of $H(t)$ include all *tree positions* $v \in 2^*$. We distinguish between (≥ 2)-*positions*, for which $t(v) \downarrow_2 \geq 2$, and 1-*positions*, for which $t(w) \downarrow_2 = 1$. The (≥ 2)-positions are assigned to Eve or Adam depending on the value of $t(w) \downarrow_1$; the 1-positions are always assigned to Eve. Additionally, there are *strategy positions* of the form (v, S), where v is a 1-position and S is a local strategy from v; they are assigned to Adam. The moves from the (≥ 2)-positions are the same as in $G(t)$ (to successors). From a 1-position v, there is a move to each strategy position (v, S). From a strategy position (v, S), there is a move to each tree position w, such that w is a leaf of S (note that w is then a (≥ 2)-position). The rank of a tree position with $t(v) \downarrow_2 = 3$ is 1; all other positions have rank 0. It is straightforward to see that Eve has a winning strategy in $G(t)$ iff she has a winning strategy in $H(t)$. In order to present $W_{1,3}$ in the form required in Definition 2, we have to make sure that the players move in alternation, (which needs not be the case in $G(t)$); this can be easily achieved by inserting some trivial moves. Then we define the relation A as the set of pairs (t, π), such that t is a tree in $T_{1,3}$ encoded as element of 2^ω, and π is a winning path in $H(t)$ encoded as element of ω^ω. Note that the winning condition in $H(t)$ requires that the (new) rank 1 is encountered only finitely often. Therefore, we can present A as the union of sets A_n, where A_n consists of those pairs (t, π), where π encounters 1 at most n times. The last set is closed (provided that we encode the positions of the game by natural numbers, not by sequences of bits.) Hence A is F_σ, as required. $\square$

Let us notice, that as in the case of $\mathbf{\Sigma}^1_1$–inductive sets holds the following

Lemma 2. *If C, D are Cantor spaces and B is a preimage of A under a continuous mapping $\phi : D \to C$ and A is game–definable, then B is also game–definable.*

Proof. Indeed, if $A = \Game(R)$ then $B = \Game(\{(x, y) : (\phi(x), y) \in R\})$. The second relation is itself a continuous inverse image of R, hence its complexity is not higher than that of R. $\square$

From Section 5 we know that $W_{1,3}$ is complete in the family of all $\mathbf{\Sigma}^1_1$–inductive sets and from Corollary 1 we know, that tree languages recognized by alternating automata of index $(1, 3)$ are $\mathbf{\Sigma}^1_1$–inductive. Hence we are getting the following known result (see [15, Exercise 7.C.10] and [24]):

Corollary 2. *Every $\mathbf{\Sigma}^1_1$–inductive set is game–definable. In particular, tree languages recognized by alternating automata of index $(1, 3)$ are game–definable.*

Our next goal is to show that the set $W_{1,3}$ is actually complete in the class of game–definable sets. To this end, we consider another variant of topological games, where the players select bits instead of natural numbers at the expense of a more complex winning criterion. The following concept will be useful.

Definition 3. A *parity coloring* (or coloring, for short) over a finite set Σ is a mapping $K : \Sigma^* \to \omega$, taking only a finite number of values. This coloring defines a set

$$[K] = \{u \in \Sigma^\omega : \limsup_{n\to\infty} K(u \restriction n) \textit{ is even } \}$$

(where $u \restriction n = u(0) \ldots u(n-1)$).

Recognition by coloring generalizes recognition by deterministic parity automata on infinite words. A related concept for finite words has been considered by Séverine Fratani [9] under the name of *automates à oracles*. A more general concept of Borel automata appears in [20]. The formulation in Definition 3 comes from the Master thesis of Michał Skrzypczak [25].

If the underlying set is a product, e.g., $\Sigma = 2 \times 2$, we identify a set $A \subseteq \Sigma^\omega$ with the relation

$$\{(x, y) \in 2^\omega \times 2^\omega : (x_0, y_0), (x_1, y_1), \ldots \in A\}.$$

Hence, a coloring $K : (2 \times 2)^* \to \omega$ induces the set $\Game([K])$ which, by definition, consists of those x, for which Player I can ensure that the result y of the play satisfies

$$\limsup_{n \to \infty} K(x \restriction n, y \restriction n) \text{ is even.}$$

Theorem 3. *The following conditions are equivalent for $A \subseteq 2^\omega$.*

1. *A is game–definable,*
2. *$A = \Game([K])$ for a coloring $K : (2 \times 2)^* \to \omega$, which takes the values in $\{1, 2, 3\}$.*

Proof. To show (2) $\Rightarrow$ (1), we prove that any set of the form $A = \Game([K])$ of (2) can be continuously reduced to $W_{1,3}$. For $x \in 2^\omega$, we define a labeled tree $t^K_x : 2^* \to \{\exists, \forall\} \times \{1, 2, 3\}$, by

$$t^K_x(v) \quad = \quad \begin{cases} (\exists, K(x \restriction |v|, v)) & \text{for } |v| \text{ even} \\ (\forall, K(x \restriction |v|, v)) & \text{for } |v| \text{ odd.} \end{cases} \tag{2}$$

Clearly the mapping $x \mapsto t^K_x$ is continuous and it is straightforward to see that $x \in A \iff t^K_x \in W_{1,3}$; indeed the winning strategies can be transferred easily between the two games. Since the class of game–definable sets is closed under continuous reductions and we know from Proposition 1 that it contains $W_{1,3}$, it follows that it contains A as well.

To prove (1) $\Rightarrow$ (2), suppose $A = \Game(R)$, for some $R \subseteq 2^\omega \times \omega^\omega$ of the class F_σ. We use a well-known fact that, generally, a set $E \subseteq X^\omega$ is of the class F_σ iff for some set of *finite* words $Z \subseteq X^*$ the following characterization holds (see, e.g., Theorem III 3.11 in [20]):

$$u \in E \quad \Leftrightarrow \quad \{n : u \restriction n \in Z\} \text{ is finite.}$$

Let $Z \subseteq (2 \times \omega)^*$ be such a set for R. Before defining the coloring K formally, we describe the game it should induce. This game simulates the game $\Gamma(R_x)$ but, instead of natural numbers, the players choose now only bits in 2. Suppose a play in $\Gamma(R_x)$ was I: m_0, II: m_1, I: m_2, II: m_3, and so on. The choice of the number m_0 by player I is now simulated by m_0 consecutive rounds in which I chooses 0, and II answers by 0 as well. (If II violates this rule, he will be forced to loose by suitable coloring.) After this phase, I plays 1, to which II answers 0 again. Then the roles exchange: now Player I plays only 0's, to which player II answers by 0 for m_1 times, and decides to put 1 afterward, and so on. For example, a play

$$\begin{array}{cccccc} \mathrm{I} & 3 & & 2 & & \cdots \\ \mathrm{II} & & 0 & & 2 & \cdots \end{array}$$

is simulated by

$$\begin{array}{cccccccccccccccccccccccc} \mathrm{I} & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & \cdots \\ \mathrm{II} & & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & \cdots \end{array}$$

A player who deviates from the above rules, looses. So a "correct" play must be of the form

$$0^{2m_0}\,10\,0^{2m_1}\,01\,0^{2m_2}\,10\,\ldots \tag{3}$$

We say that a (finite) word $0^{2m_0}\,10\,0^{2m_1}\,01\,\ldots 0^{2m_k}\,10$, *represents* the sequence $m_0m_1\ldots m_k$ if k is even; similarly, a word $0^{2m_0}\,10\,0^{2m_1}\,01\,\ldots 0^{2m_k}\,01$ represents the sequence $m_0m_1\ldots m_k$ if k is odd.

To guarantee that our coloring game simulates the game $\Gamma(R_x)$, player I should win the play (3) exactly in the case when the (pair) sequence

$$(x,\,(m_0, m_1, m_2, \ldots))$$

has only a finite number of prefixes in Z. We now define a coloring K to ensure this property. Consider $(\alpha, \beta) \in (2 \times 2)^*$. We let $K(\alpha, \beta) = 3$, whenever β contains a prefix which violates the rules. If β represents a sequence $m_0m_1\ldots m_k$, we let $K(\alpha, \beta) = 2$ if $(\alpha \restriction (k+1), m_0m_1\ldots m_k)$ is not in Z, and $K(\alpha, \beta) = 3$, otherwise. In all other cases, we let $K(\alpha, \beta) = 1$ when we simulate m_k for even k and $K(\alpha, \beta) = 2$ when we simulate m_k for odd k. It is then straightforward to see that $x \in \Game(R)$ iff $x \in \Game([K])$, which yields the desired presentation of A. □

The equivalence of Theorem 3 along with the fact established in the proof that all sets representable as in condition (2) of Theorem 3 reduce to $W_{1,3}$ yield the following.

Corollary 3. *The set $W_{1,3}$ is complete in the class of game–definable sets.*

Finally we are getting:

Corollary 4. *The following three concepts of definability coincides for $A \subseteq 2^\omega$:*

1. *A is $\mathbf{\Sigma}^1_1$–inductive,*
2. *A is game–definable,*
3. *$A = \Game([K])$ for a coloring $K : (2 \times 2)^* \to \omega$, which takes the values in $\{1, 2, 3\}$.*

Proof. The last two conditions are equivalent according to Theorem 3. If A is $\mathbf{\Sigma}^1_1$–inductive, then from Corollary 2 it is game–definable. If A is game–definable, then according to Corollary 3 it is reducible to $W_{1,3}$, hence from Lemma 1 the set A is $\mathbf{\Sigma}^1_1$–inductive as a preimage of $\mathbf{\Sigma}^1_1$–inductive set $W_{1,3}$. □

7 Conclusion and further work

We have provided an evidence that the class of $\mathbf{\Sigma}^1_1$–inductive sets is a set–theoretical generalization of the class of regular tree languages of index $(1, 3)$. This extends the previously known relations between $(0, 1)$ vs. $\mathbf{\Pi}^1_1$ and $(1, 2)$ vs. $\mathbf{\Sigma}^1_1$. Plausibly, this characterization can go further with an appropriate extension of the concept of inductiveness (in the spirit of [7]).

A related topic is the separation property, which is one of the central issues in descriptive set theory. It is known that the class of $\mathbf{\Sigma}^1_1$–inductive fails this property ([15, Theorem 6.D.4]); that is, there exist two disjoint $\mathbf{\Sigma}^1_1$–inductive sets $A, B \subseteq \{0, 1\}^\omega$ such that there is no set C which would separate A and B and which would simultaneously satisfy the conditions that C and $\{0,1\}^\omega \setminus C$ are $\mathbf{\Sigma}_1^1$–inductive sets.

We established recently in a joint paper with André Arnold [2] the failure of separation property for all the levels (ι, n) of the alternating index hierarchy, for n odd (for the level $(0,1)$, the result was known [10]). We are planning to establish whether the pair of regular languages of index $(1,3)$ constructed in [2] could serve as an example of inseparable pair for the class of $\mathbf{\Sigma}_1^1$–inductive sets. The question if the separation property holds for the levels (ι, n) with n even remains open (it is only known to hold for $(1,2)$ [10, 22]). One may hope that the ideas from descriptive set theory may offer an insight into the problem.

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