Stability of interpolation and CASL (sub)logics

Andrzej Tarlecki

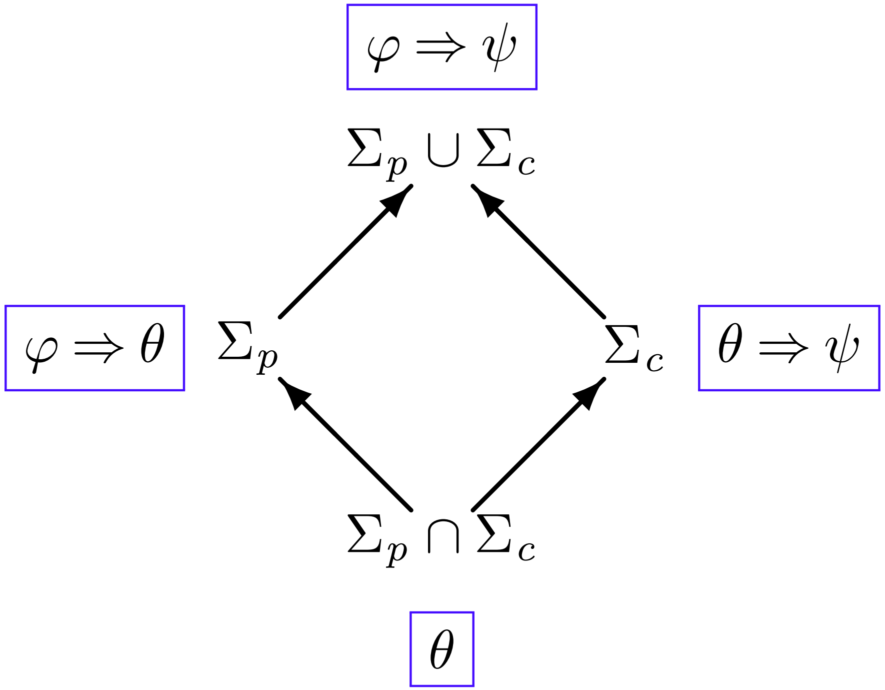
Institute of Informatics, University of Warsaw

Classical Craig's interpolation

Craig'57

In first-order logic:

Fact: *Any sentences* $\varphi \in \mathbf{Sen}(\Sigma_p)$ *and* $\psi \in \mathbf{Sen}(\Sigma_c)$ *such that* $\varphi \Rightarrow \psi$*, have an interpolant* $\theta \in \mathbf{Sen}(\Sigma_p \cap \Sigma_c)$ *such that* $\varphi \Rightarrow \theta$ *and* $\theta \Rightarrow \psi$.

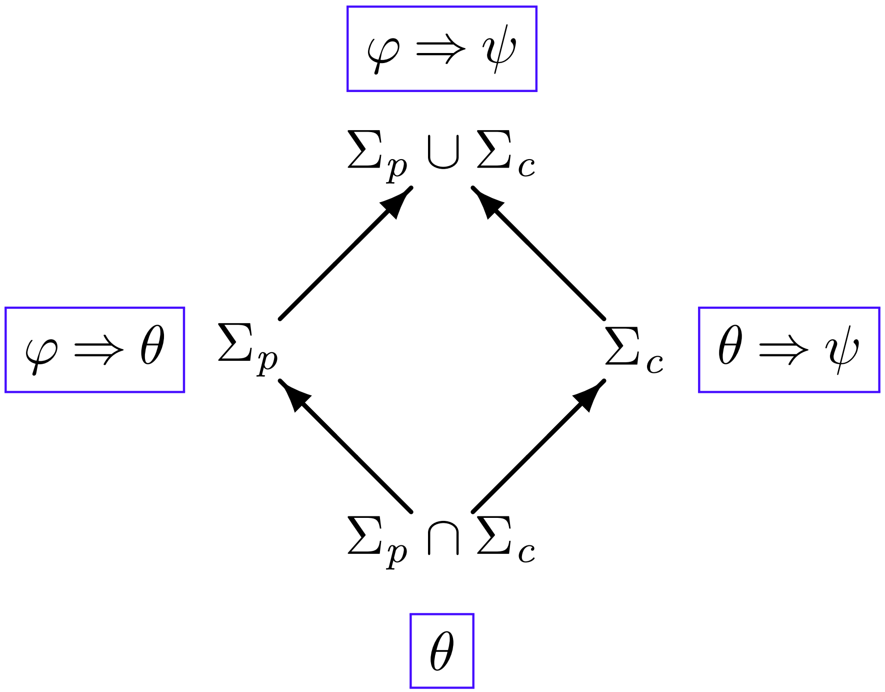


Classical Craig's interpolation

Craig'57

In first-order logic:

Fact: *Any sentences* $\varphi \in \mathbf{Sen}(\Sigma_p)$ *and* $\psi \in \mathbf{Sen}(\Sigma_c)$ *such that* $\varphi \Rightarrow \psi$, *have an interpolant* $\theta \in \mathbf{Sen}(\Sigma_p \cap \Sigma_c)$ *such that* $\varphi \Rightarrow \theta$ *and* $\theta \Rightarrow \psi$.



Numerous applications in specification & development theory:

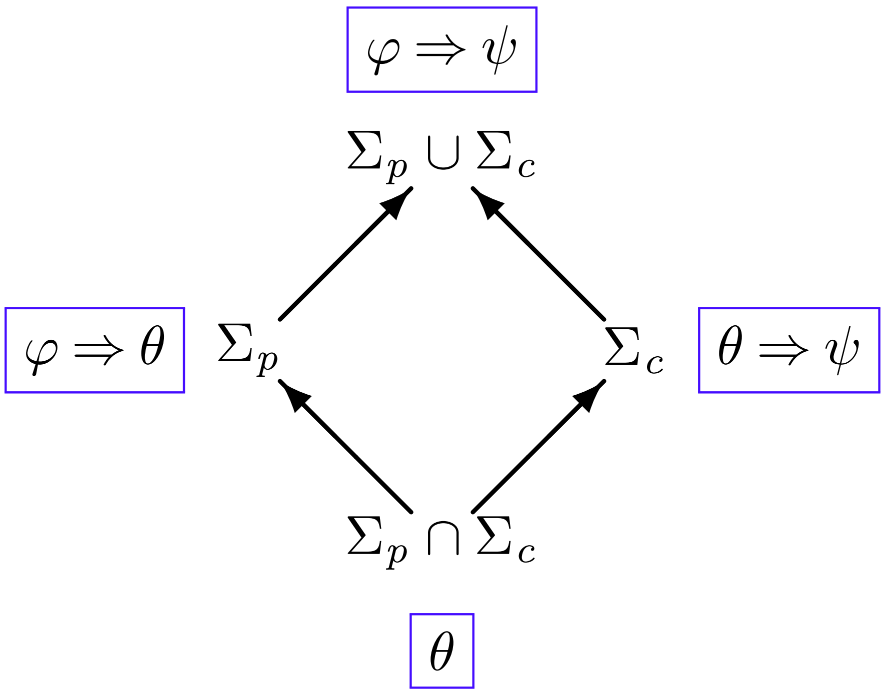
- *Maibaum, Sadler, Veloso, Dimitrakos '84–...*
- *Bergstra, Heering, Klint '90*
- *Cengarle '94, Borzyszkowski '02*
- *...*

Classical Craig's interpolation

Craig'57

In first-order logic:

Fact: *Any sentences $\varphi \in \mathbf{Sen}(\Sigma_p)$ and $\psi \in \mathbf{Sen}(\Sigma_c)$ such that $\varphi \Rightarrow \psi$, have an interpolant $\theta \in \mathbf{Sen}(\Sigma_p \cap \Sigma_c)$ such that $\varphi \Rightarrow \theta$ and $\theta \Rightarrow \psi$.*



Key related properties:

- *Robinson's consistency theorem*
- *Beth's definability theorem*

Meta-facts:

- *$\mathcal{CI}$ and $\mathcal{RC}$ are equivalent*
- *$\mathcal{CI}$ implies $\mathcal{BD}$ (not vice versa)*

"IN ESSENCE"

Institution

Goguen & Burstall: 1980 → 1992

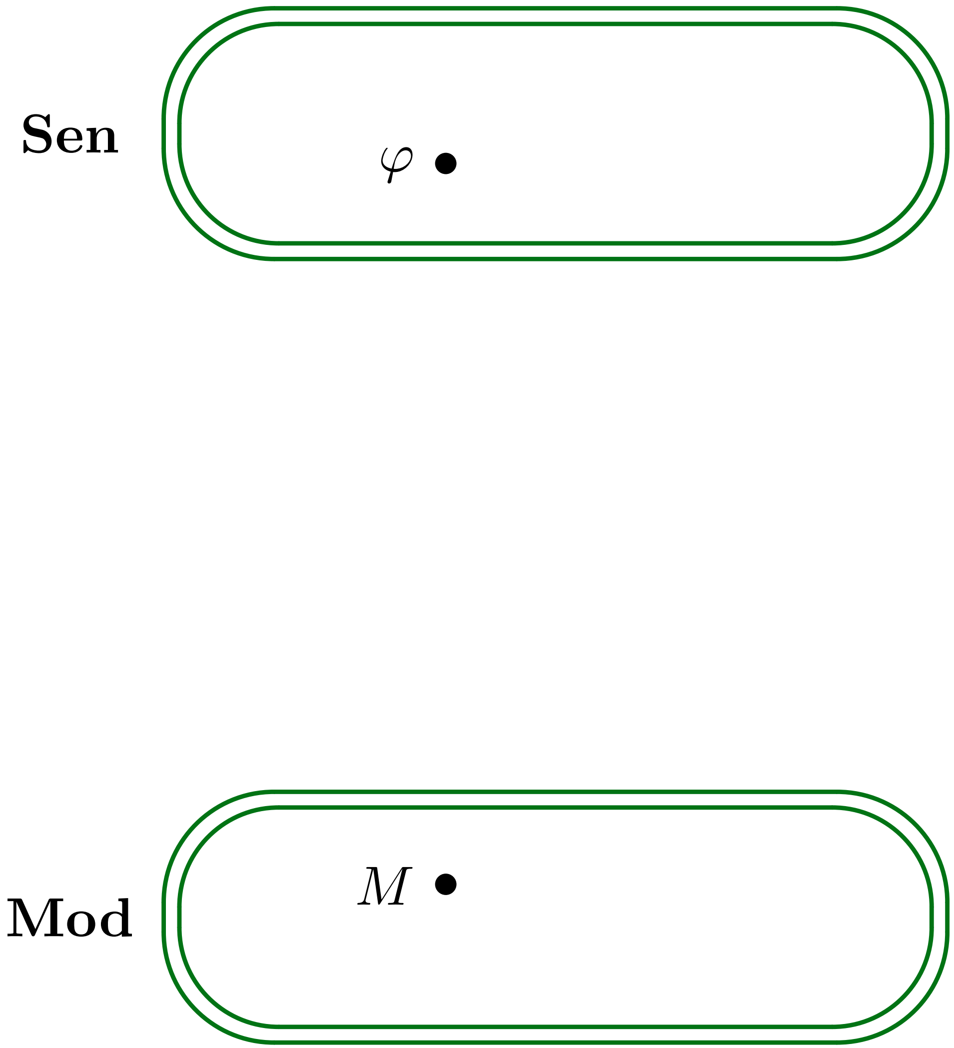
- a category **Sign** of *signatures*
- a functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$
  - $\mathbf{Sen}(\Sigma)$ is the set of Σ-*sentences*, for $\Sigma \in |\mathbf{Sign}|$
- a functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Class}$
  - $\mathbf{Mod}\Sigma$ is the category of Σ-*models*, for $\Sigma \in |\mathbf{Sign}|$
- for each $\Sigma \in |\mathbf{Sign}|$, Σ-*satisfaction relation* $\models_\Sigma \subseteq \mathbf{Mod}(\Sigma) \times \mathbf{Sen}(\Sigma)$

subject to the *satisfaction condition*:

$$M'|_\sigma \models_\Sigma \varphi \iff M' \models_{\Sigma'} \sigma(\varphi)$$

where $\sigma\colon \Sigma \to \Sigma'$ in **Sign**, $M' \in \mathbf{Mod}(\Sigma')$, $\varphi \in \mathbf{Sen}(\Sigma)$, and then $M'|_\sigma$ stands for $\mathbf{Mod}(\sigma)(M')$, and $\sigma(\varphi)$ for $\mathbf{Sen}(\sigma)(\varphi)$.

Institution: abstraction



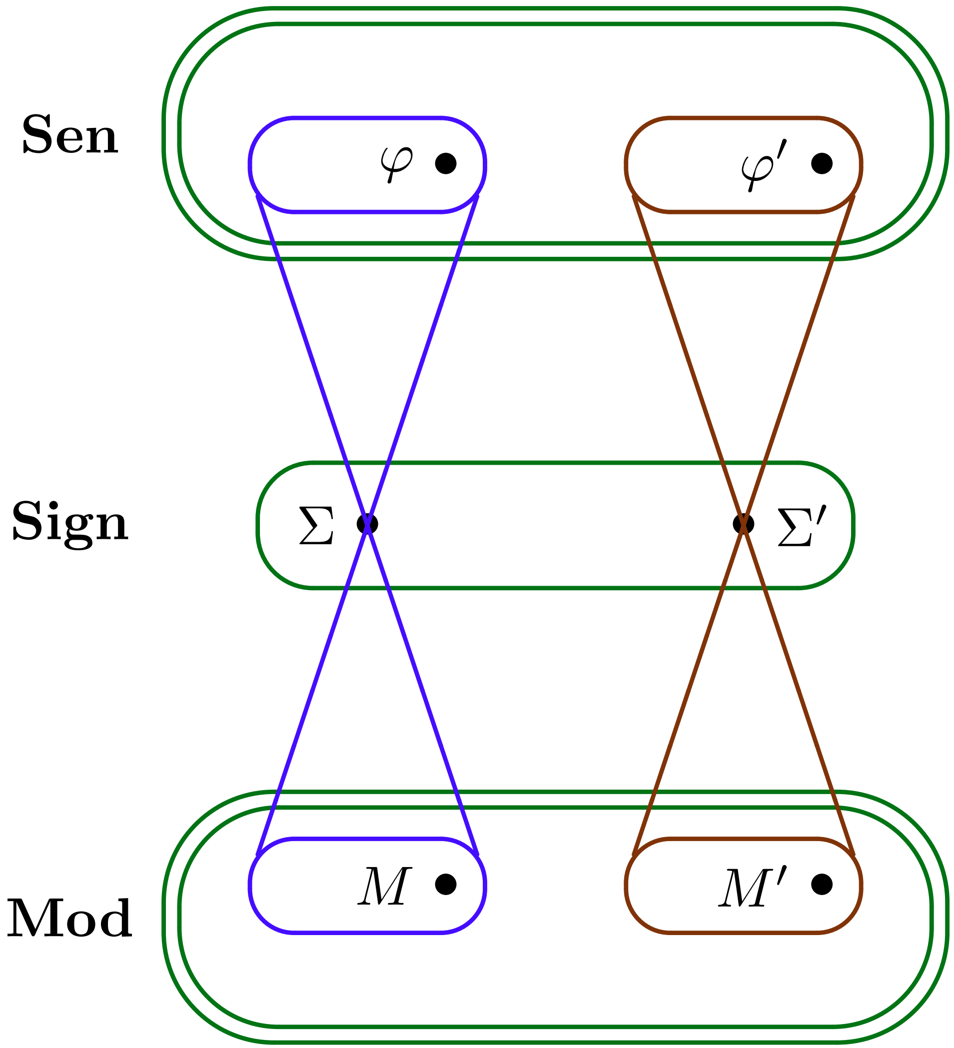
plus *satisfaction relation:*

$$\boxed{M \models \varphi}$$

and so the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod(\Phi)$, $Th(\mathcal{M})$, $Th(\Phi)$, $\Phi \models \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash \varphi$ (implying $\Phi \models \varphi$) to deal with proof-theoretic aspects.

Institution: first insight



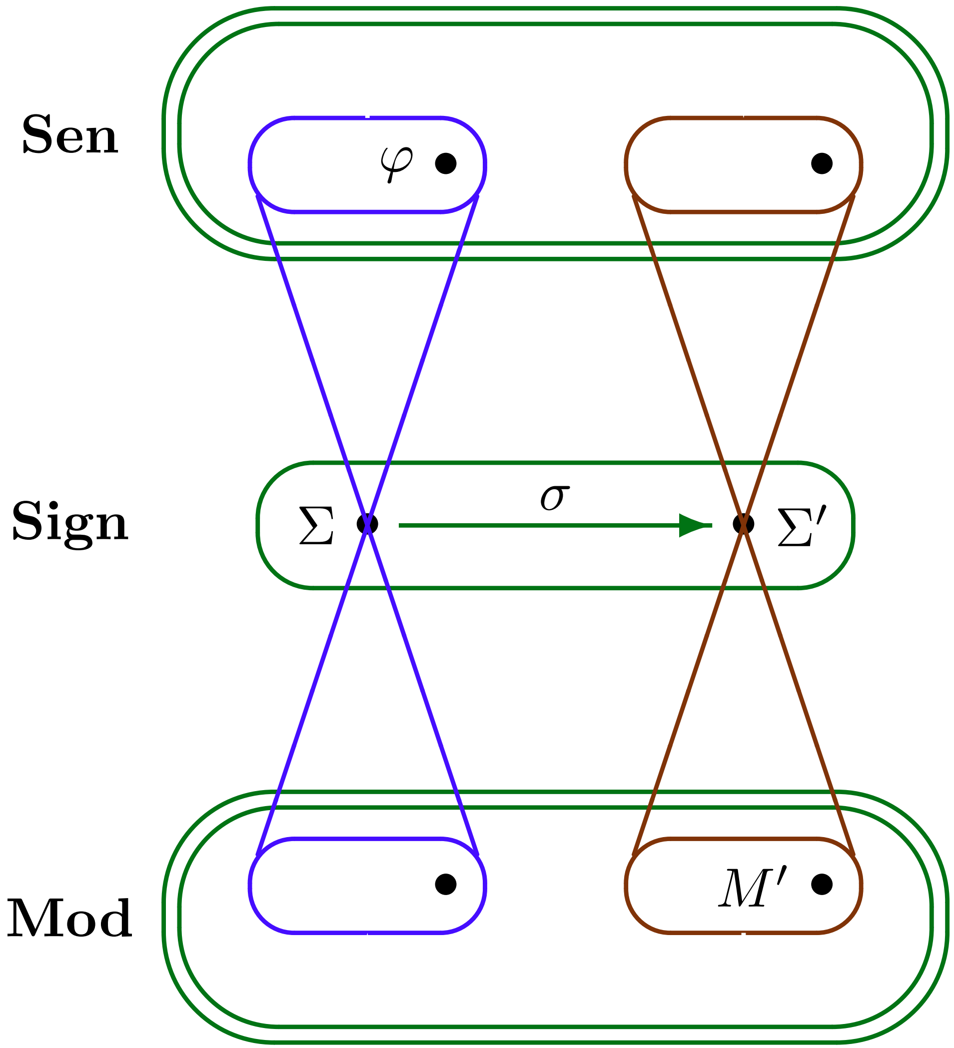
plus *satisfaction relation*, for each signature:

$$M \models_\Sigma \varphi$$

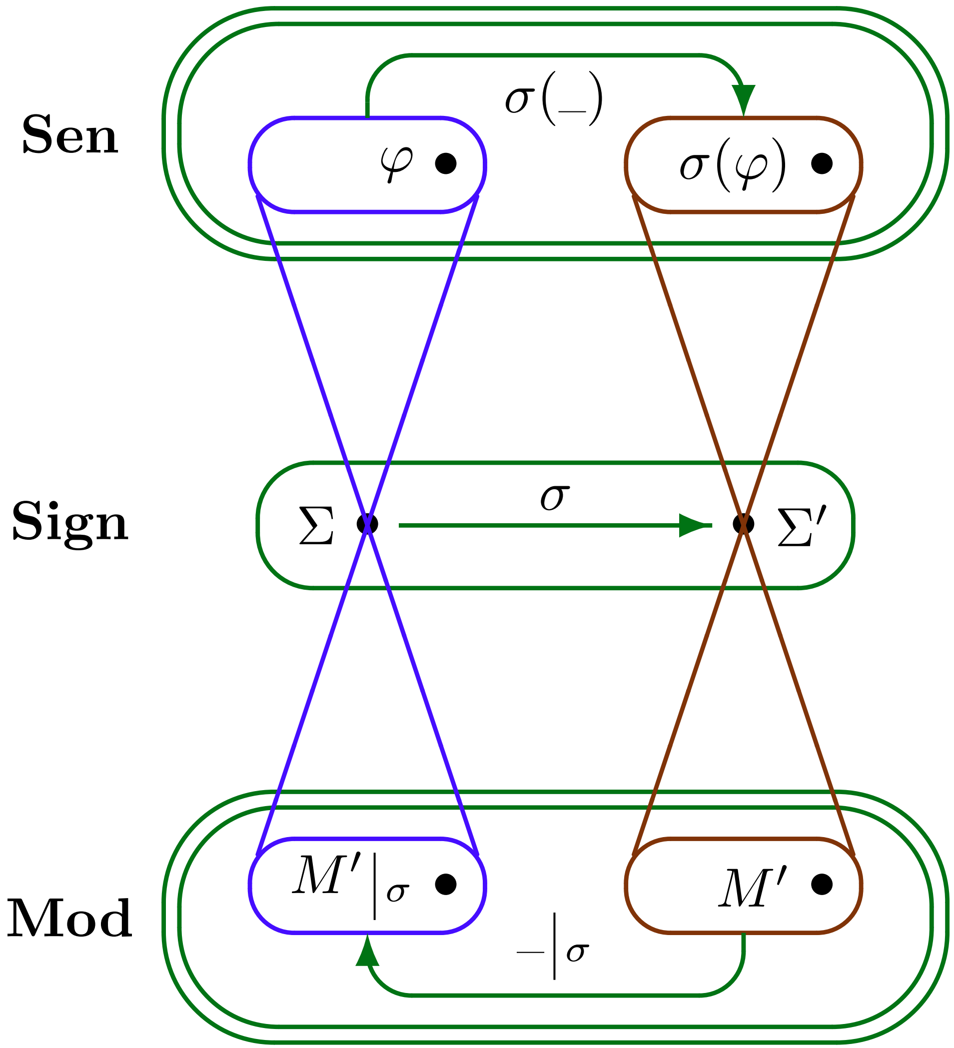
and so, for each signature, the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod_\Sigma(\Phi)$, $Th_\Sigma(\mathcal{M})$, $Th_\Sigma(\Phi)$, $\Phi \models_\Sigma \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash_\Sigma \varphi$ (implying $\Phi \models_\Sigma \varphi$) to deal with proof-theoretic aspects.

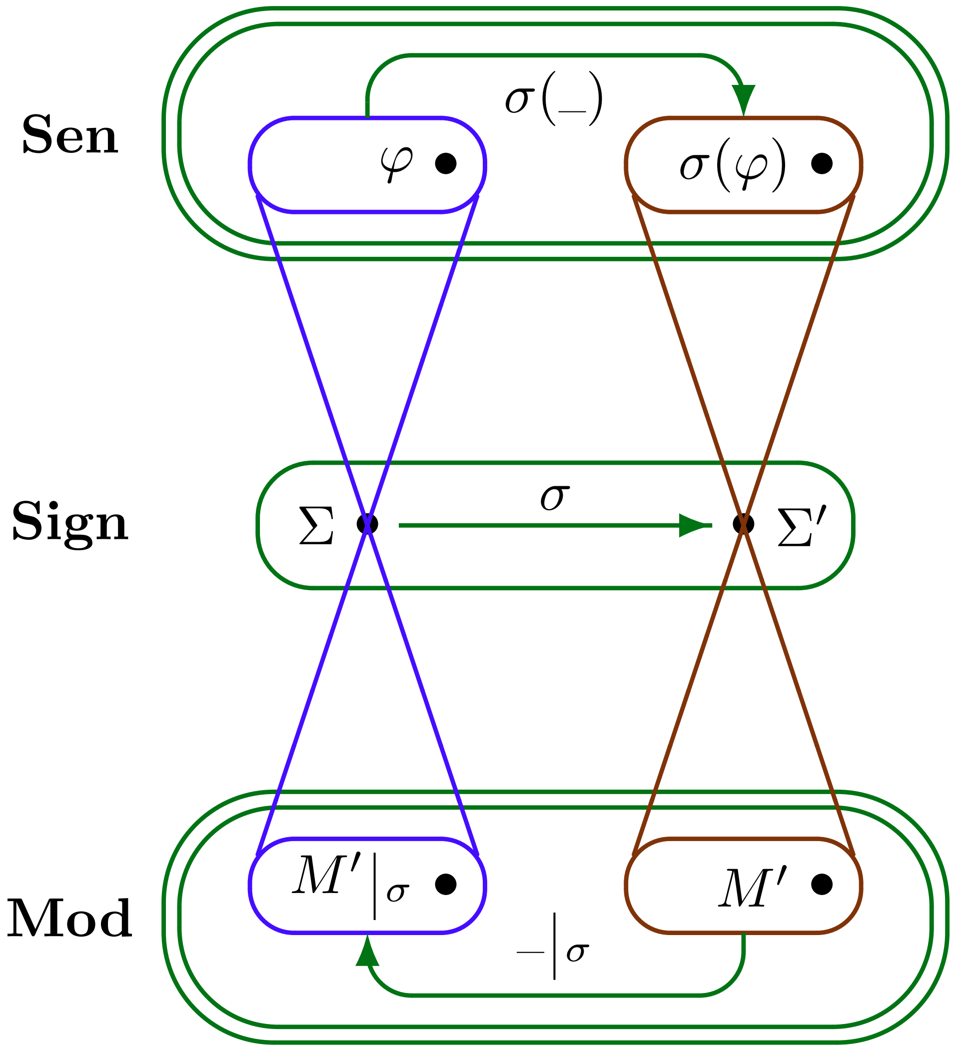
Institution: key insight



Institution: key insight



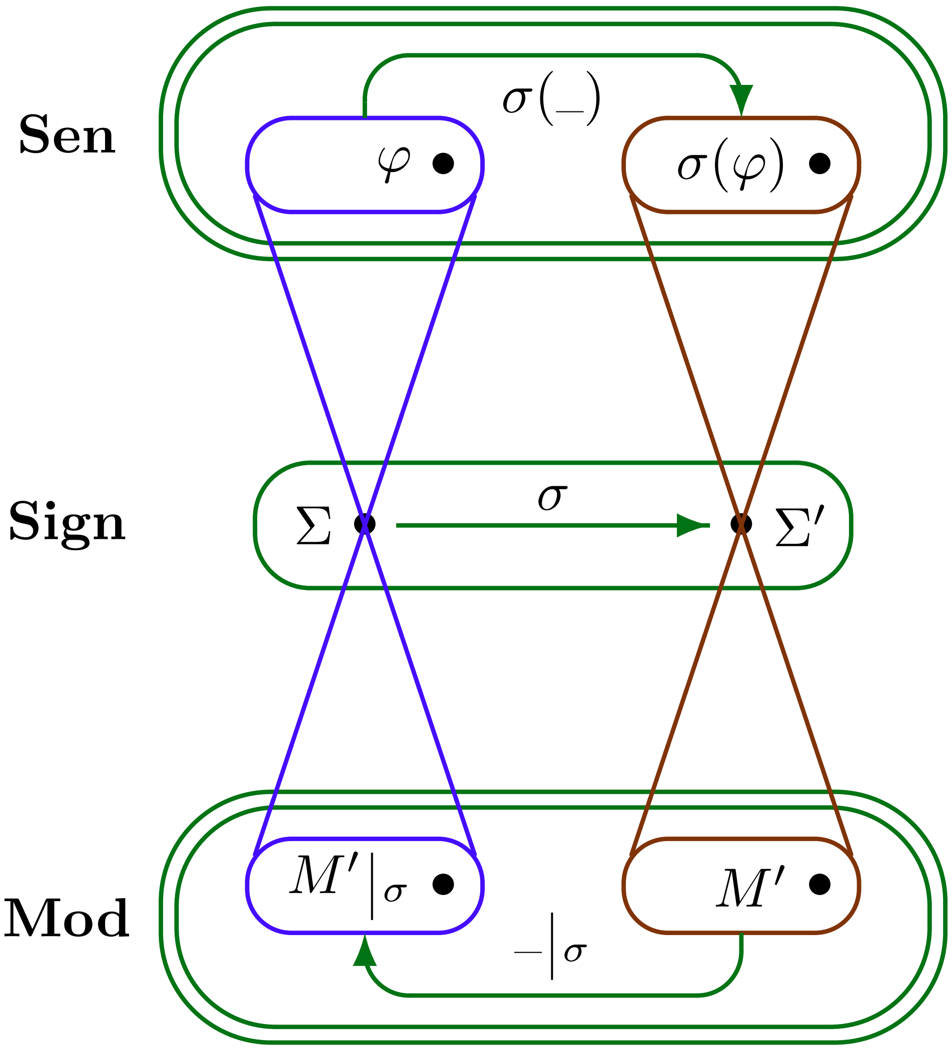
Institution: key insight



The *satisfaction condition:*

$$M' \models_{\Sigma'} \sigma(\varphi) \text{ iff } M'|_\sigma \models_\Sigma \varphi$$

Institution: key insight

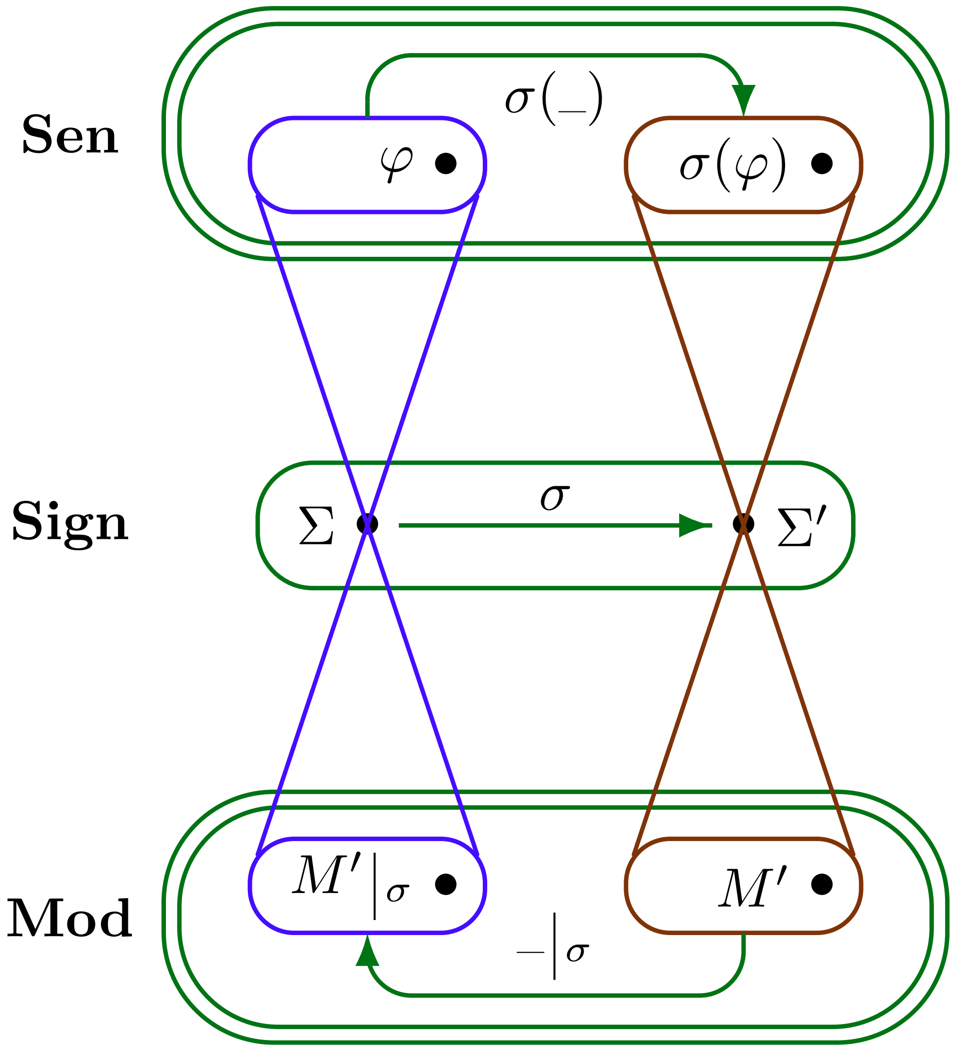


Truth is invariant
under change of notation
and independent of
additional symbols around

The *satisfaction condition:*

$$M' \models_{\Sigma'} \sigma(\varphi) \text{ iff } M'|_\sigma \models_\Sigma \varphi$$

Institution: key insight



Truth is invariant
under change of notation
and independent of
additional symbols around

The *satisfaction condition:*

$$M' \models_{\Sigma'} \sigma(\varphi) \text{ iff } M'|_\sigma \models_\Sigma \varphi$$

It follows:

$$\Phi \models_\Sigma \varphi \text{ implies } \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)$$

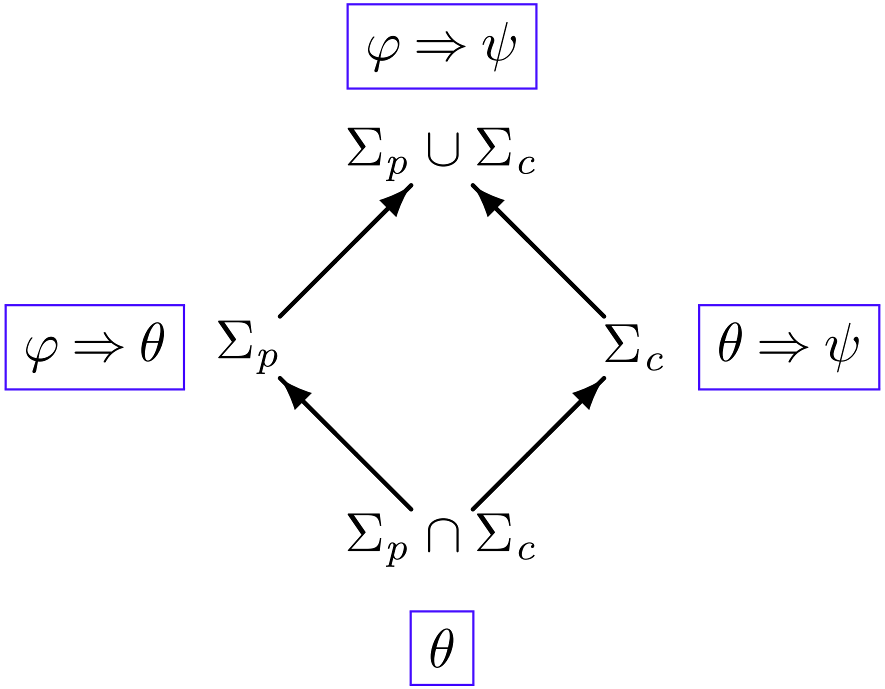
If $\_|_\sigma : \mathbf{Mod}(\Sigma') \to \mathbf{Mod}(\Sigma)$ is onto:

$$\Phi \models_\Sigma \varphi \text{ iff } \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)$$

Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

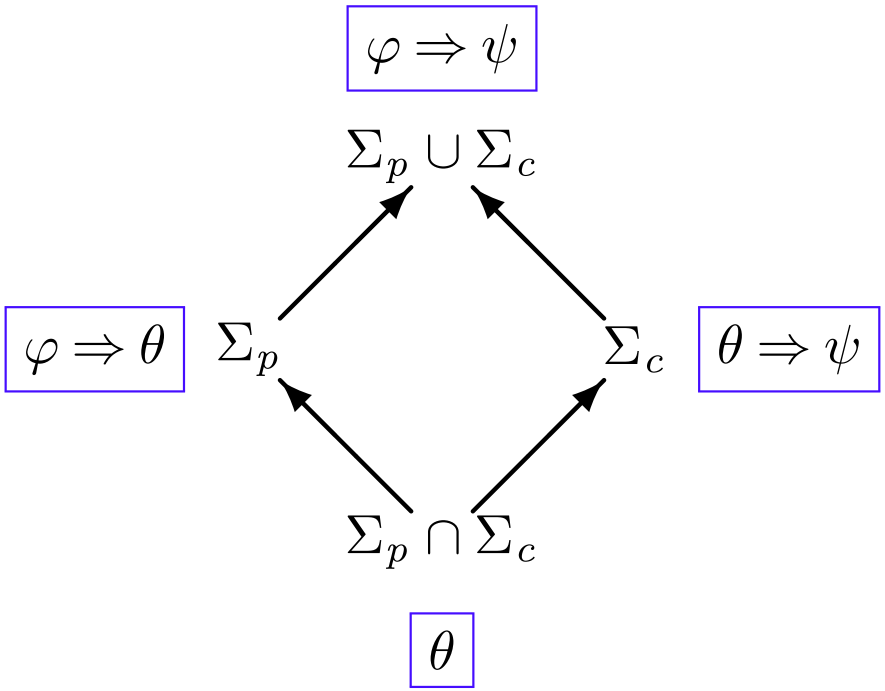
Recall:



Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

Recall:



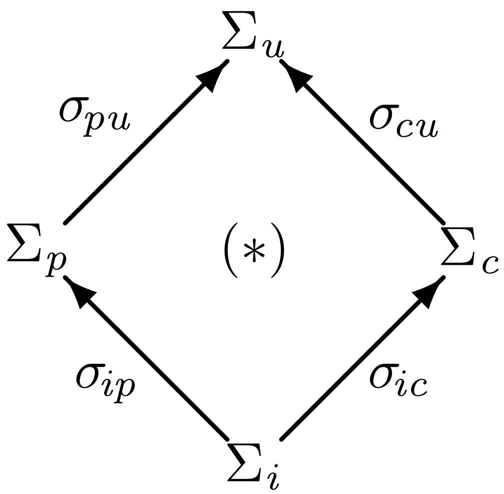
Some things don't work in $\mathbf{INS}$*:*

- *implication?*
  $\rightsquigarrow$ *entailment*
- *individual sentences?*
  $\rightsquigarrow$ *sets of sentences*
- *union/intersection square?*
  $\rightsquigarrow$ *arbitrary commutative square of signature morphisms*

Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

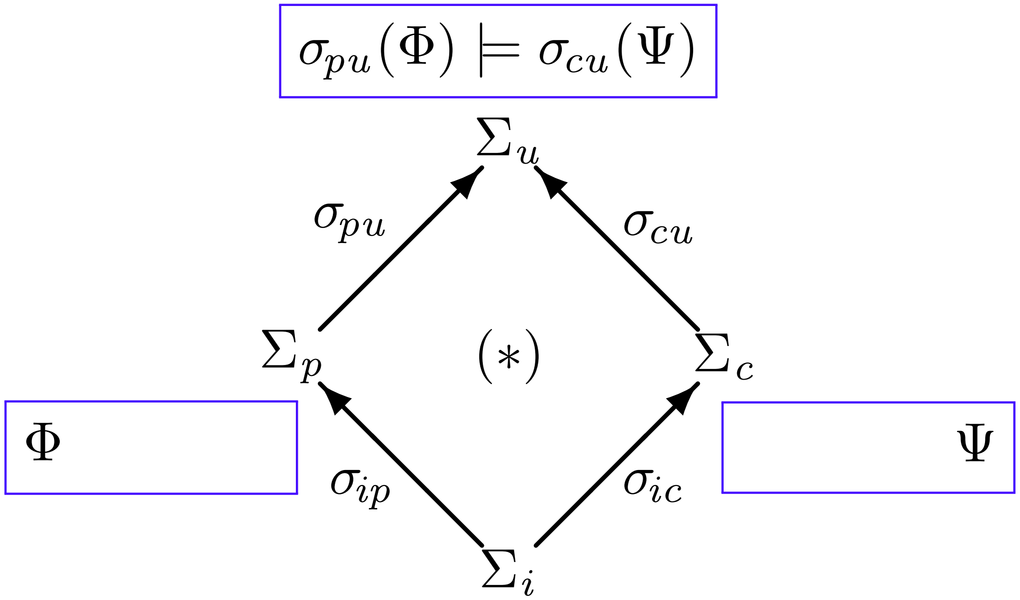
Definition:



Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

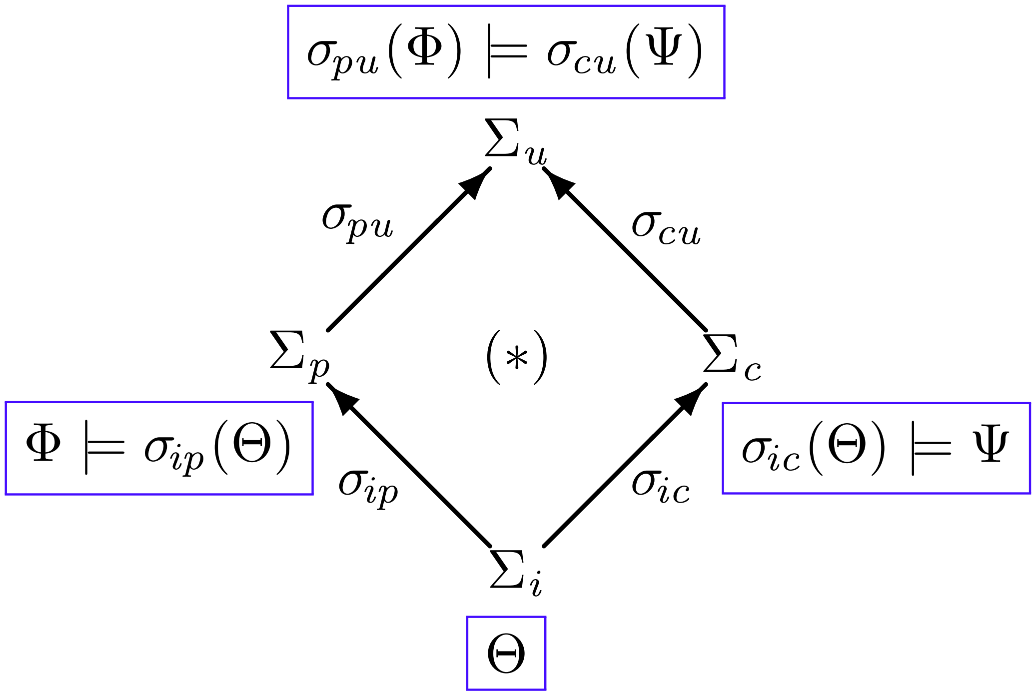
Definition:



Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

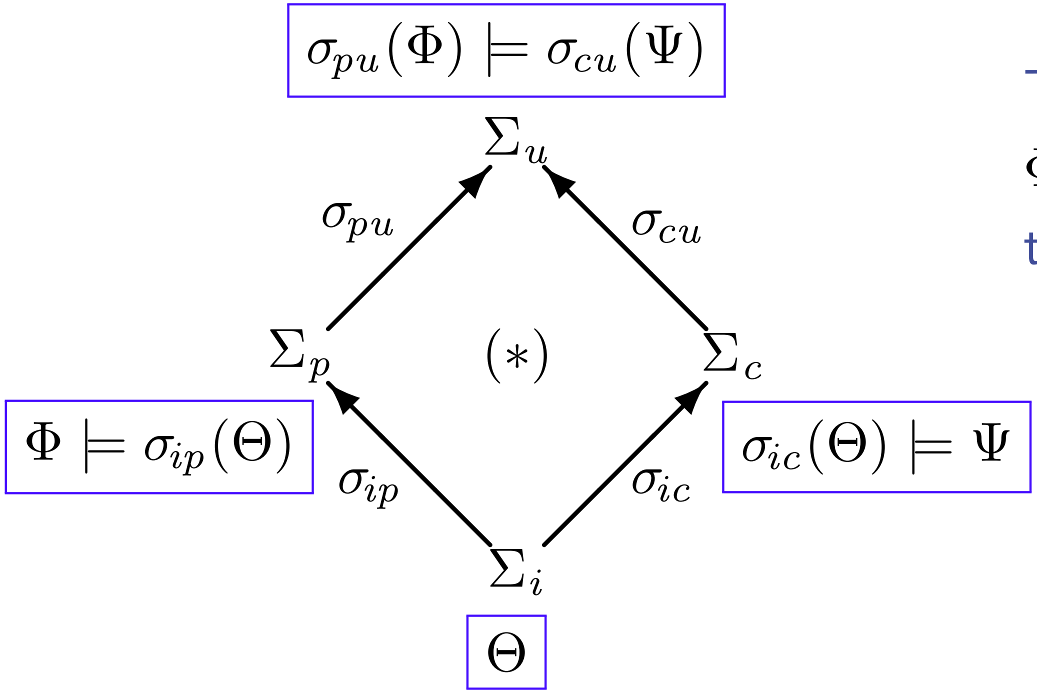
Definition: *An interpolant for $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ such that $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ is $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ such that $\Phi \models \sigma_{ip}(\Theta)$ and $\sigma_{ic}(\Theta) \models \Psi$.*



Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$:

Definition: *An interpolant for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ *such that* $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ *is* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *such that* $\Phi \models \sigma_{ip}(\Theta)$ *and* $\sigma_{ic}(\Theta) \models \Psi$.

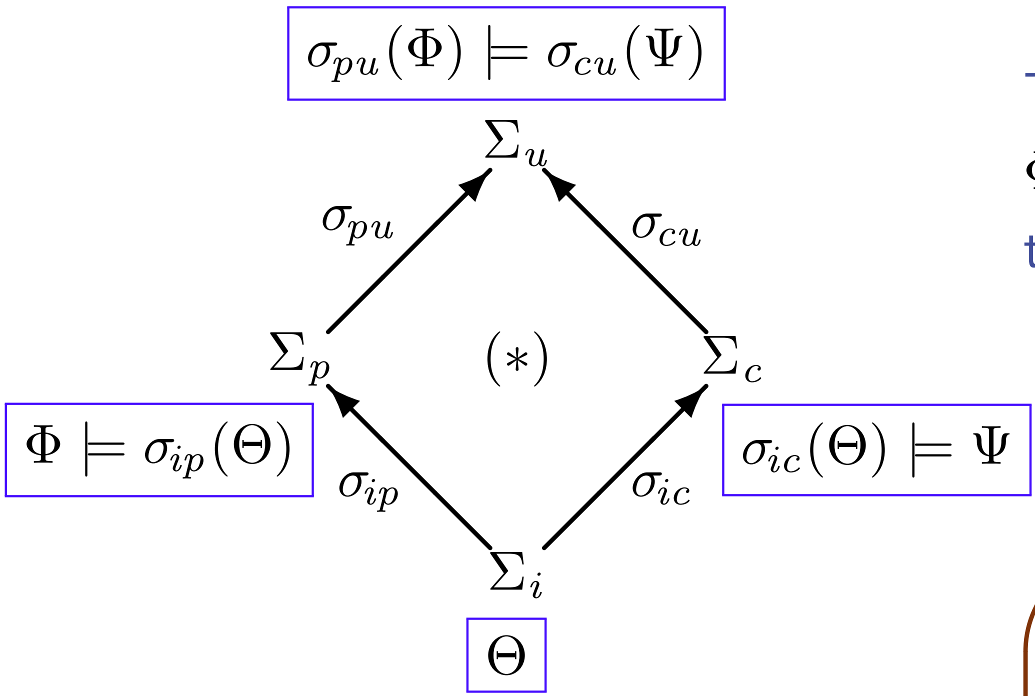


The square $(*)$ *admits interpolation* if all $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ such that $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ have an interpolant.

Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$**:**

Definition: *An interpolant for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ *such that* $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ *is* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *such that* $\Phi \models \sigma_{ip}(\Theta)$ *and* $\sigma_{ic}(\Theta) \models \Psi$*.*



The square $(*)$ *admits interpolation* if all $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ such that $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ have an interpolant.

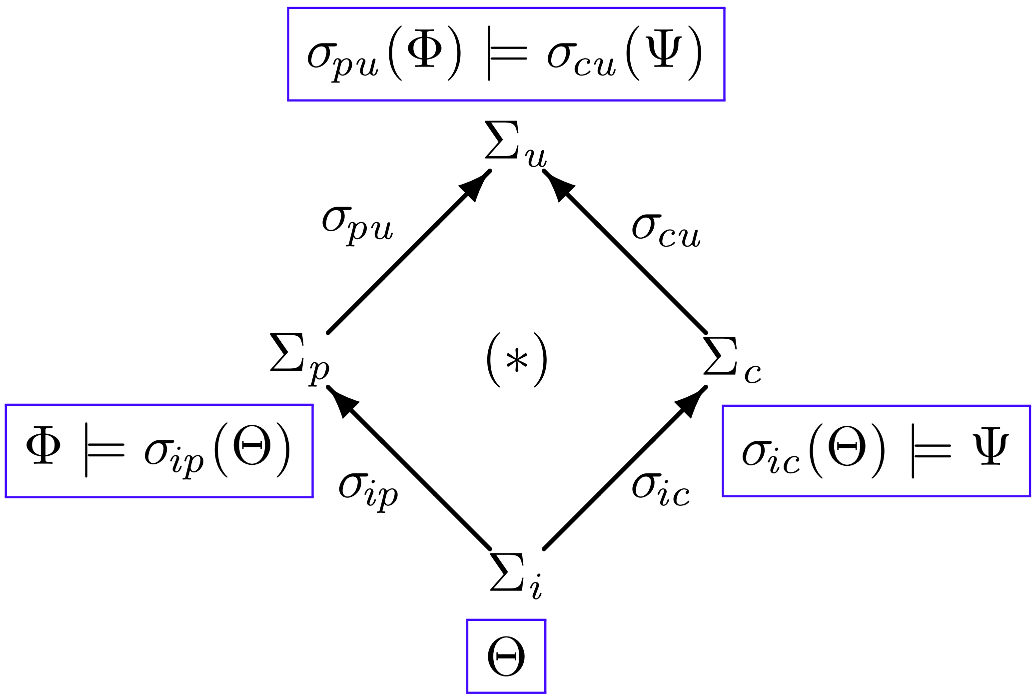
Tarlecki '86, Diaconescu *et al.* '00–. . .

(Roșu, Popescu, Șerbănuță, Găină)

Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$:

Definition: *An interpolant for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ *such that* $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ *is* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *such that* $\Phi \models \sigma_{ip}(\Theta)$ *and* $\sigma_{ic}(\Theta) \models \Psi$.

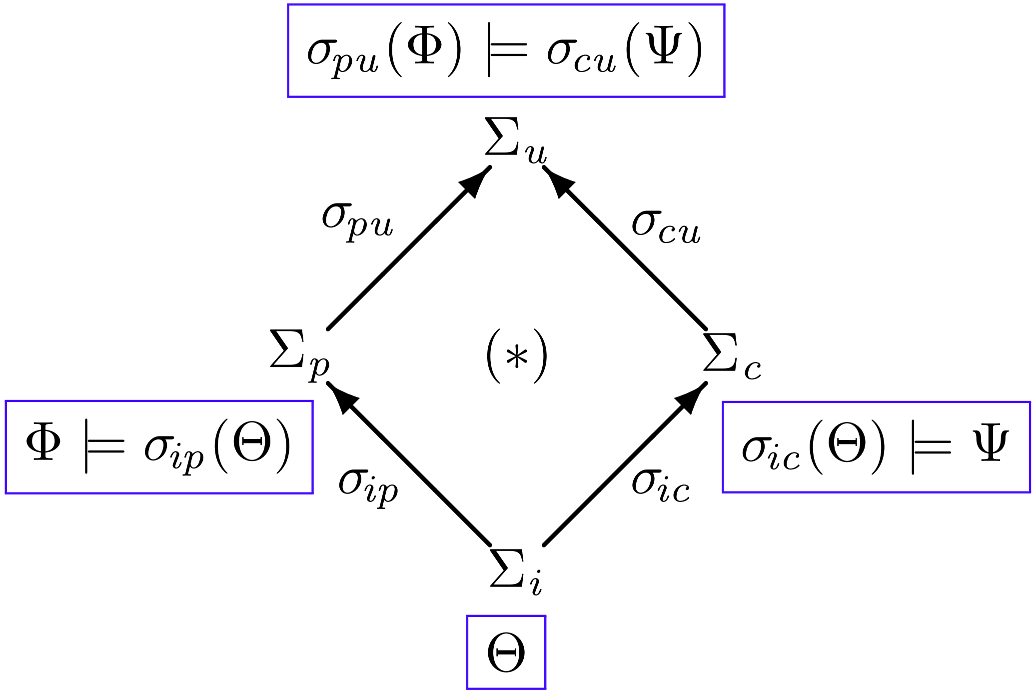


- In **PL** (propositional logic): all signature pushouts admit interpolation.
- In **FO** (many-sorted first-order logic): all signature pushouts with σ_{ip} or σ_{ic} injective on sorts admit interpolation.
- In **EQ** (many-sorted equational logic): all signature pushouts with injective σ_{ic} admit interpolation.

Craig's interpolation

In $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$:

Definition: *An interpolant for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$ *such that* $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ *is* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *such that* $\Phi \models \sigma_{ip}(\Theta)$ *and* $\sigma_{ic}(\Theta) \models \Psi$.



- In **PL** (propositional logic): all signature pushouts admit interpolation.
- In **FO** (many-sorted first-order logic): all signature pushouts with σ_{ip} or σ_{ic} injective on sorts admit interpolation.
- In **EQ** (many-sorted equational logic): all signature pushouts with injective σ_{ic} admit interpolation.

Warning: nonempty carrier sets

Interpolation in CASL sublogics

Interpolation in CASL sublogics

A pushout $(*)$ admits interpolation in:

- $\mathbf{EQ}$: σ_{ic} injective on sorts

- $\mathbf{FO}$: σ_{ip} or σ_{ic} injective on sorts

Interpolation in CASL sublogics

empty carriers permitted!

A pushout $(*)$ admits interpolation in:

- **EQ**: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*
- **FO**: σ_{ip} or σ_{ic} injective on sorts
  and

Interpolation in CASL sublogics

empty carriers permitted!

A pushout $(*)$ admits interpolation in:

- **EQ**: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*
- **FO**: σ_{ip} or σ_{ic} injective on sorts
  and *no other conditions* — BUT: *proofs to be redone!*

Interpolation in CASL sublogics

empty carriers permitted!

A pushout $(*)$ admits interpolation in:

- **EQ**: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*

- **FO**: σ_{ip} or σ_{ic} injective on sorts
  and *no other conditions* — BUT: *proofs to be redone!*

- **FO** plus partiality: as for **FO**

Interpolation in CASL sublogics

empty carriers permitted!

A pushout $(*)$ admits interpolation in:

- **EQ**: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*
- **FO**: σ_{ip} or σ_{ic} injective on sorts
  and *no other conditions* — BUT: *proofs to be redone!*
- **FO** plus partiality: as for **FO**
- **FO** plus subsorting: as for **FO** and
  each new subsorting is introduced either by σ_{ip} or by σ_{ic} (but not both)

Interpolation in CASL sublogics

A pushout $(*)$ admits interpolation in:

empty carriers permitted!

- $\mathbf{EQ}$: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*
- $\mathbf{FO}$: σ_{ip} or σ_{ic} injective on sorts
  and *no other conditions* — BUT: *proofs to be redone!*
- $\mathbf{FO}$ plus partiality: as for $\mathbf{FO}$
- $\mathbf{FO}$ plus subsorting: as for $\mathbf{FO}$ and
  each new subsorting is introduced either by σ_{ip} or by σ_{ic} (but not both)
- $\mathbf{FO}$ plus partiality and subsorting: as above

Interpolation in CASL sublogics

empty carriers permitted!

A pushout $(*)$ admits interpolation in:

- **EQ**: σ_{ic} injective on sorts
  and *does not force any old sort to be non-empty*
- **FO**: σ_{ip} or σ_{ic} injective on sorts
  and *no other conditions* — BUT: *proofs to be redone!*
- **FO** plus partiality: as for **FO**
- **FO** plus subsorting: as for **FO** and
  each new subsorting is introduced either by σ_{ip} or by σ_{ic} (but not both)
- **FO** plus partiality and subsorting: as above
- **FO** plus reachability constraints (with or without partiality and subsorting):
  one of σ_{ip} or σ_{ic} is an isomorphism (trivial cases)

Two separate problems

When building and using heterogeneous logical environments — a number of institutions linked by institution (co)morphisms or similar maps — two problems arise:

- Can interpolation properties be preserved when moving from one institution to another?

  $\leadsto$ how can we "borrow" interpolation along institution (co)morphisms?

- Can interpolation properties be spoiled when moving from one institution to another?

  $\leadsto$ how can we "spoil" interpolation along institution (co)morphisms?

In this work: **we address the latter!**

Simple institution extensions

Let $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$

Simple institution extensions

Let $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$

- *Extending* $\mathbf{INS}$ *by a new "abstract"* Σ*-model* M *with* $Th(M) \subseteq \mathbf{Sen}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$, results in $\mathbf{INS}^+ = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}^+, \langle \models^+_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}|} \rangle$:
  - $\mathbf{Mod}^+(\Sigma') = \mathbf{Mod}(\Sigma') \cup \{ \lceil M|_\tau \rceil \mid \tau\colon \Sigma' \to \Sigma \}$ — M added as $\lceil M|_{id} \rceil$
  - $\lceil M|_\tau \rceil \models^+_{\Sigma'} \varphi'$ iff $\tau(\varphi') \in Th(M)$, for $\tau\colon \Sigma' \to \Sigma$, $\varphi' \in \mathbf{Sen}(\Sigma')$

Simple institution extensions

Let $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$

- *Extending* $\mathbf{INS}$ *by a new "abstract"* Σ*-model* M *with* $Th(M) \subseteq \mathbf{Sen}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$, results in $\mathbf{INS}^+ = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}^+, \langle \models^+_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}|} \rangle$:
  - $\mathbf{Mod}^+(\Sigma') = \mathbf{Mod}(\Sigma') \cup \{ \lceil M|_\tau \rceil \mid \tau \colon \Sigma' \to \Sigma \}$ — M added as $\lceil M|_{id} \rceil$
  - $\lceil M|_\tau \rceil \models^+_{\Sigma'} \varphi'$ iff $\tau(\varphi') \in Th(M)$, for $\tau \colon \Sigma' \to \Sigma$, $\varphi' \in \mathbf{Sen}(\Sigma')$

- *Extending* $\mathbf{INS}$ *by a new "abstract"* Σ*-sentence* φ *with* $Mod(\varphi) \subseteq \mathbf{Mod}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$, results in $\mathbf{INS}^+ = \langle \mathbf{Sign}, \mathbf{Sen}^+, \mathbf{Mod}, \langle \models^+_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}|} \rangle$:
  - $\mathbf{Sen}^+(\Sigma') = \mathbf{Sen}(\Sigma') \cup \{ \lceil \tau(\varphi) \rceil \mid \tau \colon \Sigma \to \Sigma' \}$ — φ added as $\lceil id(\varphi) \rceil$
  - $M' \models^+_{\Sigma'} \lceil \tau(\varphi) \rceil$ iff $M'|_\tau \in Mod(\varphi)$, for $\tau \colon \Sigma \to \Sigma'$, $M' \in \mathbf{Mod}(\Sigma')$

Simple institution extensions

Let $\mathbf{INS} = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$

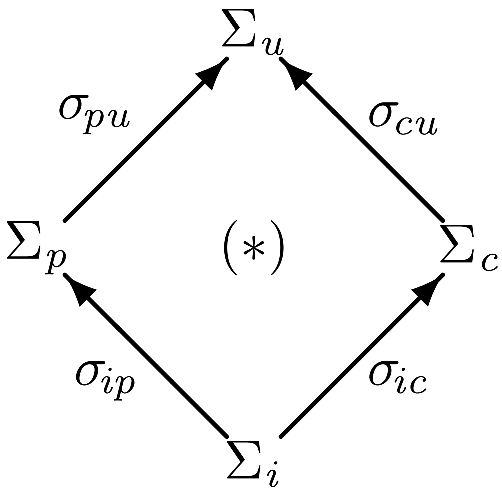
- *Extending* $\mathbf{INS}$ *by a new "abstract"* Σ*-model* M *with* $Th(M) \subseteq \mathbf{Sen}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$, results in $\mathbf{INS}^+ = \langle \mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}^+, \langle \models^+_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}|} \rangle$:
  - $\mathbf{Mod}^+(\Sigma') = \mathbf{Mod}(\Sigma') \cup \{\lceil M|_\tau \rceil \mid \tau \colon \Sigma' \to \Sigma\}$ — M added as $\lceil M|_{id} \rceil$
  - $\lceil M|_\tau \rceil \models^+_{\Sigma'} \varphi'$ iff $\tau(\varphi') \in Th(M)$, for $\tau \colon \Sigma' \to \Sigma$, $\varphi' \in \mathbf{Sen}(\Sigma')$

- *Extending* $\mathbf{INS}$ *by a new "abstract"* Σ*-sentence* φ *with* $Mod(\varphi) \subseteq \mathbf{Mod}(\Sigma)$, $\Sigma \in |\mathbf{Sign}|$, results in $\mathbf{INS}^+ = \langle \mathbf{Sign}, \mathbf{Sen}^+, \mathbf{Mod}, \langle \models^+_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}|} \rangle$:
  - $\mathbf{Sen}^+(\Sigma') = \mathbf{Sen}(\Sigma') \cup \{\lceil \tau(\varphi) \rceil \mid \tau \colon \Sigma \to \Sigma'\}$ — φ added as $\lceil id(\varphi) \rceil$
  - $M' \models^+_{\Sigma'} \lceil \tau(\varphi) \rceil$ iff $M'|_\tau \in Mod(\varphi)$, for $\tau \colon \Sigma \to \Sigma'$, $M' \in \mathbf{Mod}(\Sigma')$

Similarly for multiple models and sentences, respectively

Spoiling an interpolant by new models – easy?

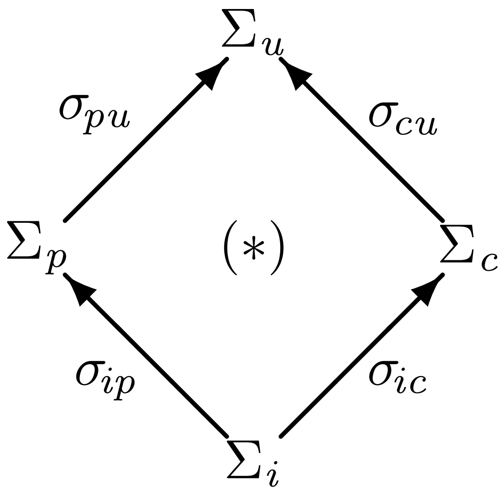
Consider an interpolant $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ for $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.



Spoiling an interpolant by new models – easy?

Consider an interpolant $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ for $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$. Apparently: any interpolant should be always easy to spoil:

- add a new Σ_p-model M such that $\Phi \subseteq Th(M)$ but $\sigma_{ip}(\Theta) \not\subseteq Th(M)$, then $\Phi \not\models \sigma_{ip}(\Theta)$; or
- add a new Σ_c-model N such that $\Psi \not\subseteq Th(N)$ but $\sigma_{ic}(\Theta) \subseteq Th(N)$, then $\sigma_{ic}(\Theta) \not\models \Psi$.

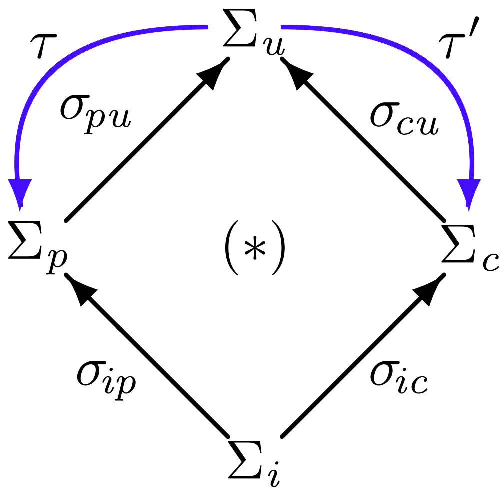


Spoiling an interpolant by new models – easy?

Consider an interpolant $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ for $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$. Apparently: any interpolant should be always easy to spoil:

- add a new Σ_p-model M such that $\Phi \subseteq Th(M)$ but $\sigma_{ip}(\Theta) \not\subseteq Th(M)$, then $\Phi \not\models \sigma_{ip}(\Theta)$; or
- add a new Σ_c-model N such that $\Psi \not\subseteq Th(N)$ but $\sigma_{ic}(\Theta) \subseteq Th(N)$, then $\sigma_{ic}(\Theta) \not\models \Psi$.

BUT:

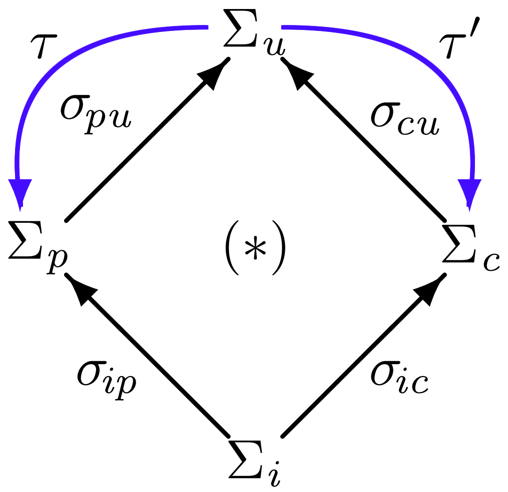


Spoiling an interpolant by new models – easy?

Consider an interpolant $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ for $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$. Apparently: any interpolant should be always easy to spoil:

- add a new Σ_p-model M such that $\Phi \subseteq Th(M)$ but $\sigma_{ip}(\Theta) \not\subseteq Th(M)$, then $\Phi \not\models \sigma_{ip}(\Theta)$; or
- add a new Σ_c-model N such that $\Psi \not\subseteq Th(N)$ but $\sigma_{ic}(\Theta) \subseteq Th(N)$, then $\sigma_{ic}(\Theta) \not\models \Psi$.

BUT:



- $\lceil M|_\tau \rceil \in \mathbf{Mod}^+(\Sigma_u)$ for $\tau \colon \Sigma_u \to \Sigma_p$
- $\lceil N|_{\tau'} \rceil \in \mathbf{Mod}^+(\Sigma_u)$ for $\tau' \colon \Sigma_u \to \Sigma_c$

may spoil $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$ ...

Spoiling an interpolant by new models

Fact: *An interpolant* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$*,* $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$*, may be spoiled by extending* $\mathbf{INS}$ *by new models if*

- *there is* $\Phi^\bullet \subseteq \mathbf{Sen}(\Sigma_p)$ *such that:*
  - $\Phi \subseteq \Phi^\bullet$*,* $\sigma_{ip}(\Theta) \not\subseteq \Phi^\bullet$ *and*
  - *for all* $\tau \colon \Sigma_u \to \Sigma_p$*, if* $\tau(\sigma_{pu}(\Phi)) \subseteq \Phi^\bullet$ *then* $\tau(\sigma_{cu}(\Psi)) \subseteq \Phi^\bullet$

or

- *there is* $\Psi^\circ \subseteq \mathbf{Sen}(\Sigma_c)$ *such that:*
  - $\sigma_{ic}(\Theta) \subseteq \Psi^\circ$*,* $\Psi \not\subseteq \Psi^\circ$ *and*
  - *for all* $\tau' \colon \Sigma_u \to \Sigma_c$*, if* $\tau'(\sigma_{pu}(\Phi)) \subseteq \Psi^\circ$ *then* $\tau'(\sigma_{cu}(\Psi)) \subseteq \Psi^\circ$

Syntactic separation

- $\Phi^\bullet \subseteq \mathbf{Sen}(\Sigma)$ *never separates* $\Phi' \subseteq \mathbf{Sen}(\Sigma')$ *from* $\Psi' \subseteq \mathbf{Sen}(\Sigma')$ when for all $\tau\colon \Sigma' \to \Sigma$, if $\tau(\Phi') \subseteq \Phi^\bullet$ then $\tau(\Psi') \subseteq \Phi^\bullet$.
- for $\Phi \subseteq \mathbf{Sen}(\Sigma)$ and $\Phi', \Psi' \subseteq \mathbf{Sen}(\Sigma')$, let

$$[\Phi' \overset{\Sigma'}{\underset{\Sigma}{\leadsto}} \Psi'](\Phi)$$

be the least set of Σ-sentences that contains Φ and never separates Φ' from Ψ'.

Spoiling an interpolant by new models

Syntactic separation

- $\Phi^\bullet \subseteq \mathbf{Sen}(\Sigma)$ *never separates* $\Phi' \subseteq \mathbf{Sen}(\Sigma')$ *from* $\Psi' \subseteq \mathbf{Sen}(\Sigma')$ when for all $\tau \colon \Sigma' \to \Sigma$, if $\tau(\Phi') \subseteq \Phi^\bullet$ then $\tau(\Psi') \subseteq \Phi^\bullet$.
- for $\Phi \subseteq \mathbf{Sen}(\Sigma)$ and $\Phi', \Psi' \subseteq \mathbf{Sen}(\Sigma')$, let

$$[\Phi' \overset{\Sigma'}{\underset{\Sigma}{\leadsto}} \Psi'](\Phi)$$

be the least set of Σ-sentences that contains Φ and never separates Φ' from Ψ'.

Fact: *An interpolant* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$, *may be spoiled by extending* $\mathbf{INS}$ *by new models* **if**

- $\sigma_{ip}(\Theta) \not\subseteq [\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\leadsto}} \sigma_{cu}(\Psi)](\Phi)$ **or**
- $\Psi \not\subseteq [\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_c}{\leadsto}} \sigma_{cu}(\Psi)](\sigma_{ic}(\Theta))$

Spoiling an interpolant by new models

Syntactic separation

- $\Phi^\bullet \subseteq \mathbf{Sen}(\Sigma)$ *never separates* $\Phi' \subseteq \mathbf{Sen}(\Sigma')$ *from* $\Psi' \subseteq \mathbf{Sen}(\Sigma')$ when for all $\tau \colon \Sigma' \to \Sigma$, if $\tau(\Phi') \subseteq \Phi^\bullet$ then $\tau(\Psi') \subseteq \Phi^\bullet$.
- for $\Phi \subseteq \mathbf{Sen}(\Sigma)$ and $\Phi', \Psi' \subseteq \mathbf{Sen}(\Sigma')$, let

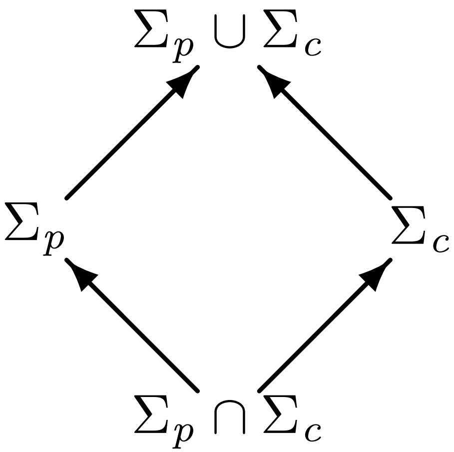
$$[\Phi' \underset{\Sigma}{\overset{\Sigma'}{\rightsquigarrow}} \Psi'](\Phi)$$

be the least set of Σ-sentences that contains Φ and never separates Φ' from Ψ'.

Fact: *An interpolant* $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$ *for* $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ *and* $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$, *may be spoiled by extending* $\mathbf{INS}$ *by new models* **iff**

- $\sigma_{ip}(\Theta) \not\subseteq [\sigma_{pu}(\Phi) \underset{\Sigma_p}{\overset{\Sigma_u}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\Phi)$ **or**
- $\Psi \not\subseteq [\sigma_{pu}(\Phi) \underset{\Sigma_c}{\overset{\Sigma_u}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\sigma_{ic}(\Theta))$

In propositional logic: examples

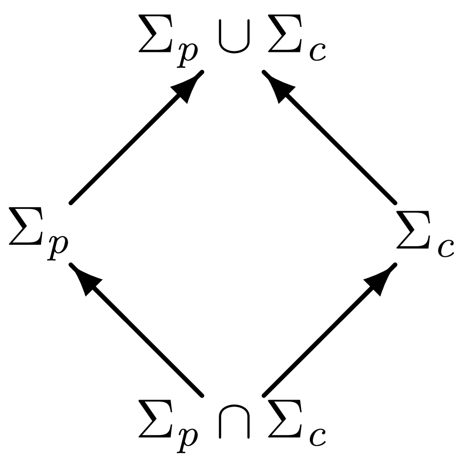


Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{r \wedge p}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{q \vee p}$

Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include:
p, $p \vee p$, $p \wedge p$, $(p \vee p) \wedge (p \vee \neg p)$, ...

In propositional logic: examples



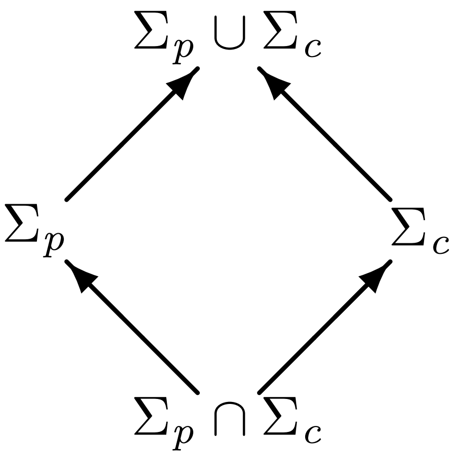
Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{r \wedge p}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{q \vee p}$

Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include: p, $p \vee p$, $p \wedge p$, $(p \vee p) \wedge (p \vee \neg p)$, ...

Fact: *No interpolant for φ and ψ is stable under extensions of $\mathbf{PL}$ by new models.*

In propositional logic: examples



Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{r \land p}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{q \lor p}$

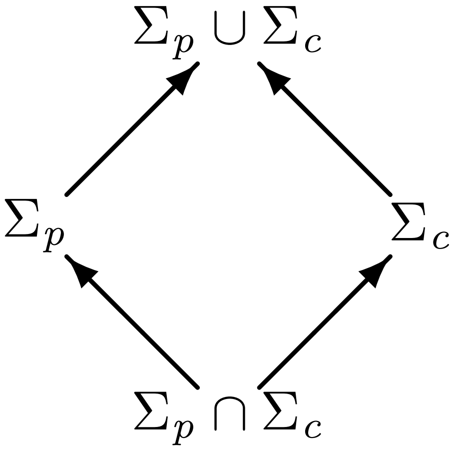
Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include:
p, $p \lor p$, $p \land p$, $(p \lor p) \land (p \lor \neg p)$, ...

Fact: *No interpolant for φ and ψ is stable under extensions of $\mathbf{PL}$ by new models.*

This follows since:

- $[r \land p \overset{\Sigma_p \cup \Sigma_c}{\underset{\Sigma_p}{\leadsto}} q \lor p](r \land p) = \{r \land p, r \lor p, p \lor p\}$, and
- $[r \land p \overset{\Sigma_p \cup \Sigma_c}{\underset{\Sigma_c}{\leadsto}} q \lor p](p \lor p) = \{p \lor p\}$

Examples in propositional logic

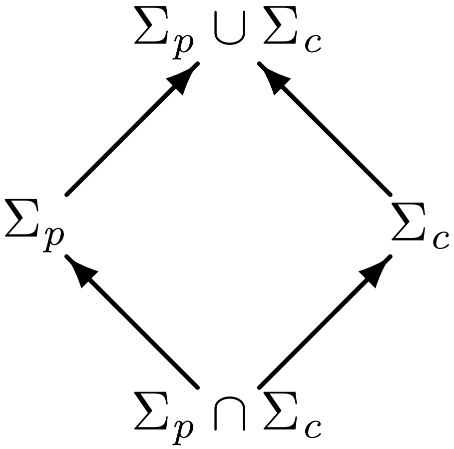


Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{(p \vee r) \wedge (p \vee \neg r)}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{(p \vee q) \wedge (p \vee \neg q)}$

Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include:
p, $p \vee p$, $p \wedge p$, $(p \vee p) \wedge (p \vee \neg p)$, $\ldots$

Examples in propositional logic



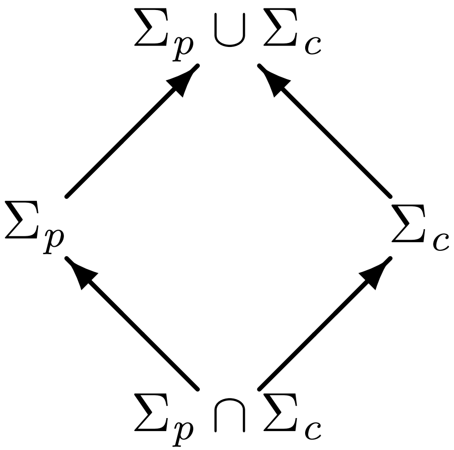
Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{(p \vee r) \wedge (p \vee \neg r)}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{(p \vee q) \wedge (p \vee \neg q)}$

Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include: p, $p \vee p$, $p \wedge p$, $(p \vee p) \wedge (p \vee \neg p)$, $\ldots$

Fact: *The interpolant* $(p \vee p) \wedge (p \vee \neg p)$ *is stable under extensions of* $\mathbf{PL}$ *by new models.*

Examples in propositional logic



Put:

- $\Sigma_p = \{p, r\}$, $\varphi = \boxed{(p \vee r) \wedge (p \vee \neg r)}$
- $\Sigma_c = \{p, q\}$, $\psi = \boxed{(p \vee q) \wedge (p \vee \neg q)}$

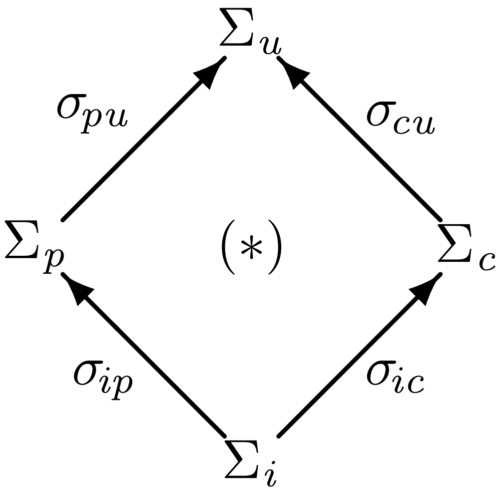
Clearly, $\varphi \models \psi$. Interpolants for φ and ψ include:
p, $p \vee p$, $p \wedge p$, $(p \vee p) \wedge (p \vee \neg p)$, $\ldots$

Fact: *The interpolant* $(p \vee p) \wedge (p \vee \neg p)$ *is stable under extensions of* $\mathbf{PL}$ *by new models.*

This follows since:

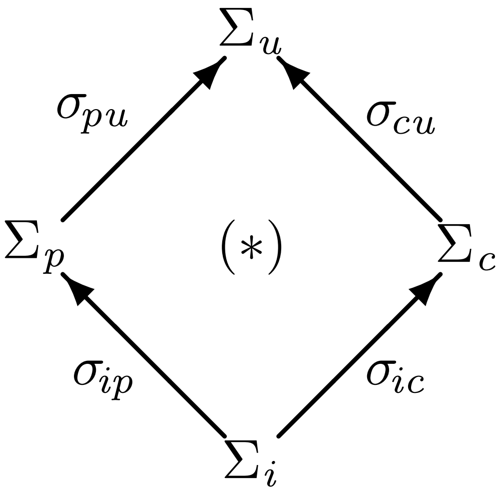
- $(p \vee p) \wedge (p \vee \neg p) \in [\varphi \overset{\Sigma_p \cup \Sigma_c}{\underset{\Sigma_p}{\rightsquigarrow}} \psi]((p \vee r) \wedge (p \vee \neg r))$, and
- $(p \vee q) \wedge (p \vee \neg q) \in [\varphi \overset{\Sigma_p \cup \Sigma_c}{\underset{\Sigma_c}{\rightsquigarrow}} \psi]((p \vee p) \wedge (p \vee \neg p))$

Spoiling interpolation by new models



Consider $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.

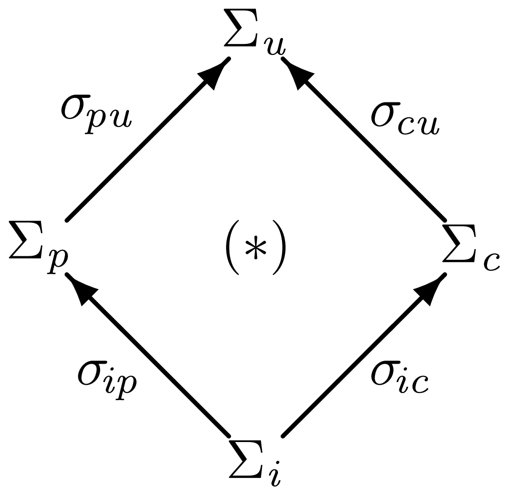
Spoiling interpolation by new models



Consider $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.

Can all interpolants for Φ and Ψ be spoiled by new models?

Spoiling interpolation by new models

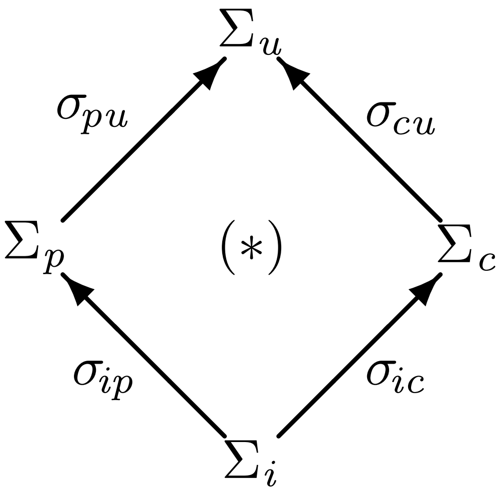


Consider $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.

Can all interpolants for Φ and Ψ be spoiled by new models?

Fact: Φ *and* Ψ *have no interpolant in some extension of* $\mathbf{INS}$ *by new models if* $\Psi \not\subseteq \sigma_{ic}(\sigma_{ip}^{-1}([\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\leadsto}} \sigma_{cu}(\Psi)](\Phi)))$.

Spoiling interpolation by new models



Consider $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.

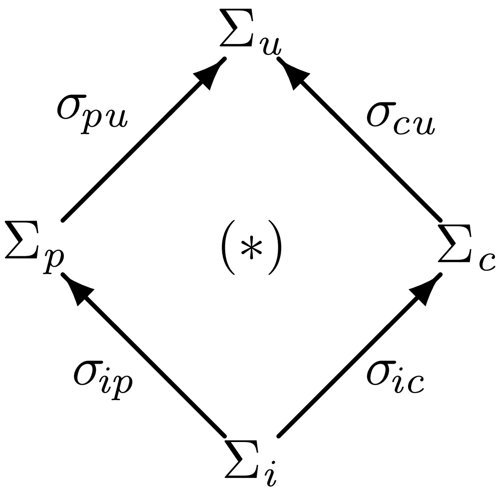
Can all interpolants for Φ and Ψ be spoiled by new models?

Fact: Φ *and* Ψ *have no interpolant in some extension of* $\mathbf{INS}$ *by new models if* $\Psi \not\subseteq \sigma_{ic}(\sigma_{ip}^{-1}([\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\Phi)))$.

Define:

$$\Theta^* = \sigma_{ip}^{-1}\left([\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\Phi) \cap Th(\Phi)\right) \subseteq \mathbf{Sen}(\Sigma_i)$$

Spoiling interpolation by new models



Consider $\Phi \subseteq \mathbf{Sen}(\Sigma_p)$ and $\Psi \subseteq \mathbf{Sen}(\Sigma_c)$, $\sigma_{pu}(\Phi) \models \sigma_{cu}(\Psi)$.

Can all interpolants for Φ and Ψ be spoiled by new models?

Fact: Φ *and* Ψ *have no interpolant in some extension of* $\mathbf{INS}$ *by new models if* $\Psi \not\subseteq \sigma_{ic}(\sigma_{ip}^{-1}([\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\Phi)))$.

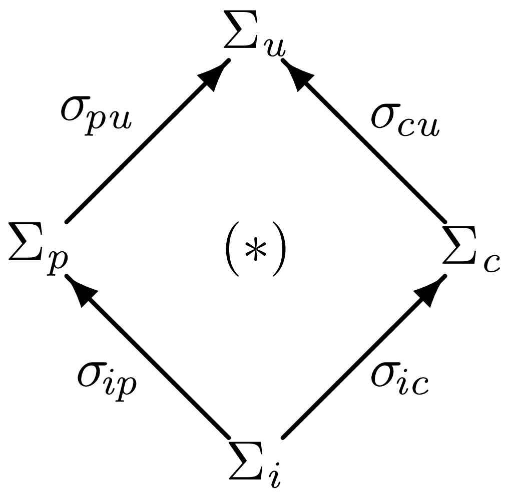
Define:

$$\Theta^* = \sigma_{ip}^{-1}\left([\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_p}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\Phi) \cap Th(\Phi) \right) \subseteq \mathbf{Sen}(\Sigma_i)$$

Fact: Φ *and* Ψ *have an interpolant in every extension of* $\mathbf{INS}$ *by new models* **iff**

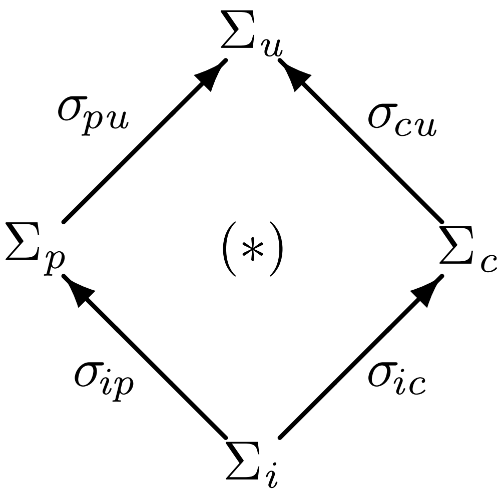
$$\Psi \subseteq [\sigma_{pu}(\Phi) \overset{\Sigma_u}{\underset{\Sigma_c}{\rightsquigarrow}} \sigma_{cu}(\Psi)](\sigma_{ic}(\Theta^*)) \text{ and } \sigma_{ic}(\Theta^*) \models \Psi$$

Spoiling interpolation by new sentences



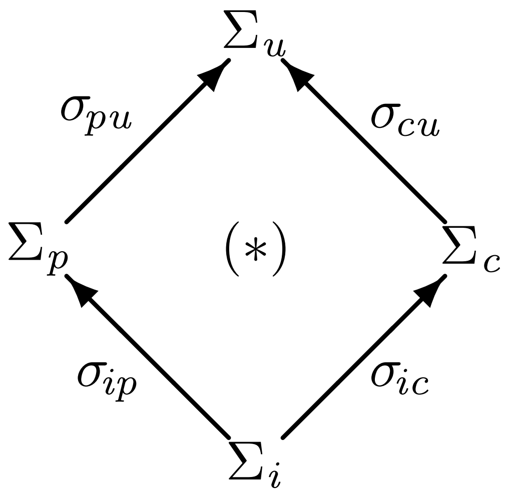
Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff**

Spoiling interpolation by new sentences



Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff** *for all classes* $\mathcal{M} \subseteq \mathbf{Mod}(\Sigma_p)$ *and* $\mathcal{N} \subseteq \mathbf{Mod}(\Sigma_c)$ *such that* $\mathcal{M}|_{\sigma_{pu}}^{-1} \subseteq \mathcal{N}|_{\sigma_{cu}}^{-1}$

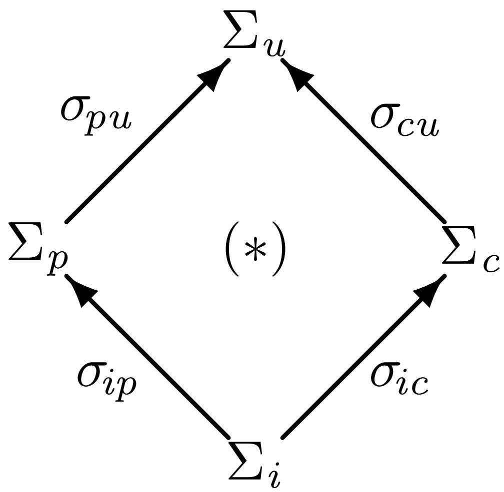
Spoiling interpolation by new sentences



Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff** *for all classes* $\mathcal{M} \subseteq \mathbf{Mod}(\Sigma_p)$ *and* $\mathcal{N} \subseteq \mathbf{Mod}(\Sigma_c)$ *such that* $\mathcal{M}|_{\sigma_{pu}}^{-1} \subseteq \mathcal{N}|_{\sigma_{cu}}^{-1}$ *there is a class* $\mathcal{K} \subseteq \mathbf{Mod}(\Sigma_i)$ *such that* $\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K}$ *and* $\mathcal{K}|_{\sigma_{ic}}^{-1} \subseteq \mathcal{N}$

that is definable in $\mathbf{INS}$

Spoiling interpolation by new sentences

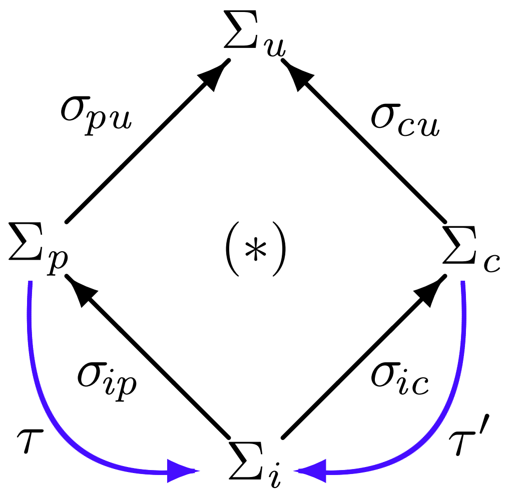


Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff** *for all classes* $\mathcal{M} \subseteq \mathbf{Mod}(\Sigma_p)$ *and* $\mathcal{N} \subseteq \mathbf{Mod}(\Sigma_c)$ *such that* $\mathcal{M}|_{\sigma_{pu}}^{-1} \subseteq \mathcal{N}|_{\sigma_{cu}}^{-1}$ *there is a class* $\mathcal{K} \subseteq \mathbf{Mod}(\Sigma_i)$ *such that* $\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K}$ *and* $\mathcal{K}|_{\sigma_{ic}}^{-1} \subseteq \mathcal{N}$, *i.e.*

$$\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K} \subseteq (\mathbf{Mod}(\Sigma_i) \setminus (\mathbf{Mod}(\Sigma_c) \setminus \mathcal{N})|_{\sigma_{ic}})$$

that is definable in $\mathbf{INS}$

Spoiling interpolation by new sentences

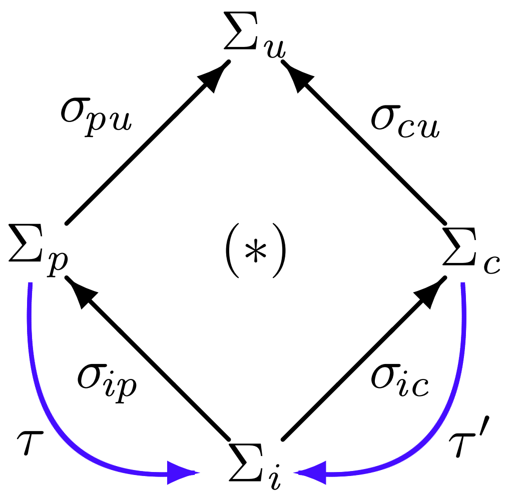


Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff** *for all classes* $\mathcal{M} \subseteq \mathbf{Mod}(\Sigma_p)$ *and* $\mathcal{N} \subseteq \mathbf{Mod}(\Sigma_c)$ *such that* $\mathcal{M}|^{-1}_{\sigma_{pu}} \subseteq \mathcal{N}|^{-1}_{\sigma_{cu}}$ *there is a class* $\mathcal{K} \subseteq \mathbf{Mod}(\Sigma_i)$ *such that* $\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K}$ *and* $\mathcal{K}|^{-1}_{\sigma_{ic}} \subseteq \mathcal{N}$, *i.e.*

$$\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K} \subseteq (\mathbf{Mod}(\Sigma_i) \setminus (\mathbf{Mod}(\Sigma_c) \setminus \mathcal{N})|_{\sigma_{ic}})$$

that is definable in $\mathbf{INS}$

Spoiling interpolation by new sentences



Fact: $(*)$ *admits interpolation in every extension of* $\mathbf{INS}$ *by new sentences* **iff** *for all classes* $\mathcal{M} \subseteq \mathbf{Mod}(\Sigma_p)$ *and* $\mathcal{N} \subseteq \mathbf{Mod}(\Sigma_c)$ *such that* $\mathcal{M}|^{-1}_{\sigma_{pu}} \subseteq \mathcal{N}|^{-1}_{\sigma_{cu}}$ *there is a class* $\mathcal{K} \subseteq \mathbf{Mod}(\Sigma_i)$ *such that* $\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K}$ *and* $\mathcal{K}|^{-1}_{\sigma_{ic}} \subseteq \mathcal{N}$*, i.e.*

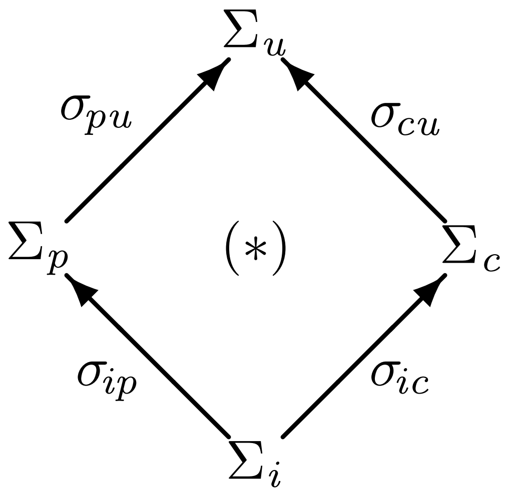
$$\mathcal{M}|_{\sigma_{ip}} \subseteq \mathcal{K} \subseteq (\mathbf{Mod}(\Sigma_i) \setminus (\mathbf{Mod}(\Sigma_c) \setminus \mathcal{N})|_{\sigma_{ic}})$$

that is definable in $\mathbf{INS}$ *from* $\{\langle \Sigma_p, \mathcal{M}\rangle, \langle \Sigma_c, \mathcal{N}\rangle\}$.

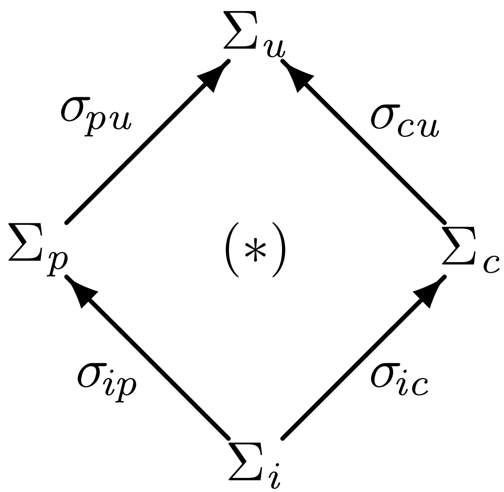
$\mathcal{K} \subseteq \mathbf{Mod}(\Sigma_i)$ *is definable in* $\mathbf{INS}$ *from* $\{\langle \Sigma_p, \mathcal{M}\rangle, \langle \Sigma_c, \mathcal{N}\rangle\}$ if there are $\Theta \subseteq \mathbf{Sen}(\Sigma_i)$, $\tau_j \colon \Sigma_p \to \Sigma_i$, $j \in \mathcal{J}_p$, and $\tau'_j \colon \Sigma_c \to \Sigma_i$, $j \in \mathcal{J}_c$ such that

$$\mathcal{K} = \bigcap_{j \in \mathcal{J}_p} \mathcal{M}|^{-1}_{\tau_j} \cap \bigcap_{j \in \mathcal{J}_c} \mathcal{N}|^{-1}_{\tau'_j} \cap Mod(\Theta)$$

Spoiling interpolation by new models and sentences



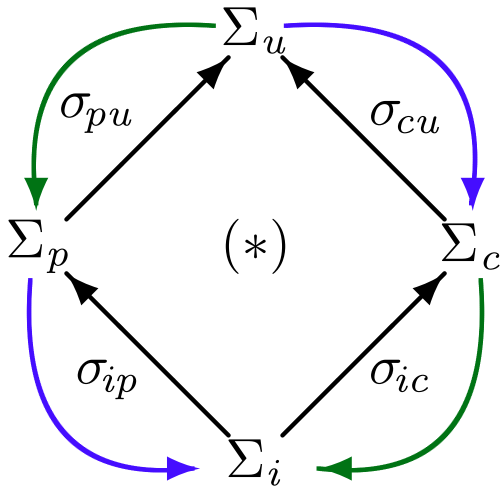
Spoiling interpolation by new models and sentences



Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *if*

- $\sigma_{ip}: \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_p)$ *is surjective and* $\sigma_{cu}: \Sigma_c \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{cu}}: \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_c)$ *is surjective*), *or*
- $\sigma_{ic}: \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_c)$ *is surjective and* $\sigma_{pu}: \Sigma_p \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{pu}}: \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_p)$ *is surjective*).

Spoiling interpolation by new models and sentences



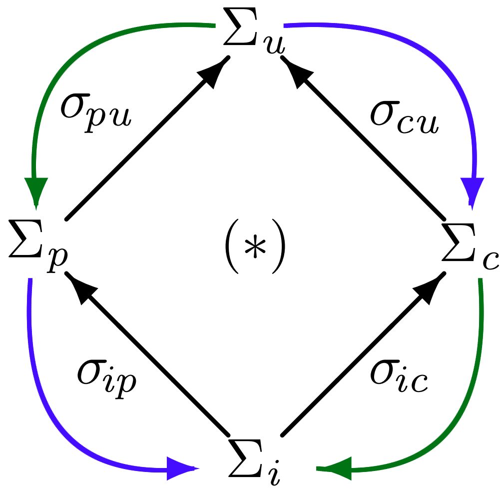
Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *if*

- $\sigma_{ip}\colon \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_p)$ *is surjective and* $\sigma_{cu}\colon \Sigma_c \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{cu}}\colon \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_c)$ *is surjective), or*
- $\sigma_{ic}\colon \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_c)$ *is surjective and* $\sigma_{pu}\colon \Sigma_p \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{pu}}\colon \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_p)$ *is surjective).*

Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$
if

- $\sigma_{ip}\colon \Sigma_i \to \Sigma_p$ *is a retraction and* $\sigma_{cu}\colon \Sigma_c \to \Sigma_u$ *is a coretraction, or*
- $\sigma_{ic}\colon \Sigma_i \to \Sigma_c$ *is a retraction and* $\sigma_{pu}\colon \Sigma_p \to \Sigma_u$ *is a coretraction.*

Spoiling interpolation by new models and sentences



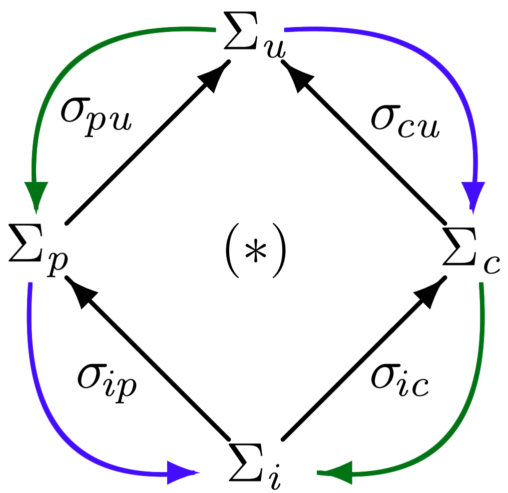
Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *if*

- $\sigma_{ip} : \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_p)$ *is surjective and* $\sigma_{cu} : \Sigma_c \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{cu}} : \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_c)$ *is surjective), or*
- $\sigma_{ic} : \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_c)$ *is surjective and* $\sigma_{pu} : \Sigma_p \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{pu}} : \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_p)$ *is surjective).*

Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *and in all its extensions by new models and sentences* **if**

- $\sigma_{ip} : \Sigma_i \to \Sigma_p$ *is a retraction and* $\sigma_{cu} : \Sigma_c \to \Sigma_u$ *is a coretraction, or*
- $\sigma_{ic} : \Sigma_i \to \Sigma_c$ *is a retraction and* $\sigma_{pu} : \Sigma_p \to \Sigma_u$ *is a coretraction.*

Spoiling interpolation by new models and sentences



Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *if*

- $\sigma_{ip} : \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_p)$ *is surjective and* $\sigma_{cu} : \Sigma_c \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{cu}} : \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_c)$ *is surjective), or*
- $\sigma_{ic} : \mathbf{Sen}(\Sigma_i) \to \mathbf{Sen}(\Sigma_c)$ *is surjective and* $\sigma_{pu} : \Sigma_p \to \Sigma_u$ *is conservative* ($\_|_{\sigma_{pu}} : \mathbf{Mod}(\Sigma_u) \to \mathbf{Mod}(\Sigma_p)$ *is surjective).*

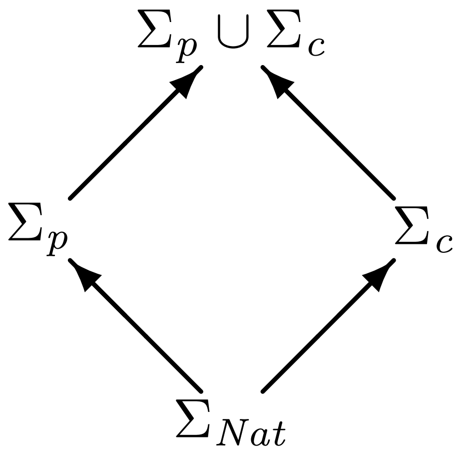
Fact: $(*)$ *admits interpolation in* $\mathbf{INS}$ *and in all its extensions by new models and sentences* **iff**

- $\sigma_{ip} : \Sigma_i \to \Sigma_p$ *is a retraction and* $\sigma_{cu} : \Sigma_c \to \Sigma_u$ *is a coretraction, or*
- $\sigma_{ic} : \Sigma_i \to \Sigma_c$ *is a retraction and* $\sigma_{pu} : \Sigma_p \to \Sigma_u$ *is a coretraction.*

Conclusion

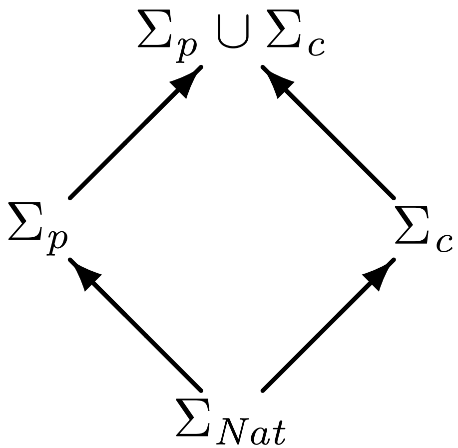
Interpolation is fragile – almost always!

Example in first-order logic



- $\Sigma_{Nat} =$ **sort** Nat **opns** $0 \colon Nat, s \colon Nat \to Nat$
- $\Sigma_p = \Sigma_{Nat}$ **then** $bop \colon Nat \times Nat \to Nat$
- $\Sigma_c = \Sigma_{Nat}$ **then** $\_+\_ \colon Nat \times Nat \to Nat$

Example in first-order logic



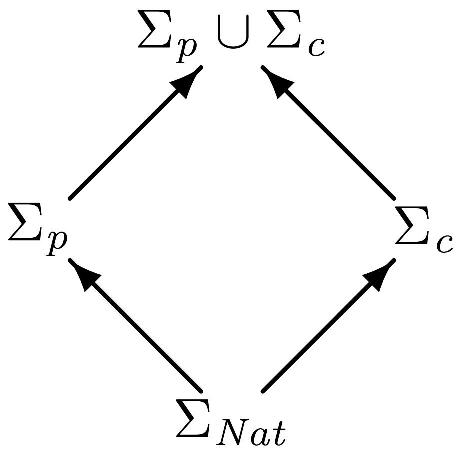
– $\Sigma_{Nat} =$ **sort** Nat **opns** $0\colon Nat, s\colon Nat \to Nat$
– $\Sigma_p = \Sigma_{Nat}$ **then** $bop\colon Nat \times Nat \to Nat$
• add a new Σ_p-sentence φ ("data constraint") with

$Mod(\varphi) = \mathcal{M} = \{A \in \mathbf{Mod}(\Sigma_p) \mid A|_{\Sigma_{Nat}} = \mathbb{N}\}$

– $\Sigma_c = \Sigma_{Nat}$ **then** $\_+\_\colon Nat \times Nat \to Nat$
• $\mathcal{N} = Mod(\psi)$, where

$$\psi \equiv (\forall x, y{:}Nat.\, x + 0 = x \wedge x + s(y) = s(x + y)) \Rightarrow \forall x, y{:}Nat.\, x + y = y + x$$

Example in first-order logic

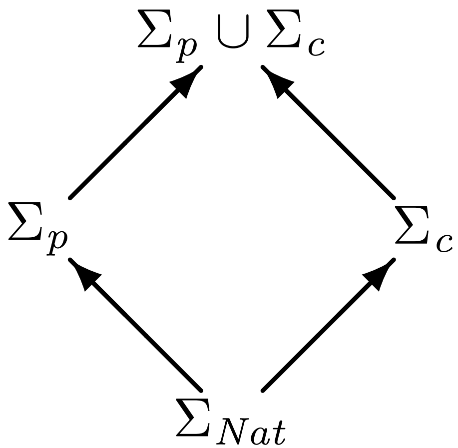


– $\Sigma_{Nat} =$ **sort** Nat **opns** $0\colon Nat, s\colon Nat \to Nat$
– $\Sigma_p = \Sigma_{Nat}$ **then** $bop\colon Nat \times Nat \to Nat$
- add a new Σ_p-sentence φ ("data constraint") with
  $Mod(\varphi) = \mathcal{M} = \{A \in \mathbf{Mod}(\Sigma_p) \mid A|_{\Sigma_{Nat}} = \mathbb{N}\}$

– $\Sigma_c = \Sigma_{Nat}$ **then** $\_+\_\colon Nat \times Nat \to Nat$
- $\mathcal{N} = Mod(\psi)$, where
  $\psi \equiv (\forall x, y{:}Nat.\, x + 0 = x \wedge x + s(y) = s(x + y)) \Rightarrow$
  $\forall x, y{:}Nat.\, x + y = y + x$

Clearly: $\varphi \models_{\Sigma_p \cup \Sigma_c} \psi$.

Example in first-order logic



- $\Sigma_{Nat} =$ **sort** Nat **opns** $0\colon Nat, s\colon Nat \to Nat$
- $\Sigma_p = \Sigma_{Nat}$ **then** $bop\colon Nat \times Nat \to Nat$
- add a new Σ_p-sentence φ ("data constraint") with

  $Mod(\varphi) = \mathcal{M} = \{A \in \mathbf{Mod}(\Sigma_p) \mid A|_{\Sigma_{Nat}} = \mathbb{N}\}$
- $\Sigma_c = \Sigma_{Nat}$ **then** $\_+\_\colon Nat \times Nat \to Nat$
- $\mathcal{N} = Mod(\psi)$, where

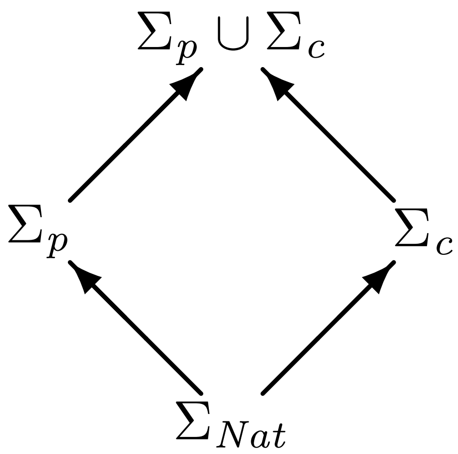
  $$\psi \equiv (\forall x, y{:}Nat.\, x + 0 = x \wedge x + s(y) = s(x + y)) \Rightarrow \forall x, y{:}Nat.\, x + y = y + x$$

Clearly: $\varphi \models_{\Sigma_p \cup \Sigma_c} \psi$.

But: there is no interpolant for φ and ψ!

(since there is no morphism from Σ_p to Σ_{Nat} and $Th(\mathbb{N}) \not\models \psi$)

Example in first-order logic



– $\Sigma_{Nat} =$ **sort** Nat **opns** $0\colon Nat, s\colon Nat \rightarrow Nat$

– $\Sigma_p =$ Σ_{Nat} **then** $uop\colon Nat \rightarrow Nat$

- add a new Σ_p-sentence φ ("data constraint") with

  $Mod(\varphi) = \mathcal{M} = \{A \in \mathbf{Mod}(\Sigma_p) \mid A|_{\Sigma_{Nat}} = \mathbb{N}\}$

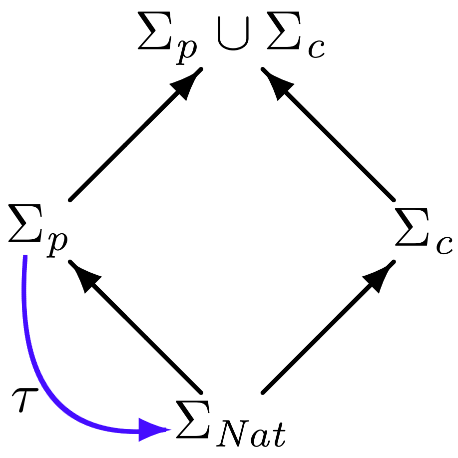
– $\Sigma_c =$ Σ_{Nat} **then** $\_+\_\colon Nat \times Nat \rightarrow Nat$

- $\mathcal{N} = Mod(\psi)$, where

  $$\psi \equiv (\forall x, y{:}Nat.\, x + 0 = x \wedge x + s(y) = s(x + y)) \Rightarrow \forall x, y{:}Nat.\, x + y = y + x$$

Clearly: $\varphi \models_{\Sigma_p \cup \Sigma_c} \psi$.

Example in first-order logic



– $\Sigma_{Nat} =$ **sort** Nat **opns** $0\colon Nat, s\colon Nat \to Nat$

– $\Sigma_p =$ Σ_{Nat} **then** $uop\colon Nat \to Nat$

- add a new Σ_p-sentence φ ("data constraint") with

  $Mod(\varphi) = \mathcal{M} = \{A \in \mathbf{Mod}(\Sigma_p) \mid A|_{\Sigma_{Nat}} = \mathbb{N}\}$

– $\Sigma_c =$ Σ_{Nat} **then** $\_+\_\colon Nat \times Nat \to Nat$

- $\mathcal{N} = Mod(\psi)$, where

  $$\psi \equiv (\forall x, y{:}Nat.\, x + 0 = x \wedge x + s(y) = s(x + y)) \Rightarrow \forall x, y{:}Nat.\, x + y = y + x$$

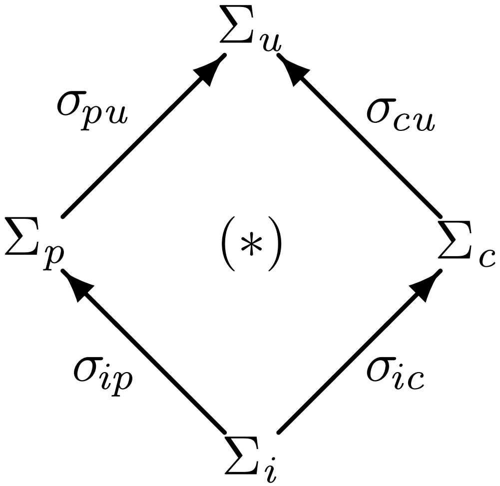
Clearly: $\varphi \models_{\Sigma_p \cup \Sigma_c} \psi$.

Now: we have $\tau\colon \Sigma_p \to \Sigma_{Nat}$, and $\tau(\varphi)$ is an interpolant for φ and ψ!

Can we spoil interpolation in propositional logic?

Can we spoil interpolation in propositional logic?

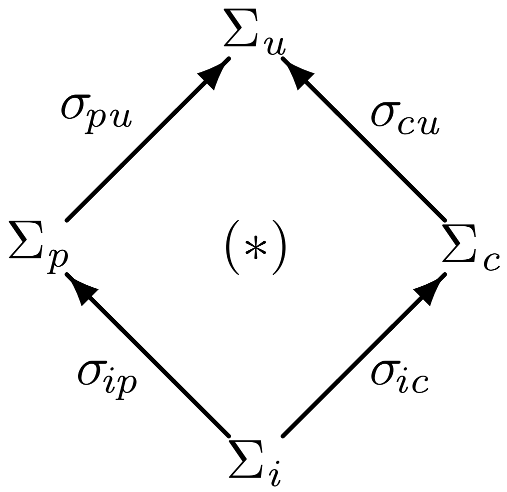
Amalgamation and interpolation



$(*)$ admits *weak amalgamation* when
for all $M \in \mathbf{Mod}(\Sigma_p)$, $N \in \mathbf{Mod}(\Sigma_c)$ with $M|_{\sigma_{ip}} = N|_{\sigma_{ic}}$
there is $K \in \mathbf{Mod}(\Sigma_u)$ such that $K|_{\sigma_{pu}} = M$ and $K|_{\sigma_{cu}} = N$.

Can we spoil interpolation in propositional logic?

Amalgamation and interpolation

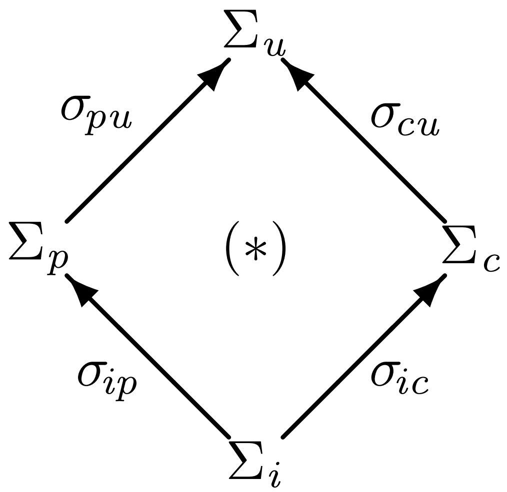


$(*)$ admits *weak amalgamation* when
for all $M \in \mathbf{Mod}(\Sigma_p)$, $N \in \mathbf{Mod}(\Sigma_c)$ with $M\big|_{\sigma_{ip}} = N\big|_{\sigma_{ic}}$
there is $K \in \mathbf{Mod}(\Sigma_u)$ such that $K\big|_{\sigma_{pu}} = M$ and $K\big|_{\sigma_{cu}} = N$.

- In **FO**, **EQ**, **PL**, and many other standard institutions: *all signature pushouts admit amalgamation*.

Can we spoil interpolation in propositional logic?

Amalgamation and interpolation



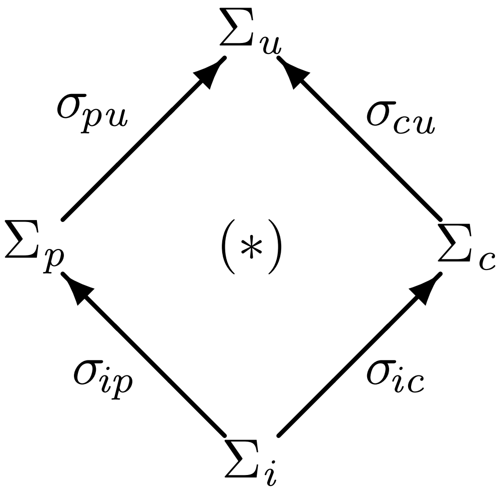
$(*)$ admits *weak amalgamation* when
for all $M \in \mathbf{Mod}(\Sigma_p)$, $N \in \mathbf{Mod}(\Sigma_c)$ with $M|_{\sigma_{ip}} = N|_{\sigma_{ic}}$
there is $K \in \mathbf{Mod}(\Sigma_u)$ such that $K|_{\sigma_{pu}} = M$ and $K|_{\sigma_{cu}} = N$.

- In $\mathbf{FO}$, $\mathbf{EQ}$, $\mathbf{PL}$, and many other standard institutions: *all signature pushouts admit amalgamation*.

Fact: *If $(*)$ admits weak amalgamation and all classes of Σ_i-models are definable then $(*)$ admits interpolation (in $\mathbf{INS}$ and in every its extension by new sentences).*

Can we spoil interpolation in propositional logic?

Amalgamation and interpolation



$(*)$ admits *weak amalgamation* when
for all $M \in \mathbf{Mod}(\Sigma_p)$, $N \in \mathbf{Mod}(\Sigma_c)$ with $M|_{\sigma_{ip}} = N|_{\sigma_{ic}}$
there is $K \in \mathbf{Mod}(\Sigma_u)$ such that $K|_{\sigma_{pu}} = M$ and $K|_{\sigma_{cu}} = N$.

- In $\mathbf{FO}$, $\mathbf{EQ}$, $\mathbf{PL}$, and many other standard institutions: *all signature pushouts admit amalgamation*.

Fact: *If $(*)$ admits weak amalgamation and all classes of Σ_i-models are definable then $(*)$ admits interpolation (in $\mathbf{INS}$ and in every its extension by new sentences).*

Fact: *If $(*)$ does not admit weak amalgamation then $(*)$ does not admit interpolation in an extension of $\mathbf{INS}$ by new sentences, and in any further its extension by new sentences.*

Further work

J. Symbolic Logic'24

- Repeat similar characterisations for Craig-Robinson (or parameterised) interpolation:
  - concepts and techniques carry over, results can be adjusted easily.
- Apply the results in the context of special commutative squares of signature morphisms used in particular applications.