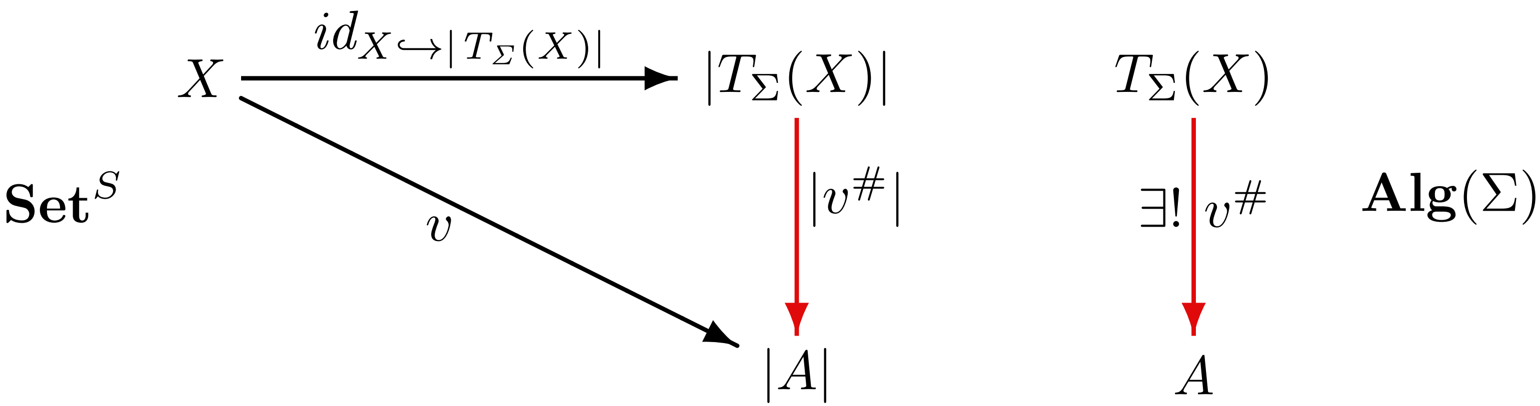
Adjunctions

Recall:

Term algebras

Theorem: *For any S-sorted set X of variables, Σ-algebra A and valuation $v\colon X \to |A|$, there is a unique Σ-homomorphism $v^{\#}\colon T_\Sigma(X) \to A$ that extends v, so that*

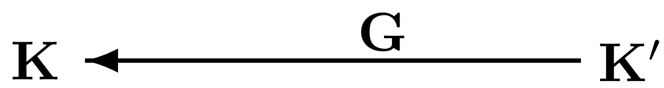
$$id_{X \hookrightarrow |T_\Sigma(X)|};v^{\#} = v$$



Free objects

Consider any functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$

Definition:



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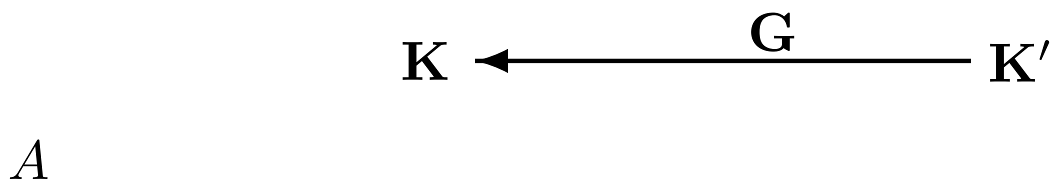

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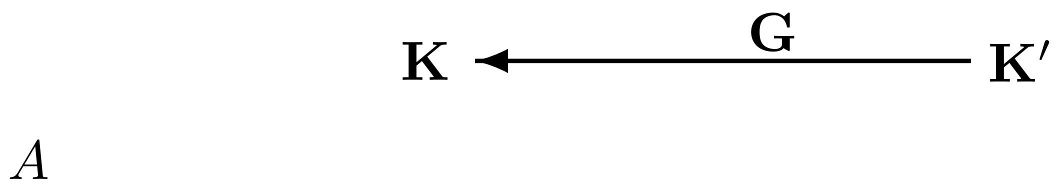

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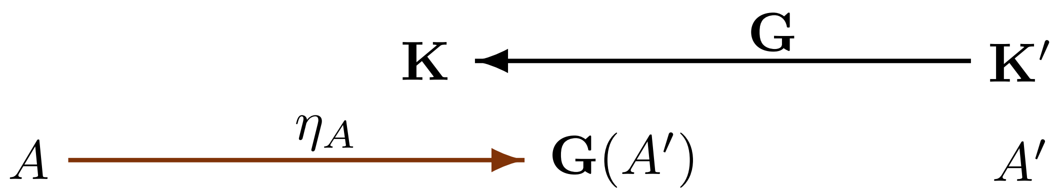

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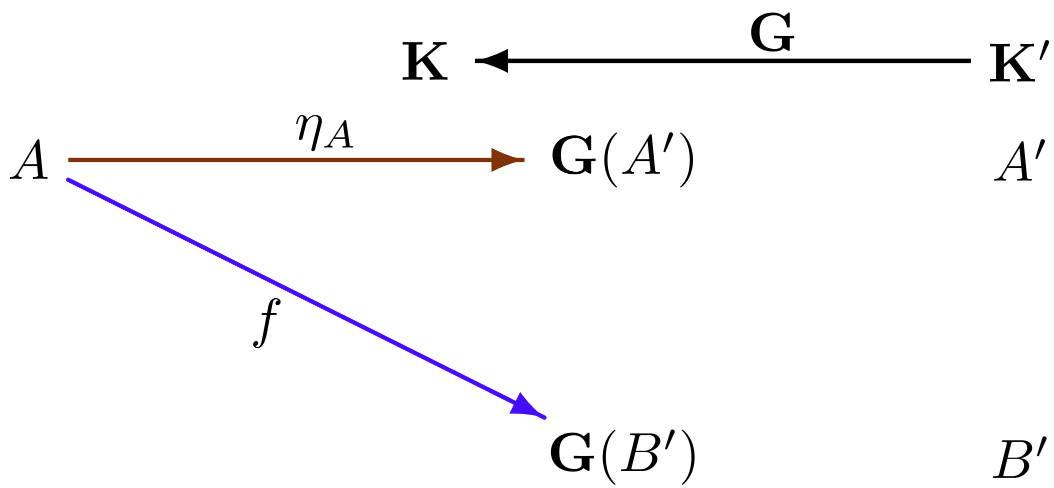

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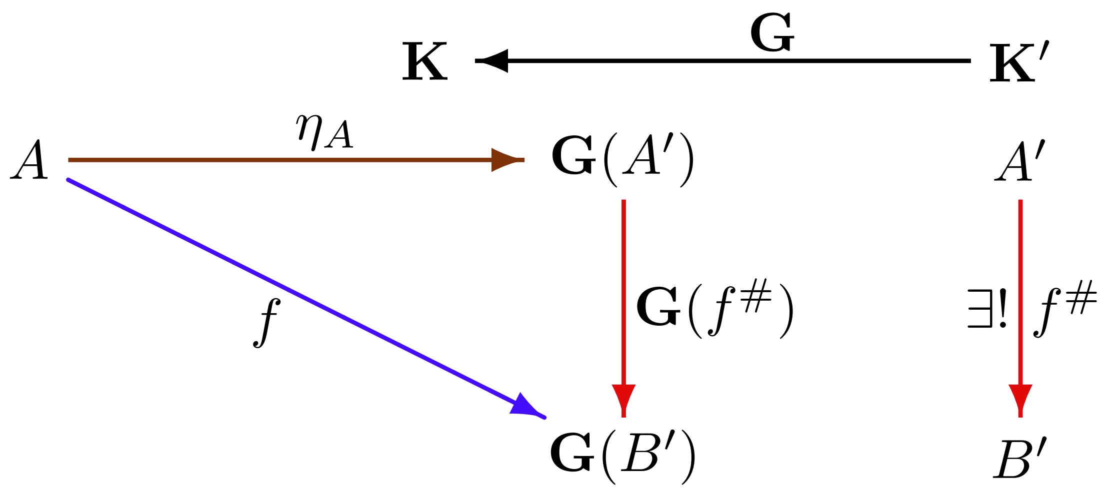
$$\boxed{\eta_A;\mathbf{G}(f^\#) = f}$$



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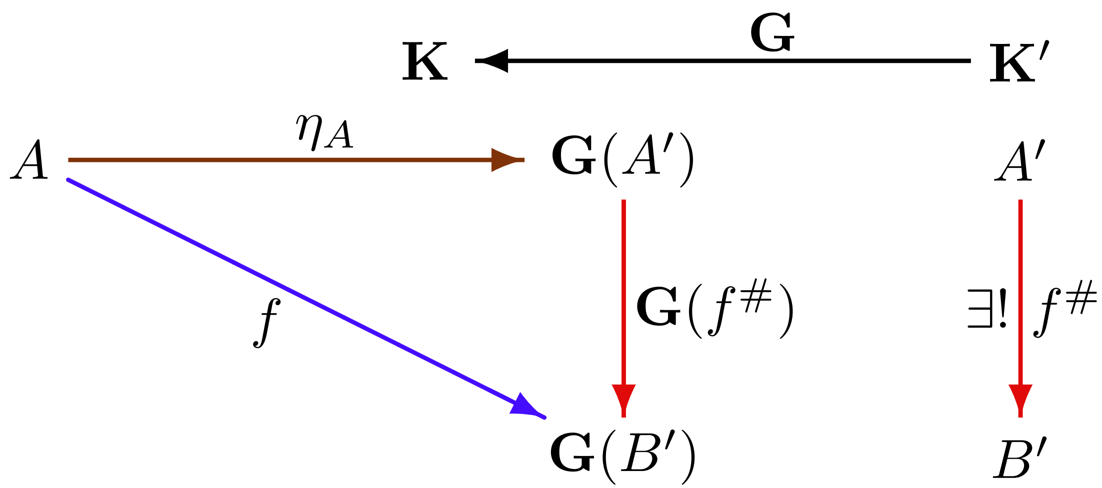
Paradigmatic example:
Term algebra $T_\Sigma(X)$ with unit $id_{X\hookrightarrow|T_\Sigma(X)|}\colon X \to |T_\Sigma(X)|$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$



Examples

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- Consider inclusion $i\colon \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them.

$$\mathbf{Real} \xleftarrow{\quad i \quad} \mathbf{Int}$$

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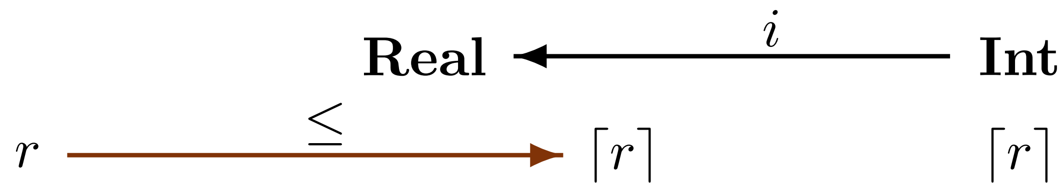

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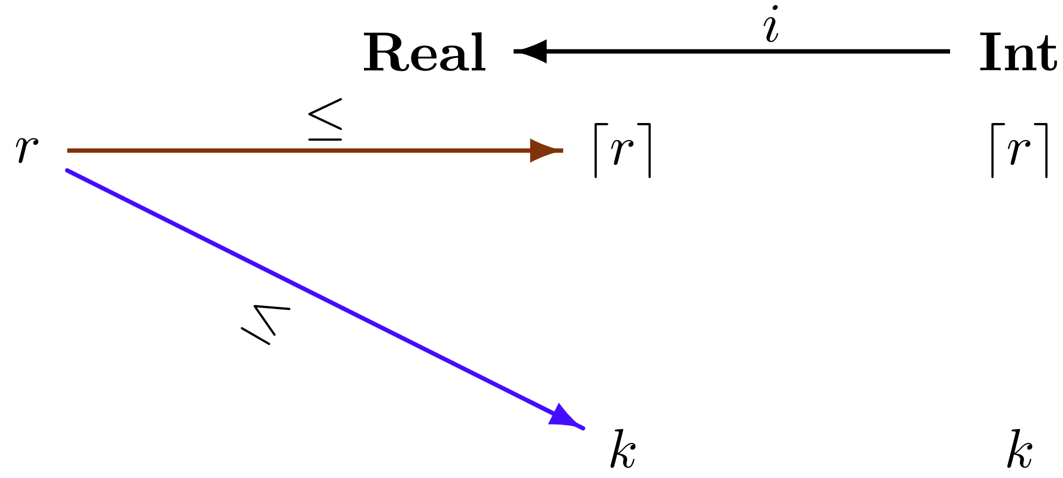

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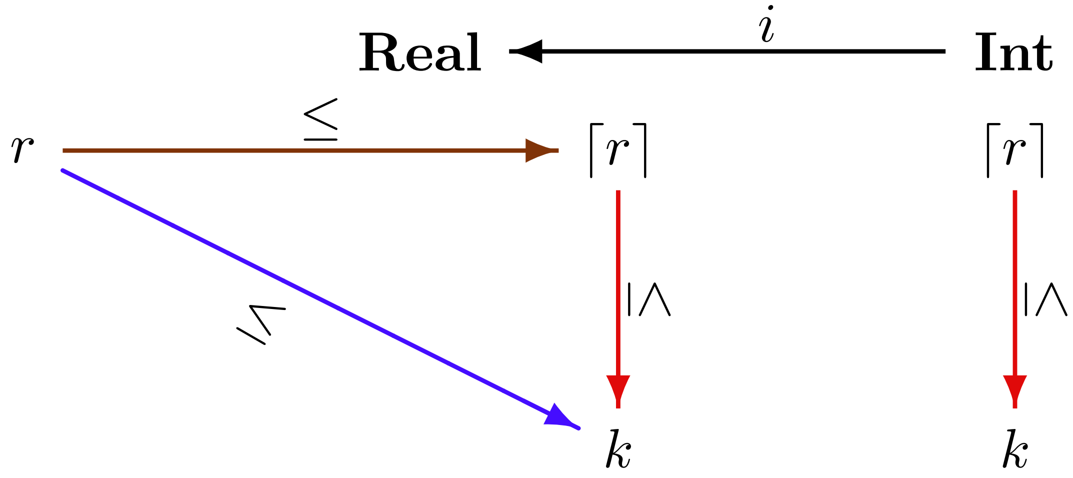

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$$\mathbf{Set} \xleftarrow{\;|\_|\;} \mathbf{Monoid}$$

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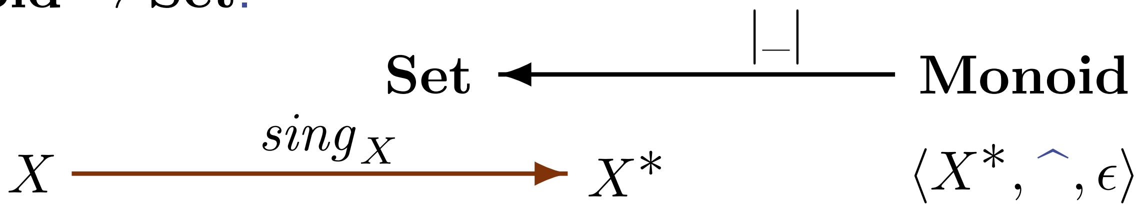

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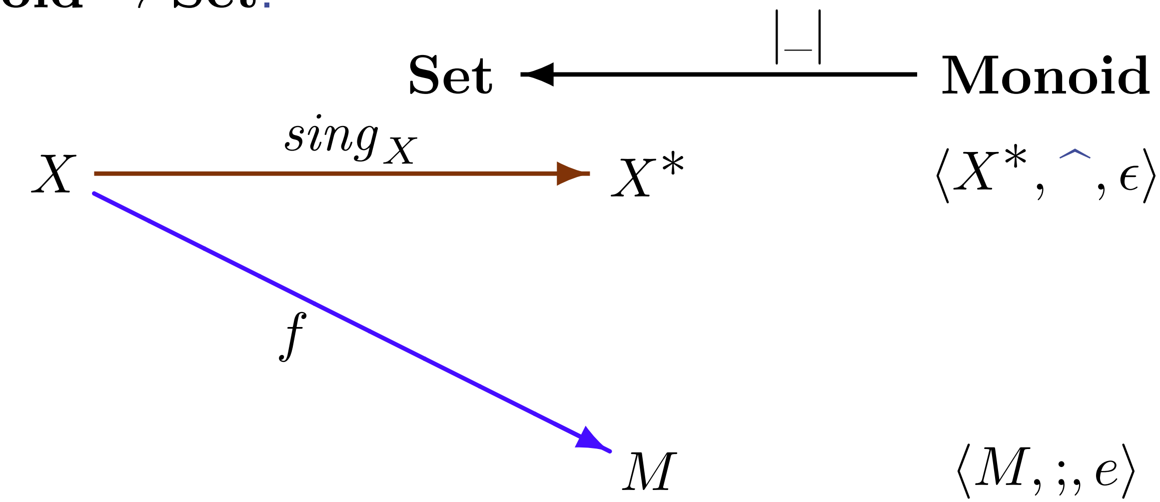

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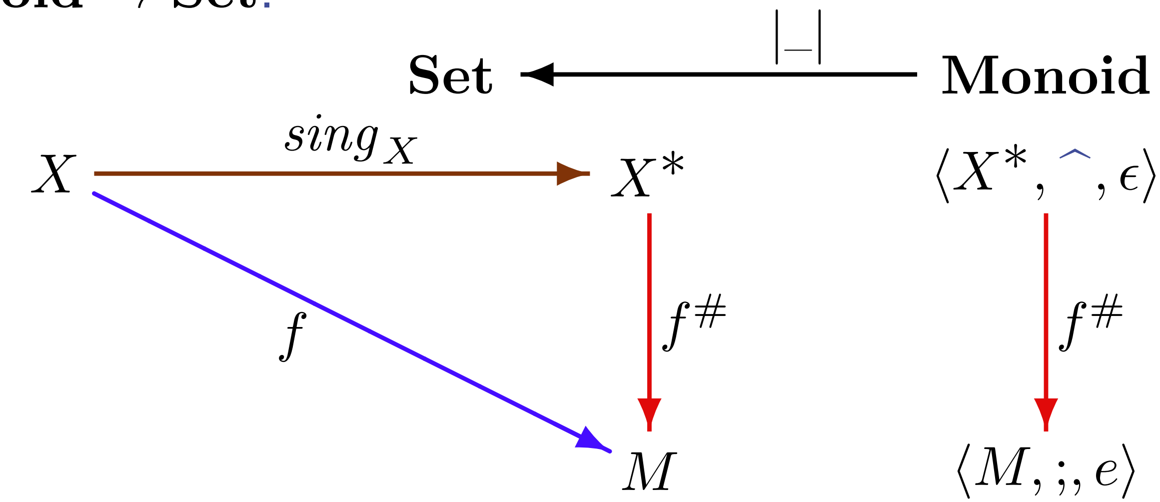

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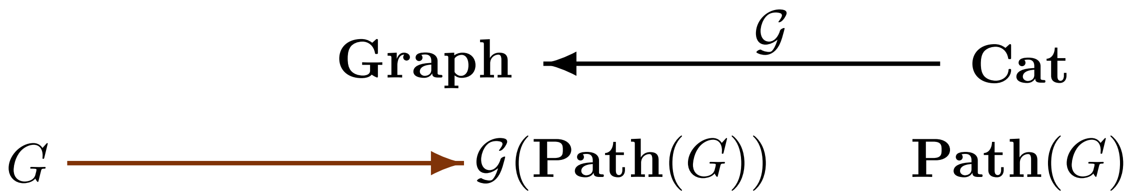

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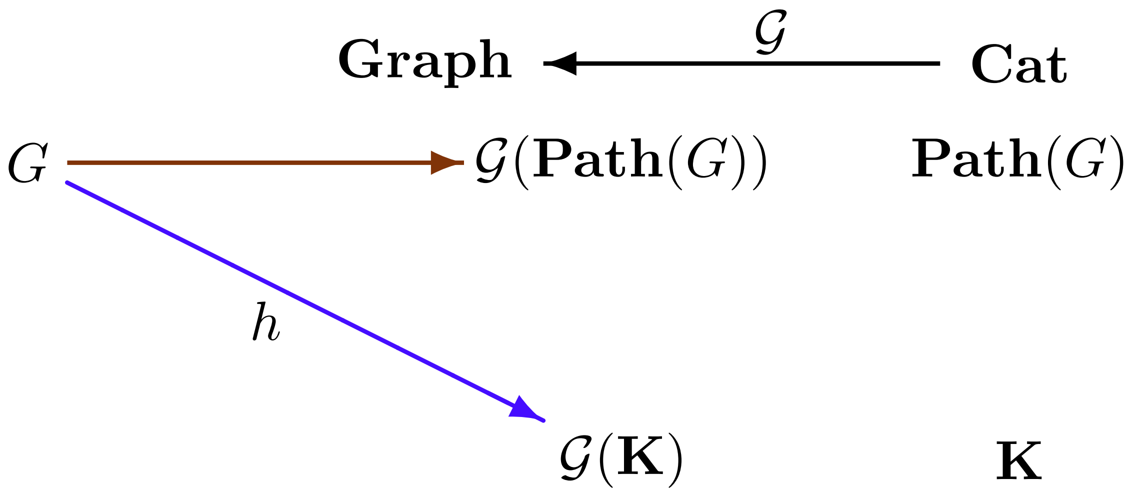

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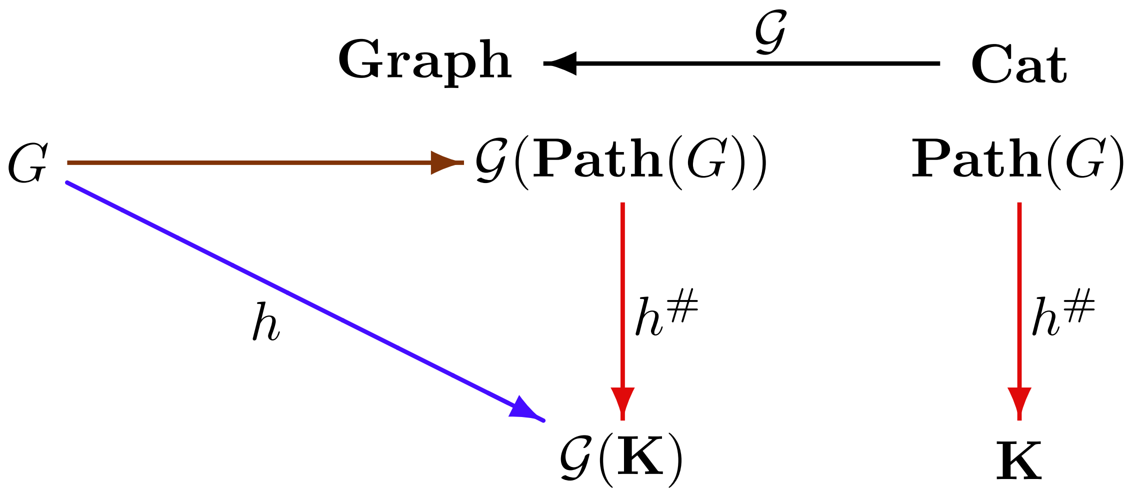

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Makes precise these and other similar examples
Indicate unit morphisms!

Free equational models

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- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $\mathbf{T}_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|: \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.

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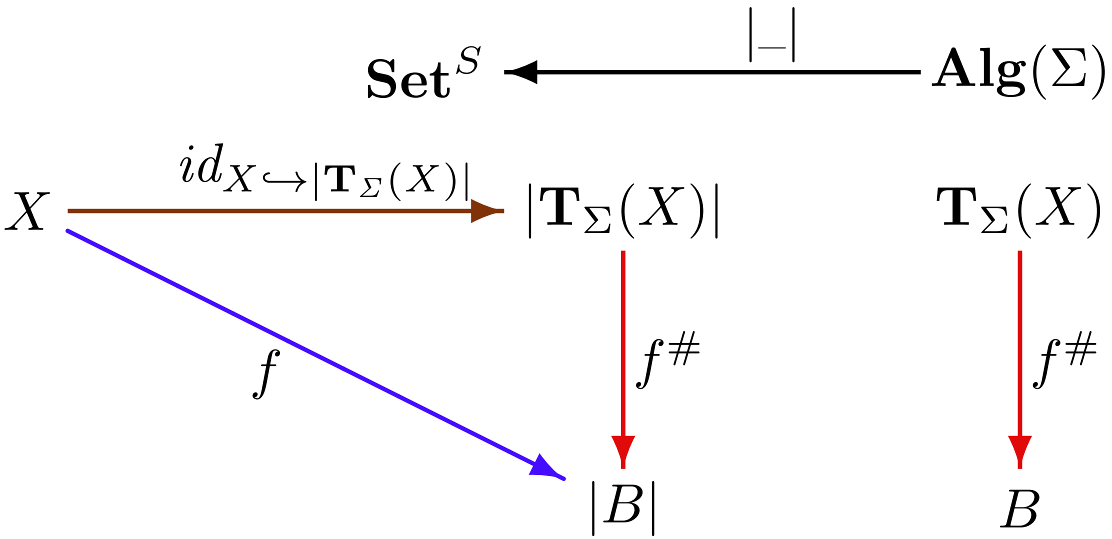

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- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}^\Phi(X) \in \mathit{Mod}(\Phi)$ that is free over X w.r.t. the carrier functor $|\_|\colon \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.

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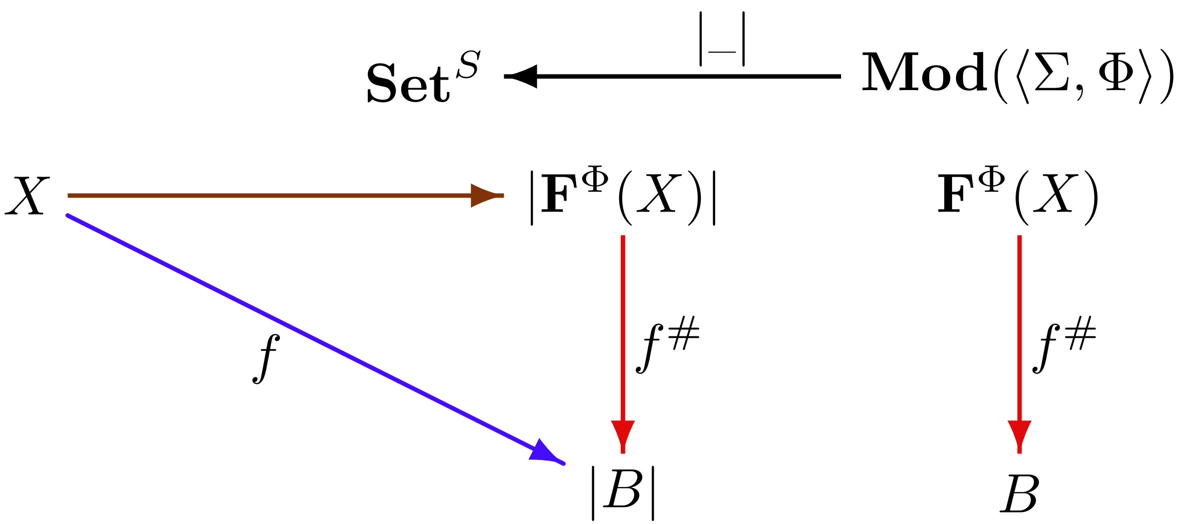

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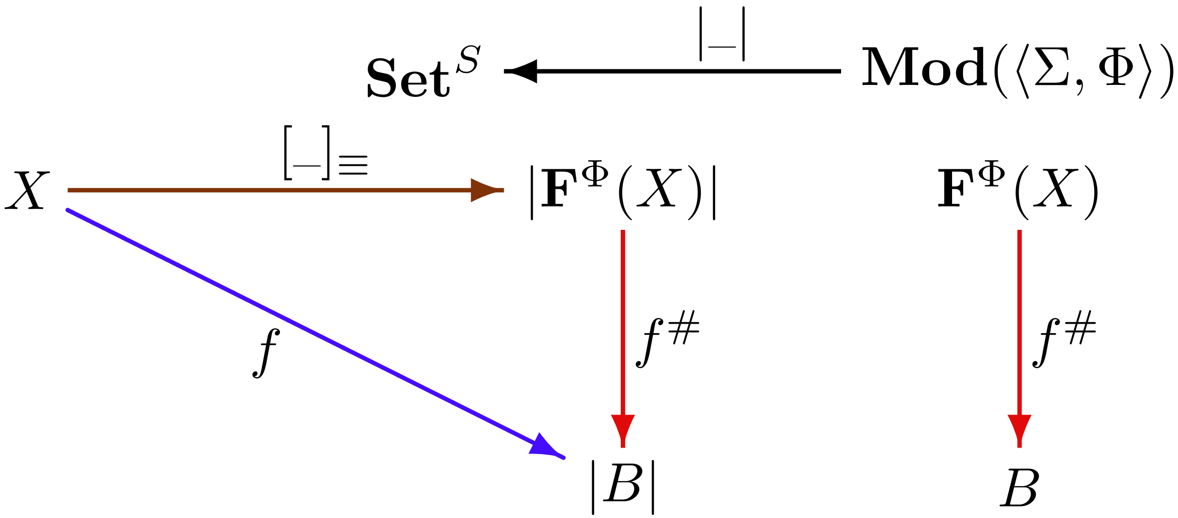

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- For any algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.

Fact: *For any algebraic signature inclusion* $\sigma\colon \Sigma \hookrightarrow \Sigma'$, *for any* Σ*-algebra* $A \in |\mathbf{Alg}(\Sigma)|$, *there exist a* Σ'*-algebra* $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ *that is free over* A *w.r.t. the reduct functor* $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.

$$\mathbf{Alg}(\Sigma) \xleftarrow{\ \_|_\sigma\ } \mathbf{Alg}(\Sigma')$$

$$A$$

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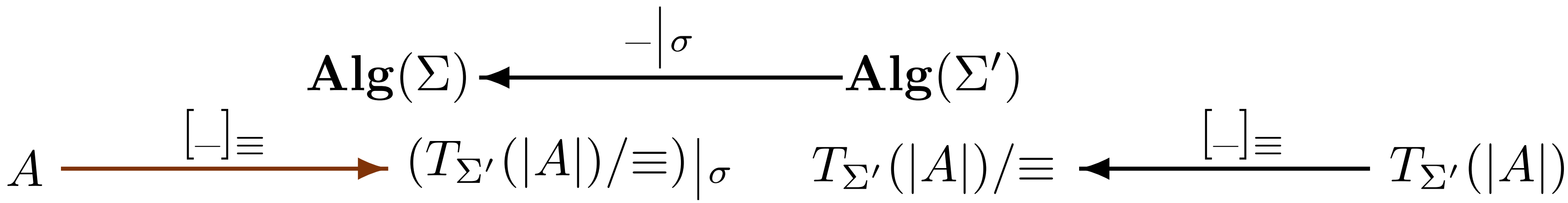
Proof (idea): Define $\mathbf{F}_\sigma(A)$ to be $T_{\Sigma'}(|A|)/{\equiv}$ with unit $[\_]_\equiv\colon A \to (T_{\Sigma'}(|A|)/{\equiv})|_\sigma$,



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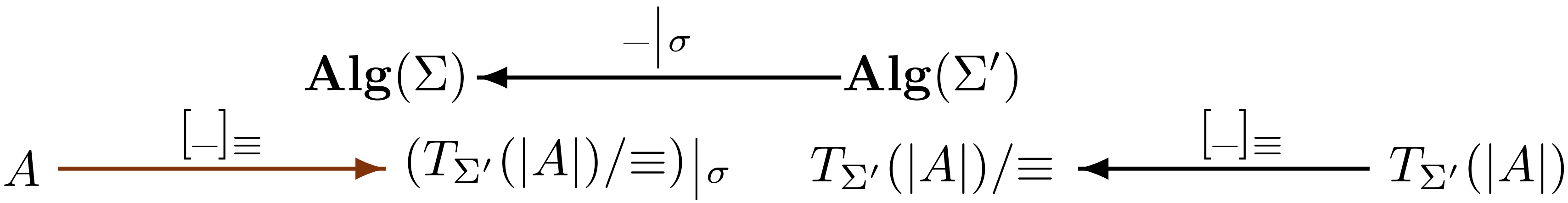

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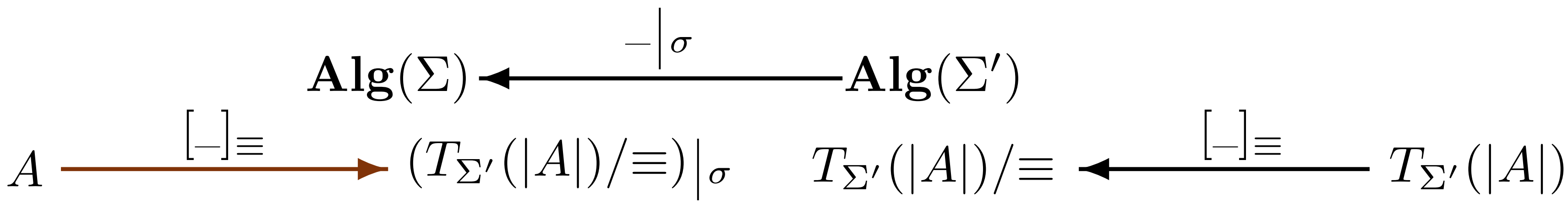

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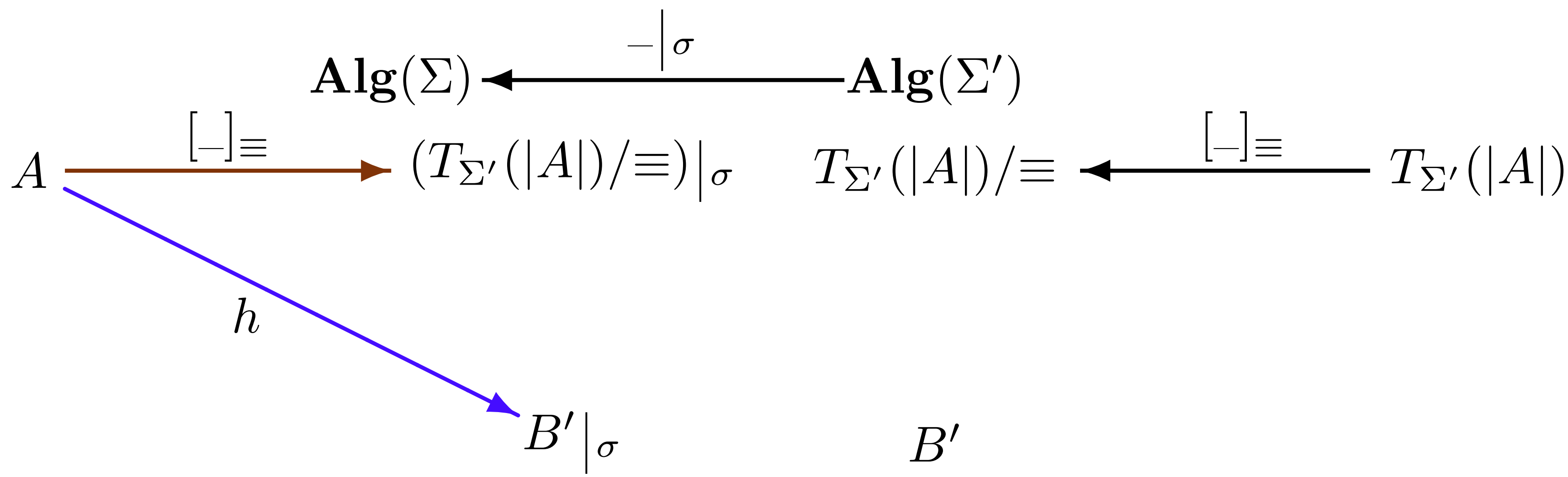

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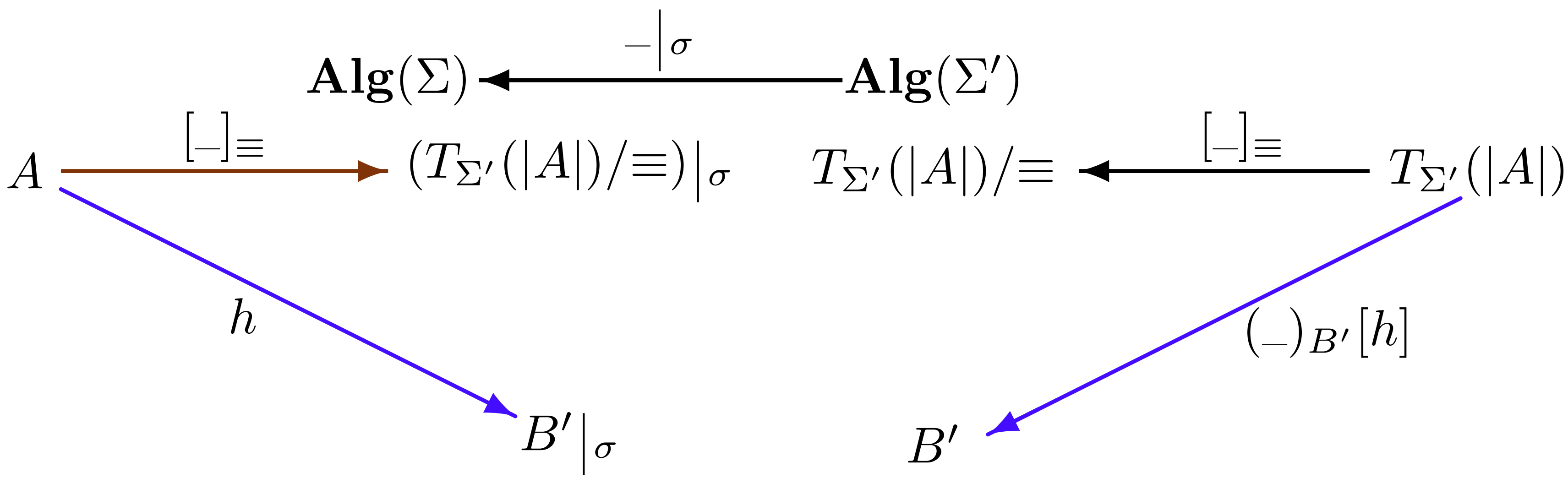

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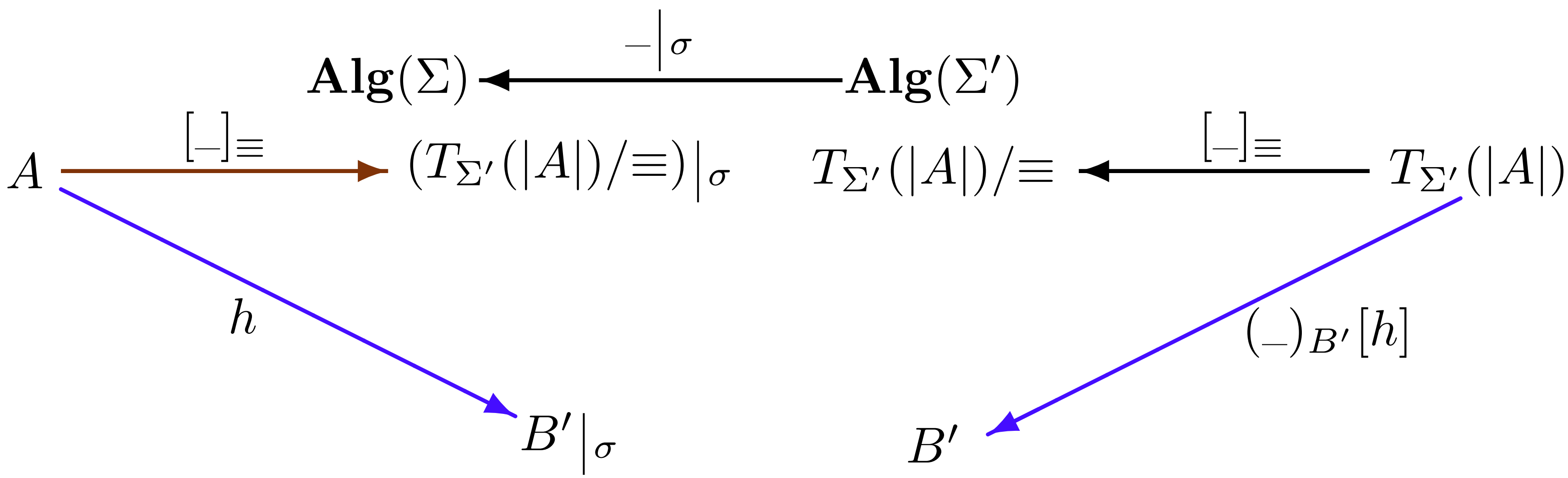

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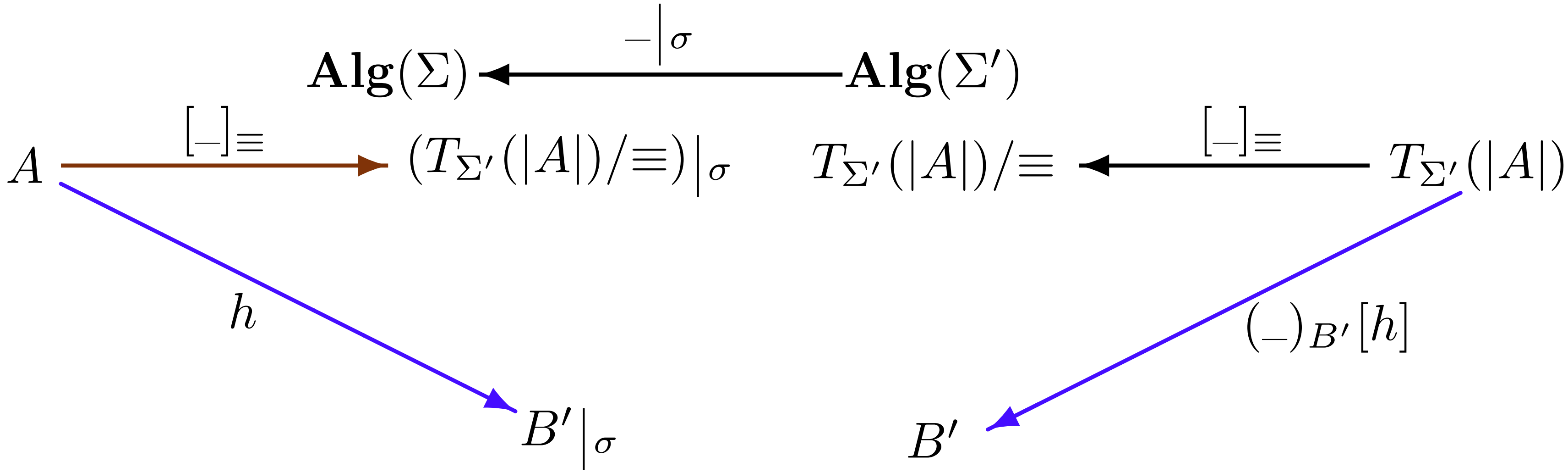
  Then $\equiv \subseteq K((\_)_{B'}[h])$, since:
  $h_s(f_A(a_1, \ldots, a_n)) = f_{B'}(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) = (f(a_1, \ldots, a_n))_{B'}[h]$



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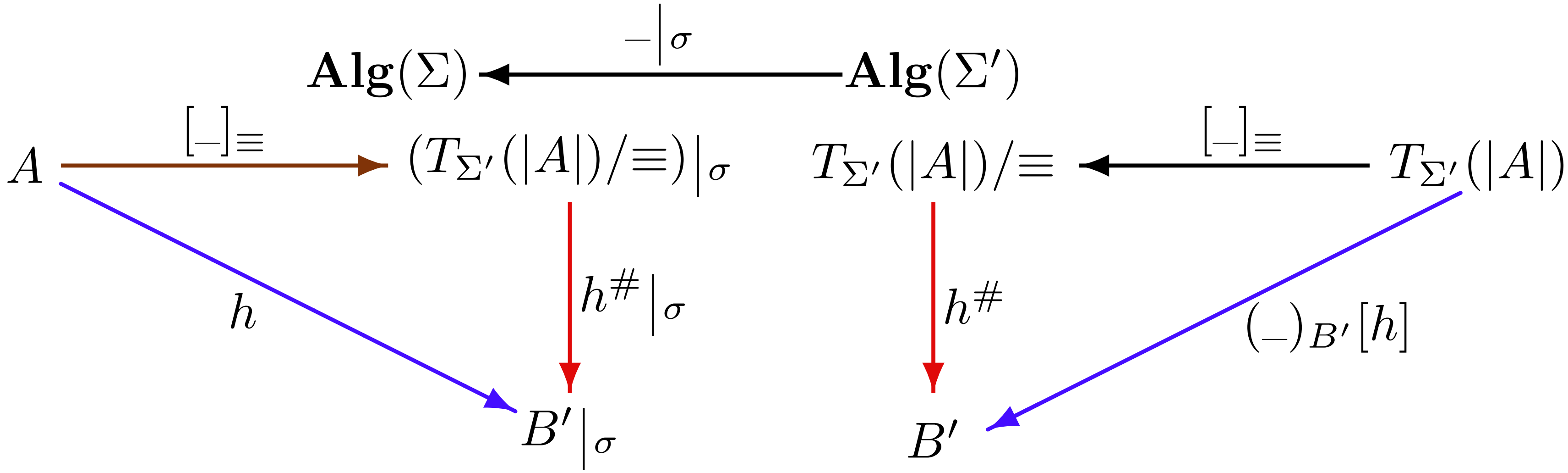

Free equational models

- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $\mathbf{T}_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.
- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}^\Phi(X) \in \mathit{Mod}(\Phi)$ that is free over X w.r.t. the carrier functor $|\_|\colon \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.
- For any algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.
- For any equational specification morphism $\sigma\colon \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$, for any model $A \in \mathit{Mod}(\Phi)$, there exist a model $\mathbf{F}_\sigma^{\Phi'}(A) \in \mathit{Mod}(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Mod}(\langle \Sigma', \Phi' \rangle) \to \mathbf{Mod}(\langle \Sigma, \Phi \rangle)$.

Prove the above.

Fact: *For any algebraic signature morphism $\sigma\colon\Sigma\to\Sigma'$ and set Φ' of Σ'-equations, for any Σ-algebra $A\in|\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma^{\Phi'}(A)\in Mod(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon\mathbf{Mod}(\langle\Sigma',\Phi'\rangle)\to\mathbf{Alg}(\Sigma)$.*

$$\mathbf{Alg}(\Sigma) \xleftarrow{\;\_|_\sigma\;} \mathbf{Mod}(\langle\Sigma',\Phi'\rangle) \quad\subseteq\quad \mathbf{Alg}(\Sigma')$$

$$A$$

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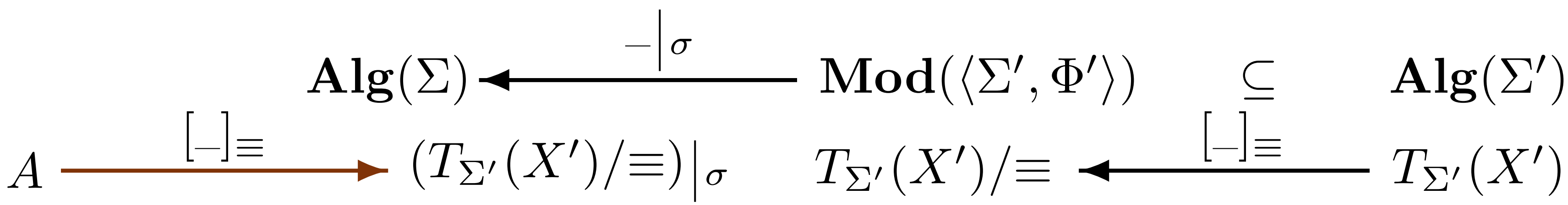

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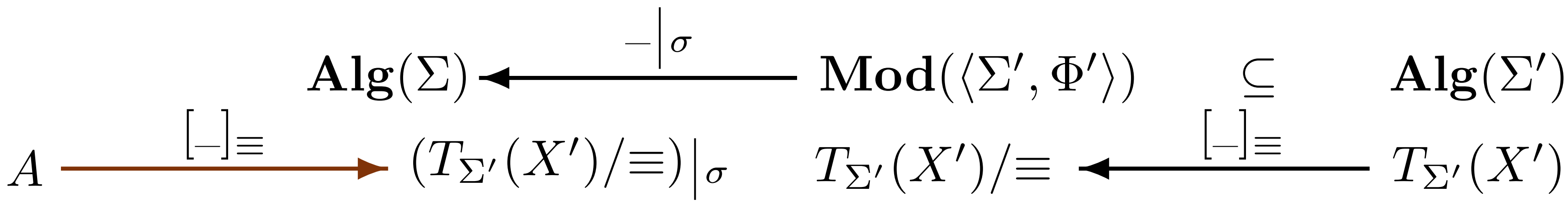

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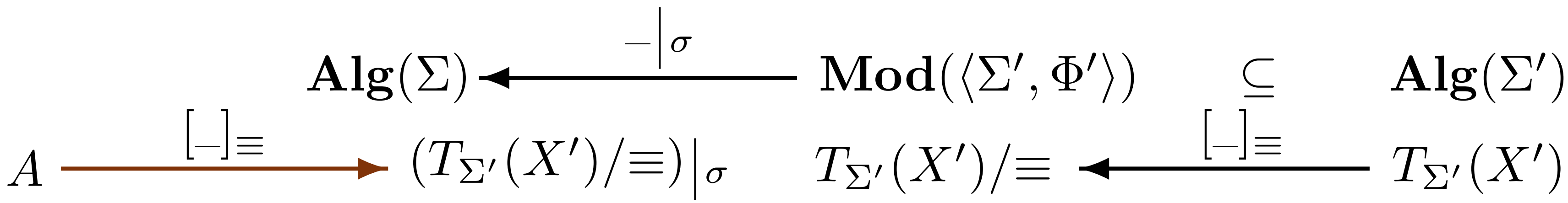

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- $T_{\Sigma'}(|A|)/{\equiv} \models \Phi'$, i.e. indeed $T_{\Sigma'}(|A|)/{\equiv} \in Mod(\Phi')$

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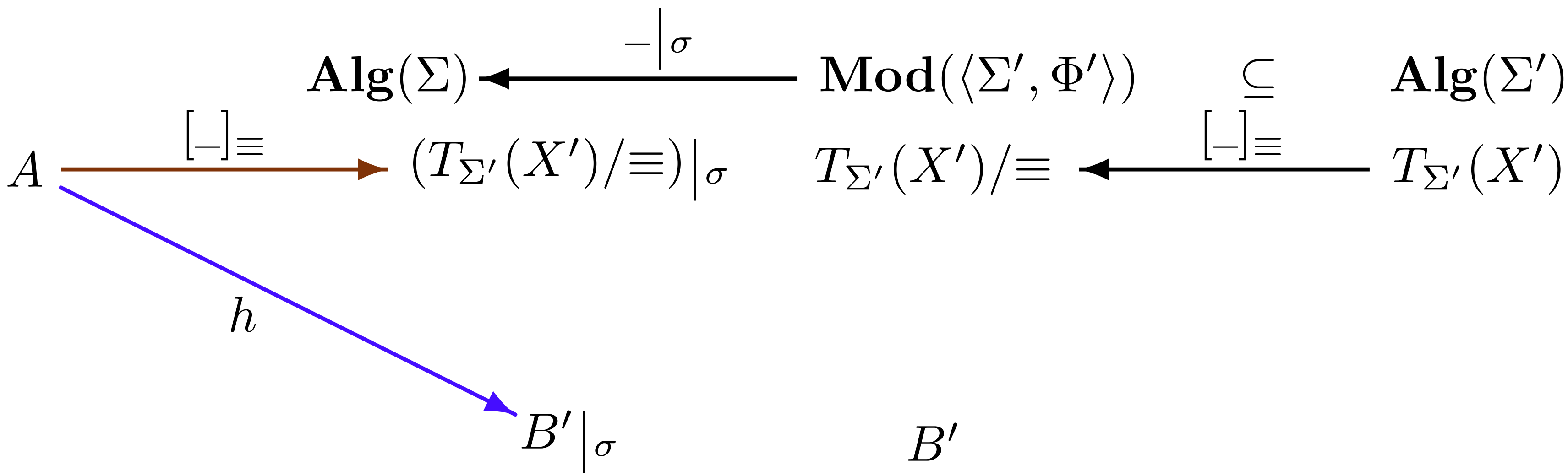

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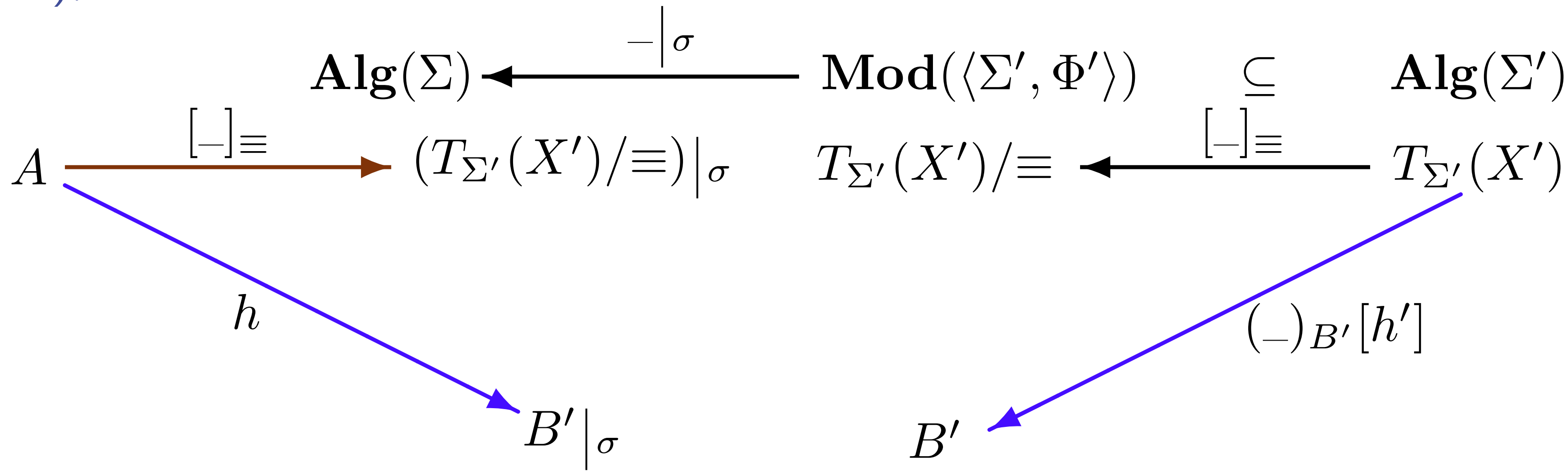

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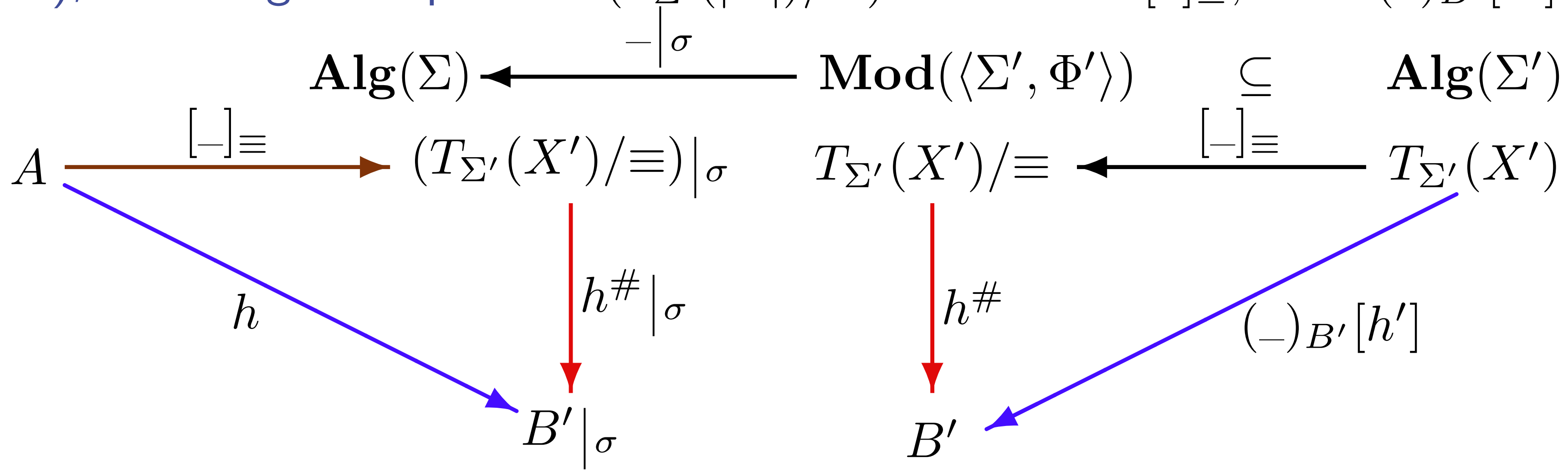

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Consider a subcategory $\mathbf{K}'' \subseteq \mathbf{K}$ *with inclusion* $\mathbf{J}\colon \mathbf{K}'' \to \mathbf{K}$ *such that* $\eta_A\colon A \to \mathbf{G}(A')$ *is in* $\mathbf{K}''$ *and we have a functor* $\mathbf{G}'\colon \mathbf{K}' \to \mathbf{K}''$ *such that* $\mathbf{G}';\mathbf{J} = \mathbf{G}$ *(i.e. the image of* $\mathbf{G}$ *is within* $\mathbf{K}''$*).*
Then $A' \in |\mathbf{K}'|$ *is free over* A *with unit* $\eta_A\colon A \to \mathbf{G}'(A')$ *w.r.t.* $\mathbf{G}'\colon \mathbf{K}' \to \mathbf{K}''$*.*

Fact: *Given a functor* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *and* $A \in |\mathbf{K}|$*, let* $A' \in |\mathbf{K}'|$ *be free over* A *with unit* $\eta_A\colon A \to \mathbf{G}(A')$ *w.r.t.* $\mathbf{G}$*.*
Consider a subcategory $\mathbf{K}'' \subseteq \mathbf{K}$ *with inclusion* $\mathbf{J}\colon \mathbf{K}'' \to \mathbf{K}$ *such that* $\eta_A\colon A \to \mathbf{G}(A')$ *is in* $\mathbf{K}''$ *and we have a functor* $\mathbf{G}'\colon \mathbf{K}' \to \mathbf{K}''$ *such that* $\mathbf{G}';\mathbf{J} = \mathbf{G}$ *(i.e. the image of* $\mathbf{G}$ *is within* $\mathbf{K}''$*).*
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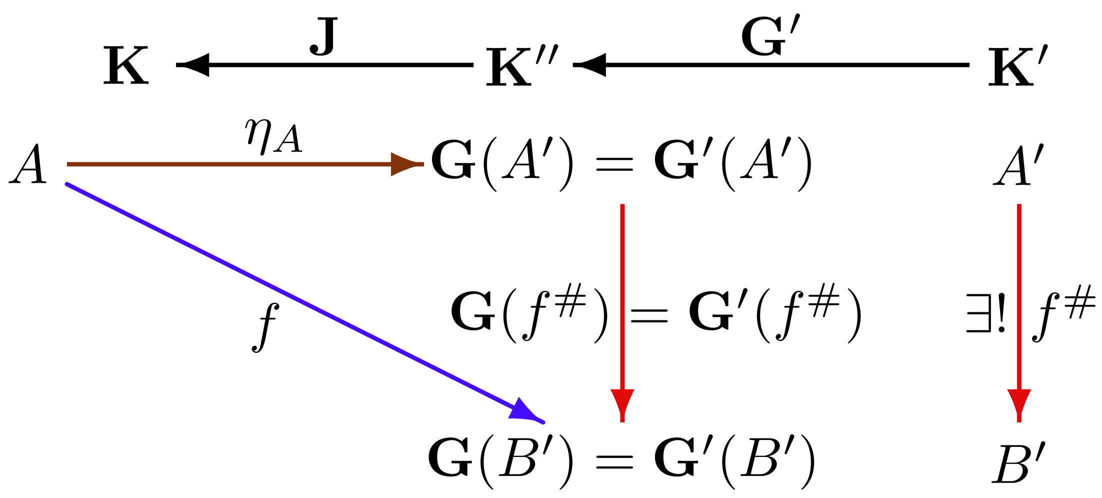
Just check:



Free equational models

- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $\mathbf{T}_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.
- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}^\Phi(X) \in \mathit{Mod}(\Phi)$ that is free over X w.r.t. the carrier functor $|\_|\colon \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.
- For any algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.
- For any equational specification morphism $\sigma\colon \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$, for any model $A \in \mathit{Mod}(\Phi)$, there exist a model $\mathbf{F}_\sigma^{\Phi'}(A) \in \mathit{Mod}(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Mod}(\langle \Sigma', \Phi' \rangle) \to \mathbf{Mod}(\langle \Sigma, \Phi \rangle)$.

Prove the above.

Facts

Consider a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$, and object $A \in |\mathbf{K}|$, and an object $A' \in |\mathbf{K}'|$ free over A w.r.t. $\mathbf{G}$ with unit $\eta_A\colon A \to \mathbf{G}(A')$.

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$$\mathbf{K} \xleftarrow{\ \mathbf{G}\ } \mathbf{K}'$$

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & \mathbf{G}(A') & \qquad & A' \\ & \searrow^{f} & \downarrow{\scriptstyle \mathbf{G}(f^\#)} & & \exists!\,\downarrow{\scriptstyle f^\#} \\ & & \mathbf{G}(B') & & B' \end{array}$$

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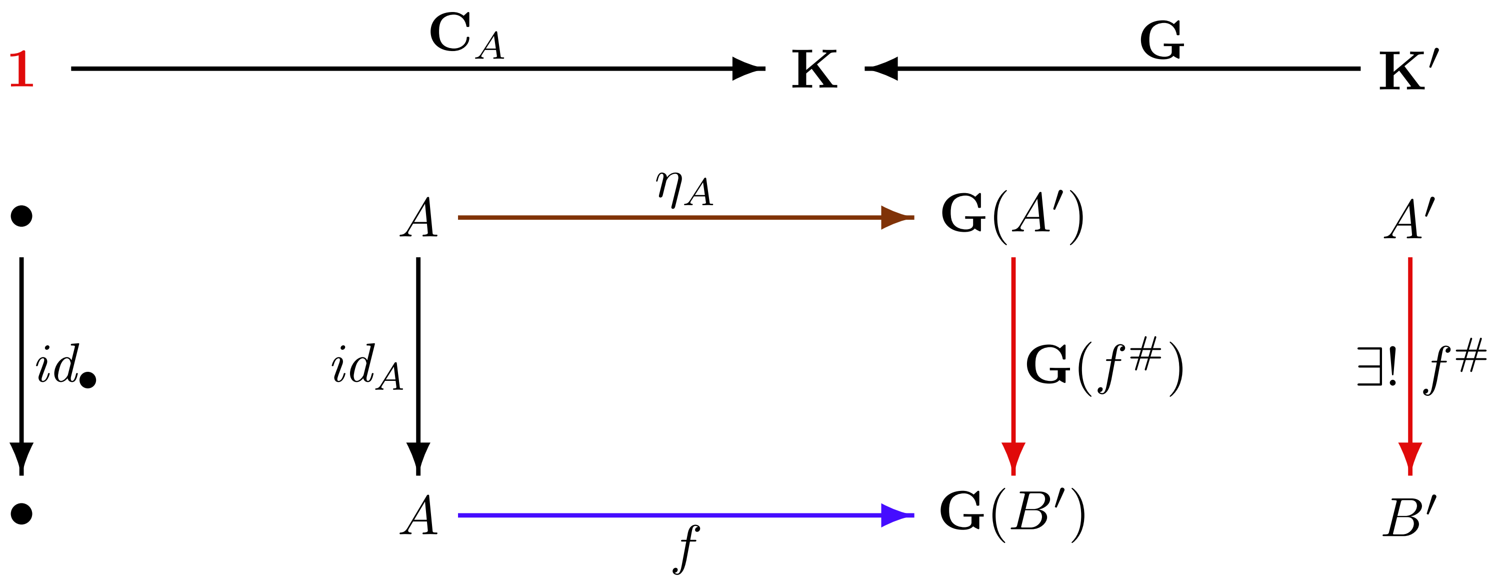

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  - $((\_)^{\#})^{-1} = \eta_A;\mathbf{G}(\_)\colon \mathbf{K}'(A', B') \to \mathbf{K}(A, \mathbf{G}(B'))$, i.e.:
    - $f = \eta_A;\mathbf{G}(f^{\#})$ for $f\colon A \to \mathbf{G}(B')$ in $\mathbf{K}$
    - $g = (\eta_A;\mathbf{G}(g))^{\#}$ for $g\colon A' \to B'$ in $\mathbf{K}'$

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Colimits as free objects

Theorem: *In a category $\mathbf{K}$, given a diagram D of shape $\mathcal{G}(D)$, the colimit of D in $\mathbf{K}$ is a free object over D w.r.t. the diagonal functor $\Delta_{\mathbf{K}}^{\mathcal{G}(D)}\colon \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{\mathcal{G}(D)}$.*

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Proof (idea): Cocones $\alpha\colon D \to X$ are diagram morphisms $\alpha\colon D \to \Delta_{\mathbf{K}}^{\mathcal{G}(D)}(X)$.

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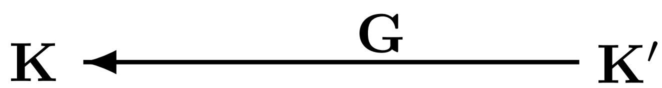
Spell this out for initial objects, coproducts, coequalisers, and pushouts

Left adjoints

Consider a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$.



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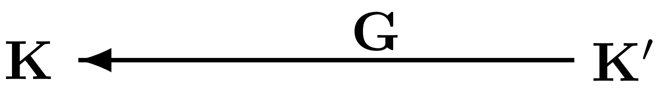

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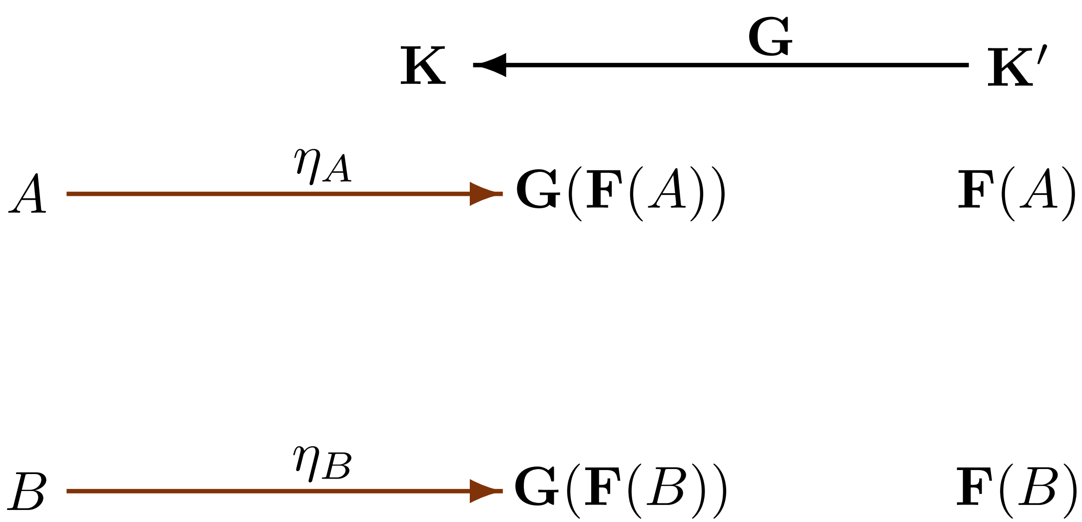

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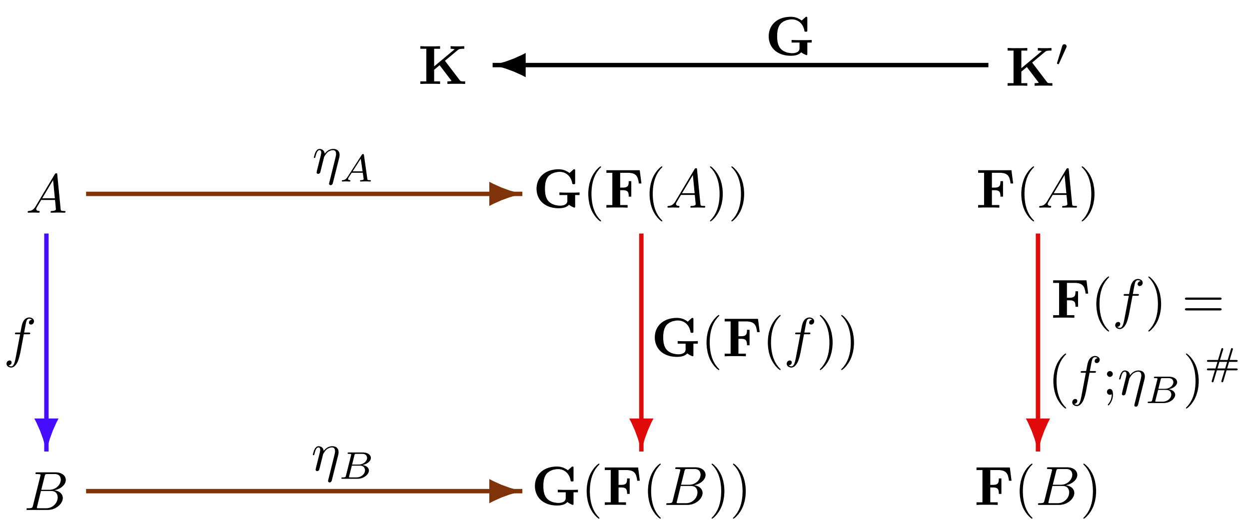

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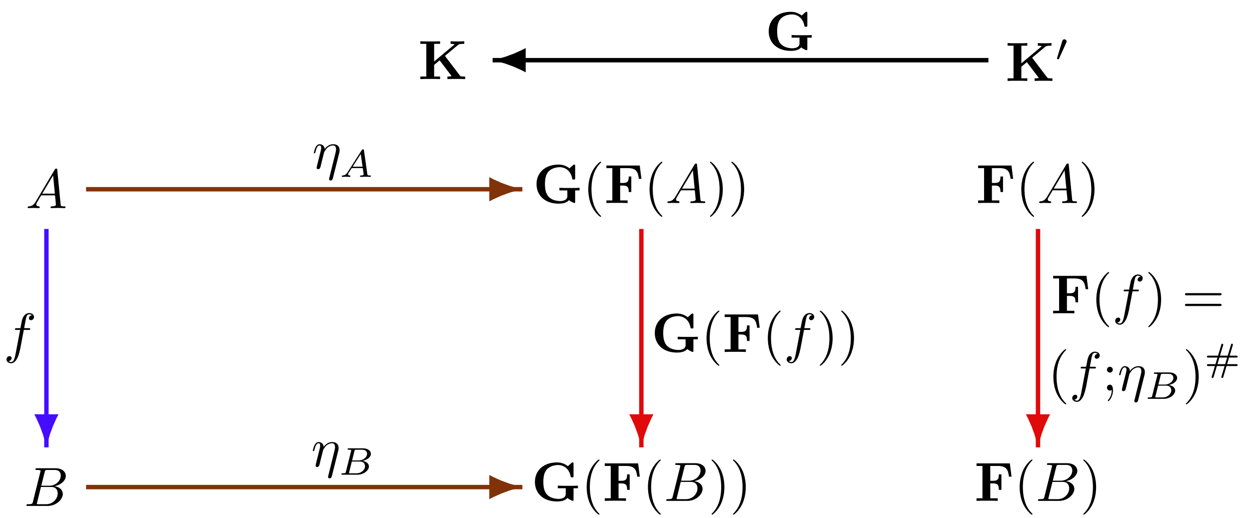

Proof

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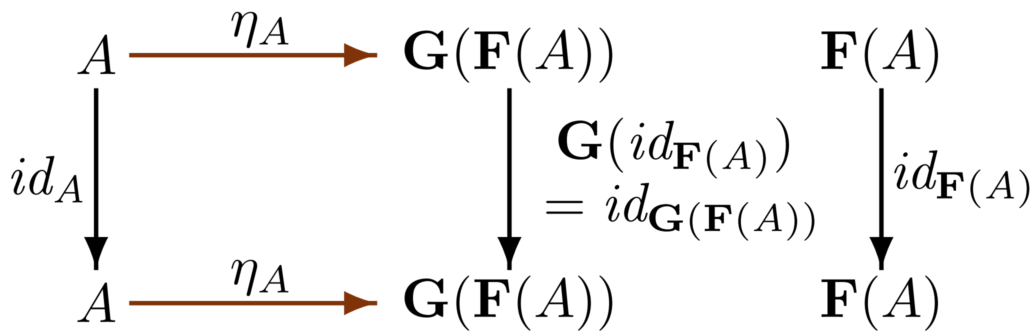

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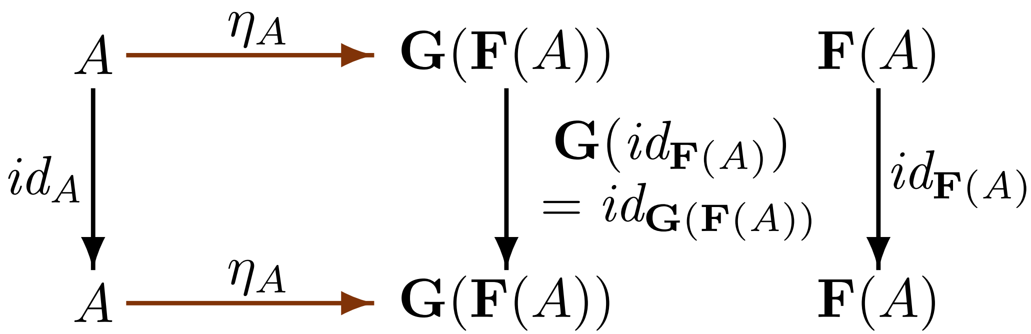

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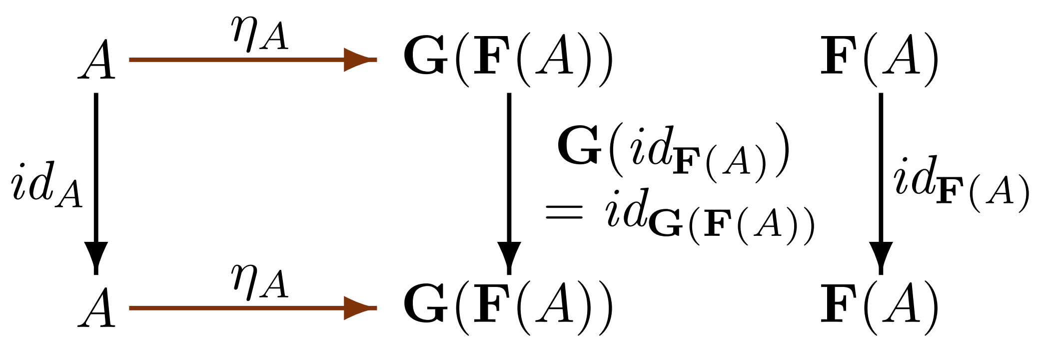

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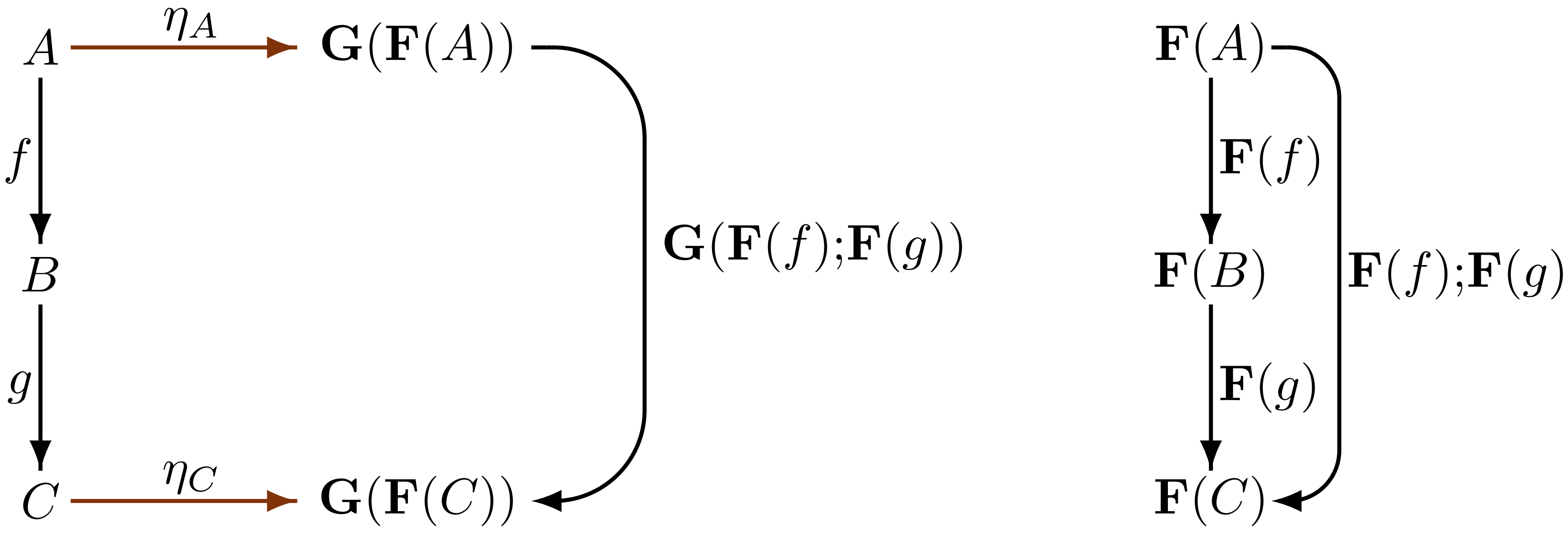

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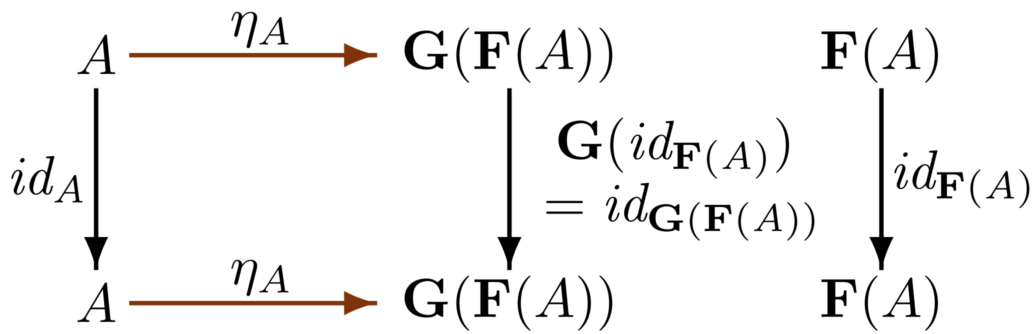

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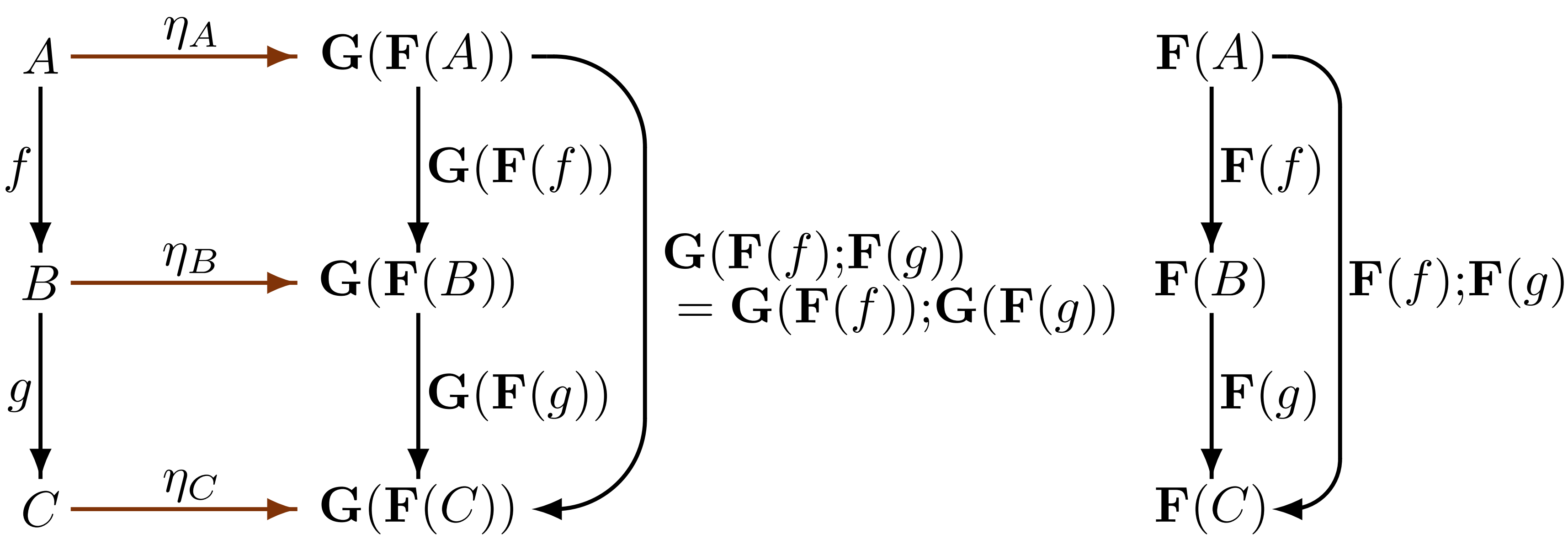

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- The term-algebra functor $T_\Sigma\colon \mathbf{Set}^S \to \mathbf{Alg}(\Sigma)$ is left adjoint to the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$.

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- ... other examples given by the examples of free objects above ...

Uniqueness of left adjoints

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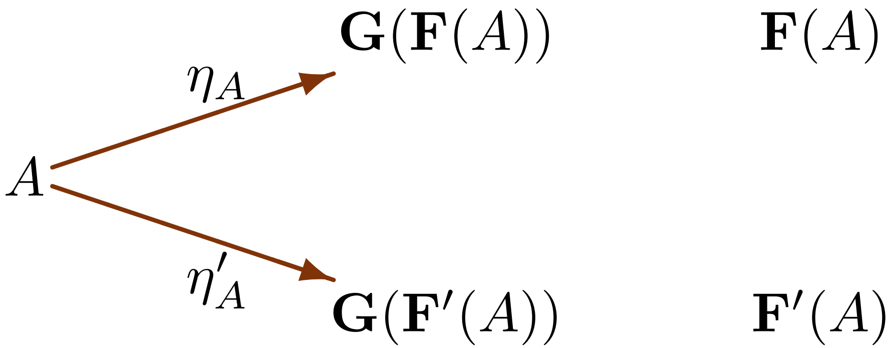

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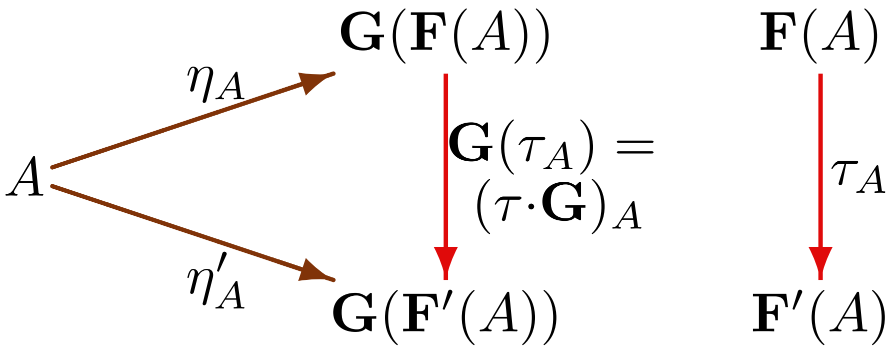

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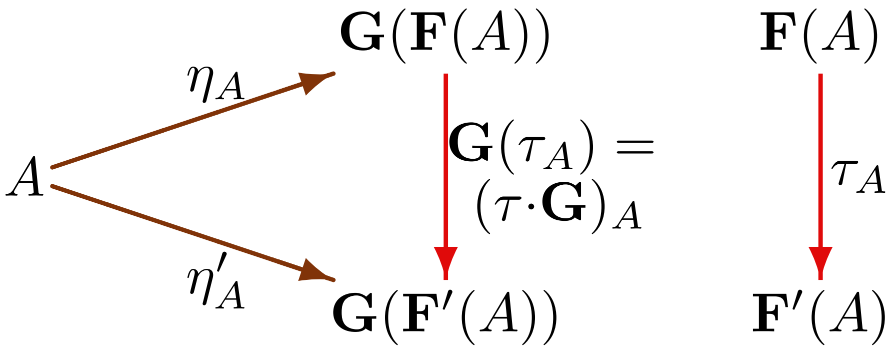

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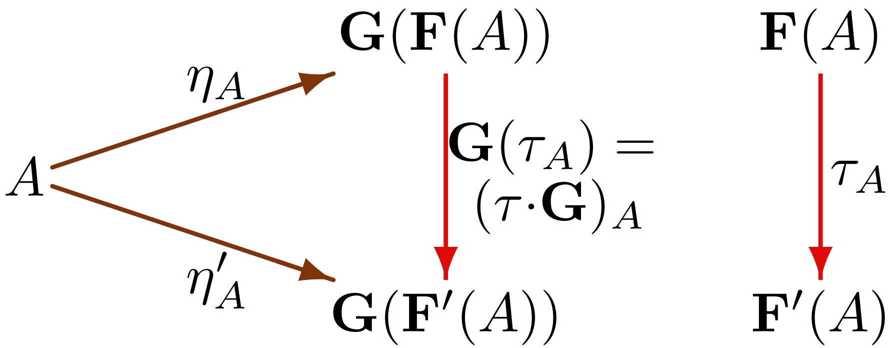

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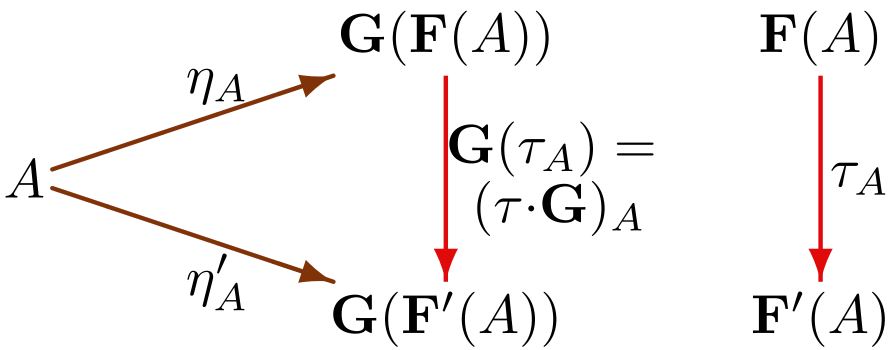

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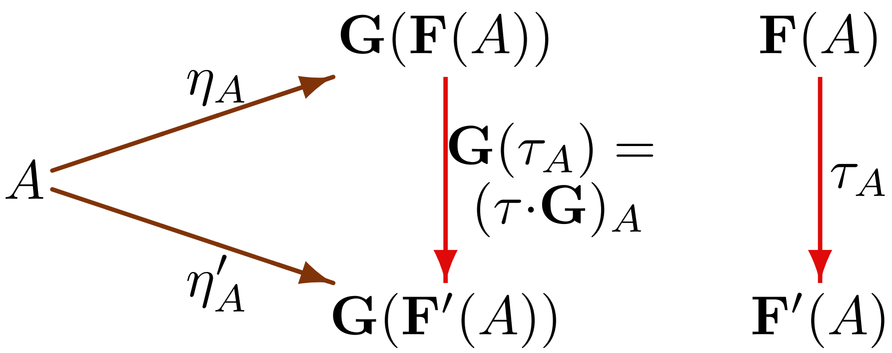

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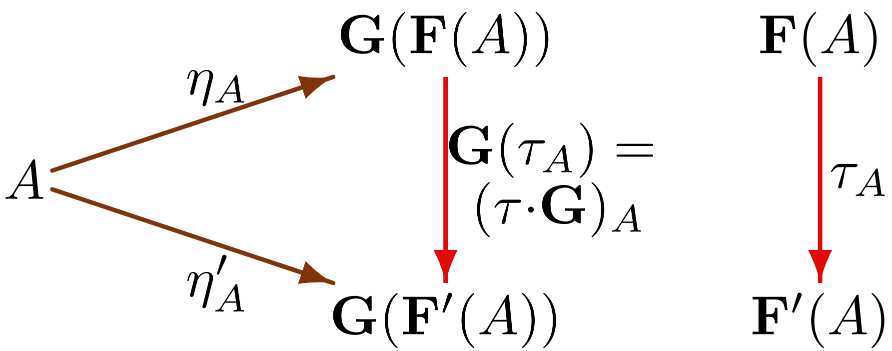

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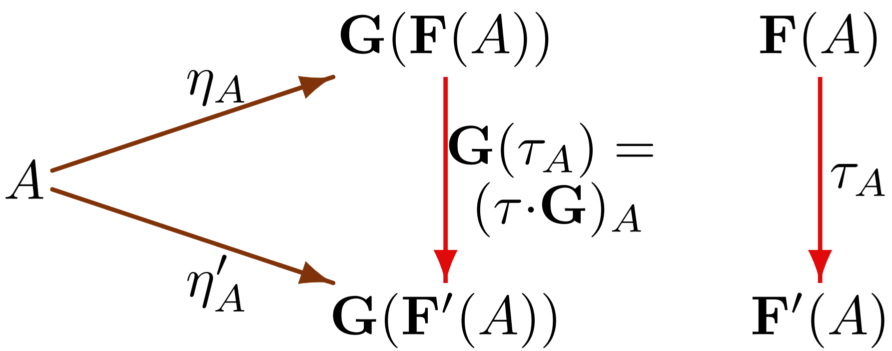

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Left adjoints and colimits

Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ with unit $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.

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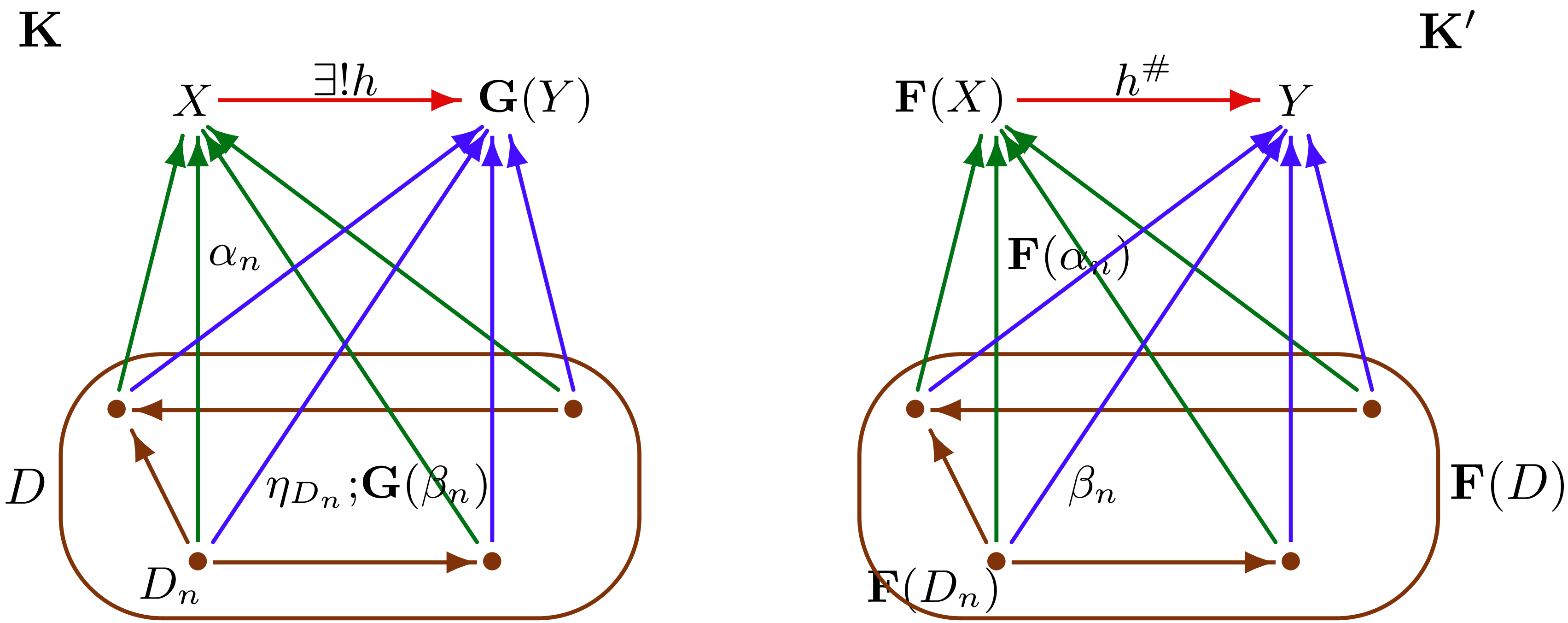

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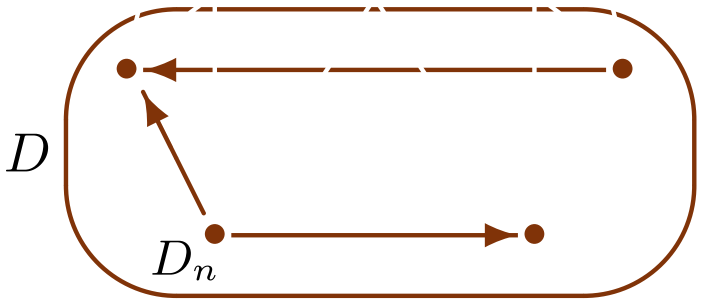

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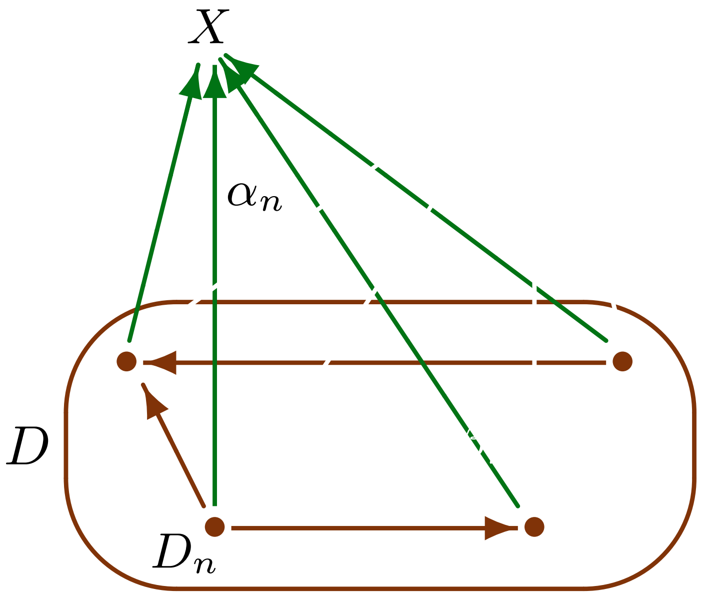

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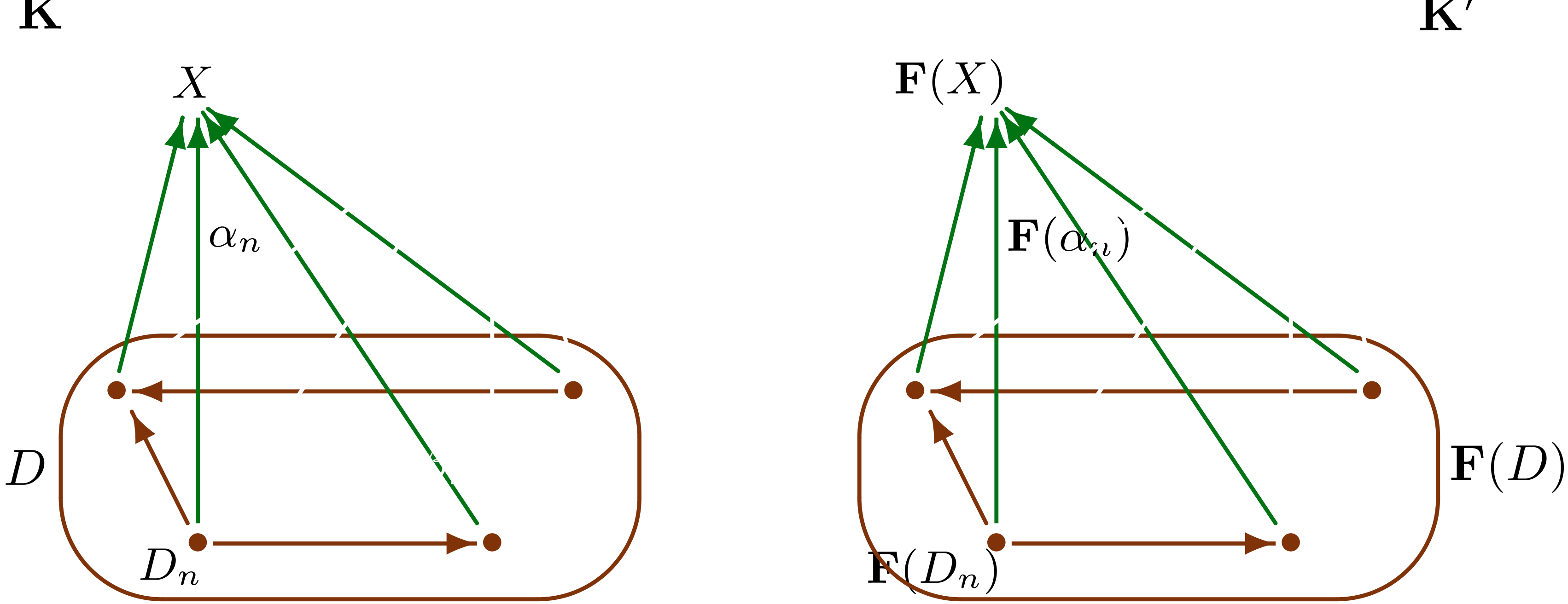

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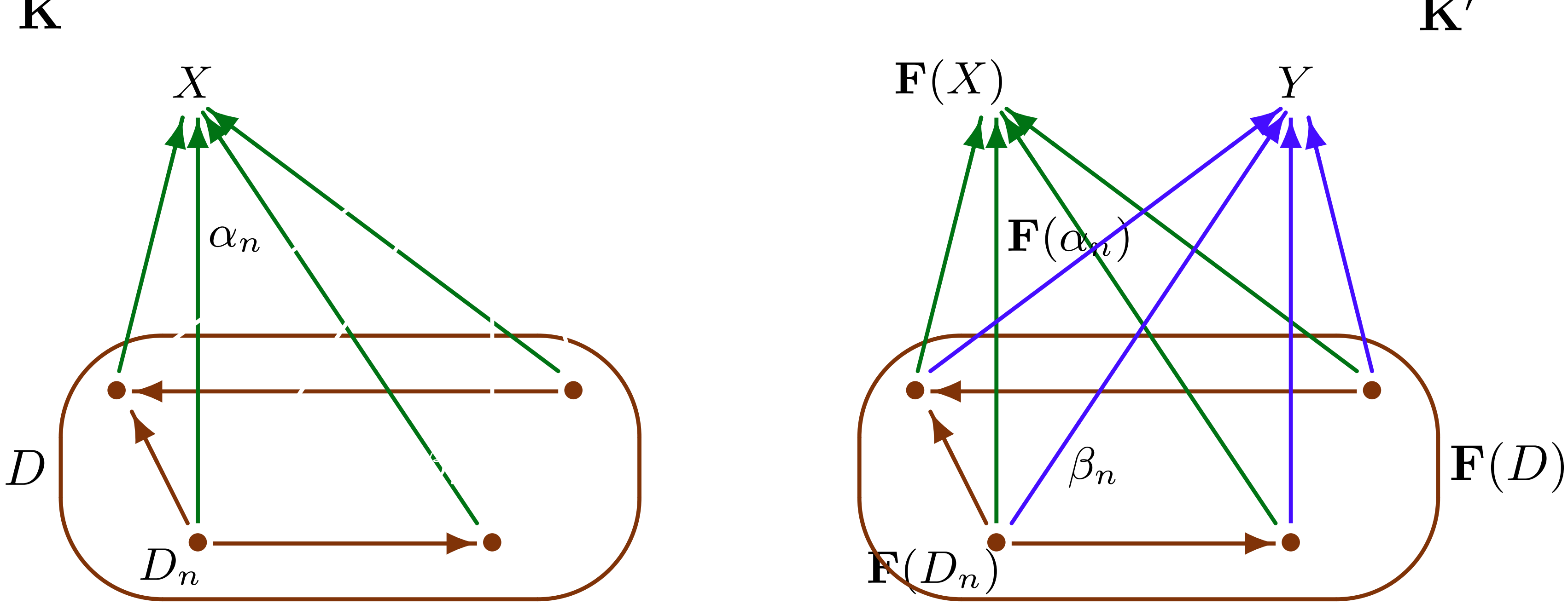

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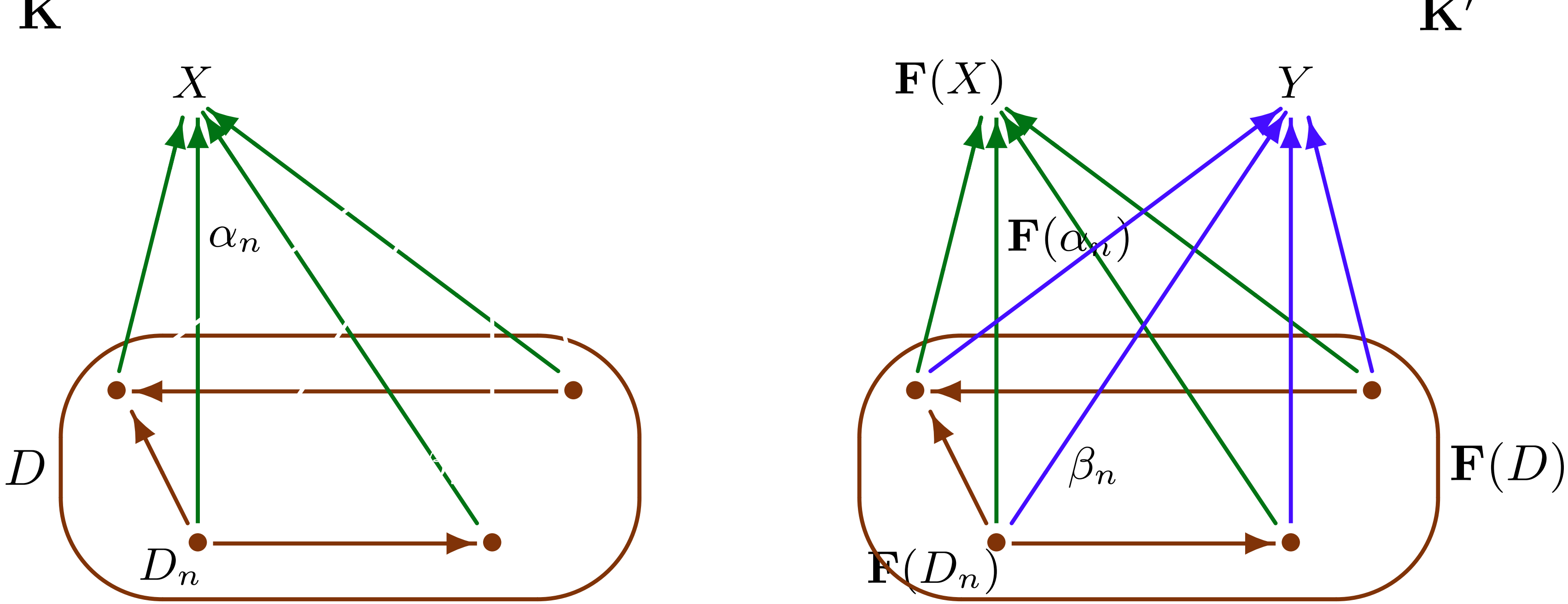

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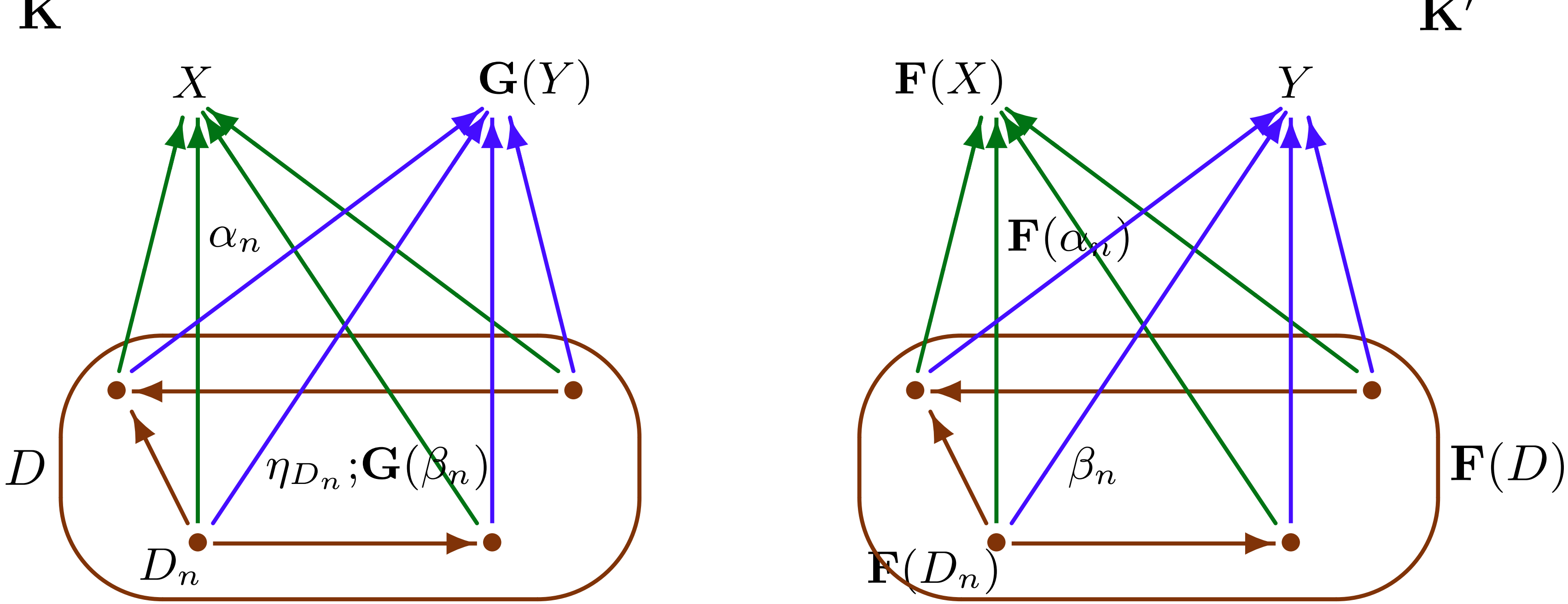

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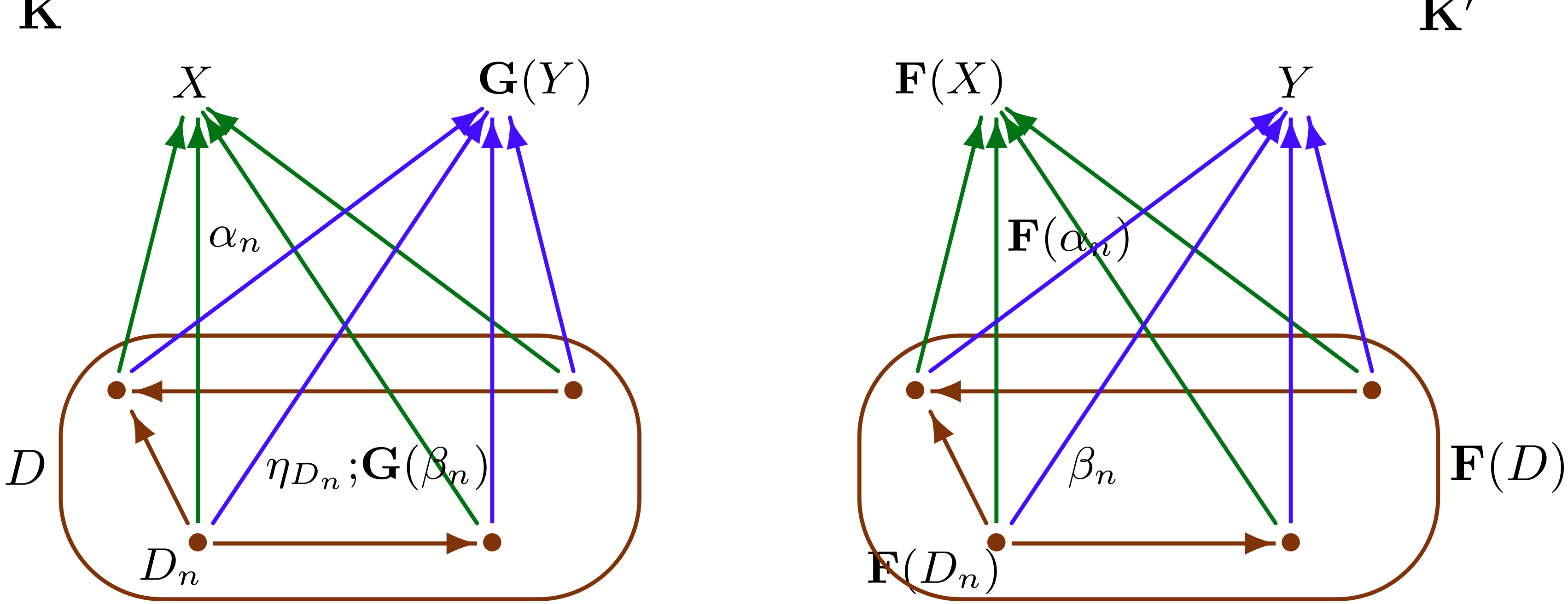

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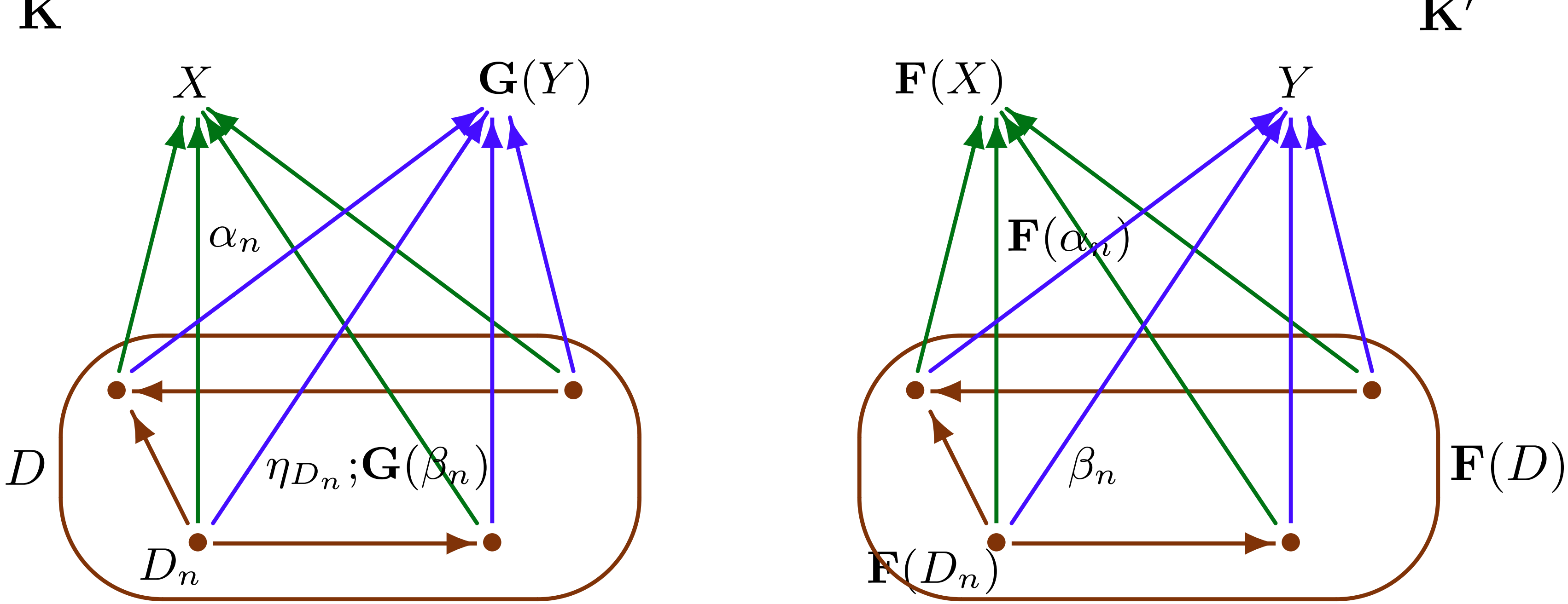

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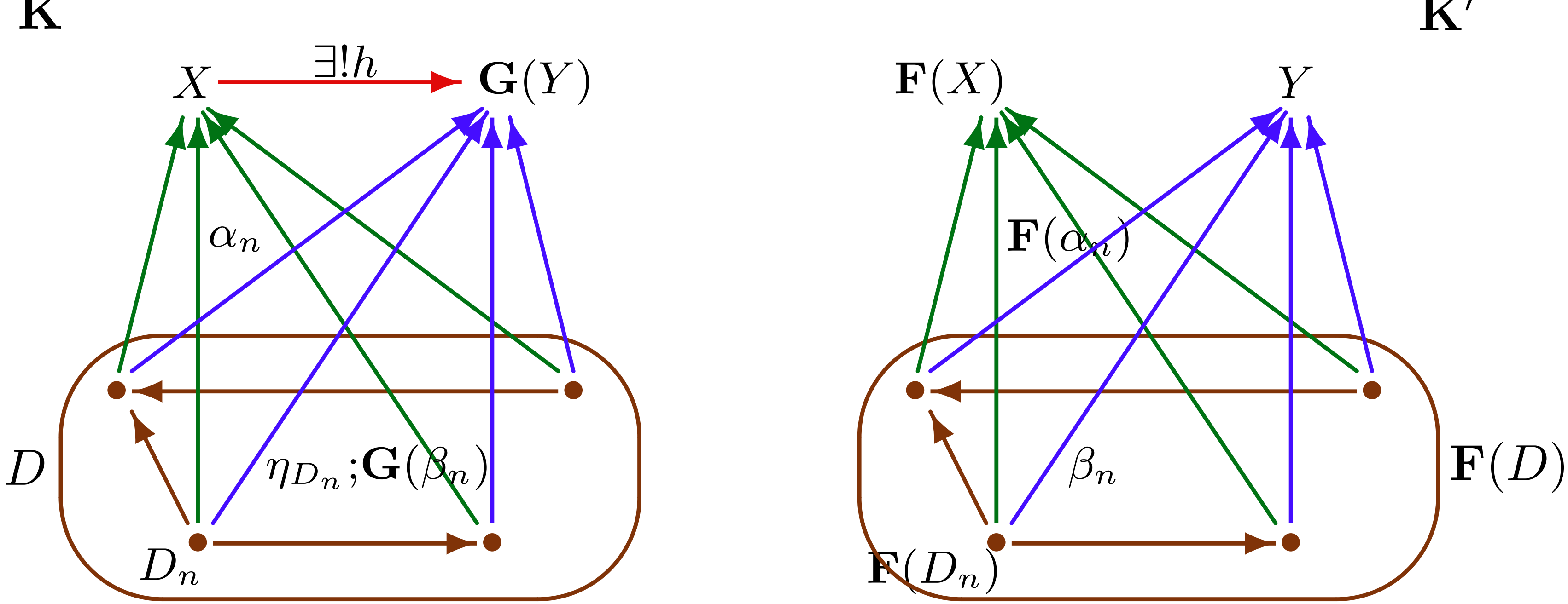

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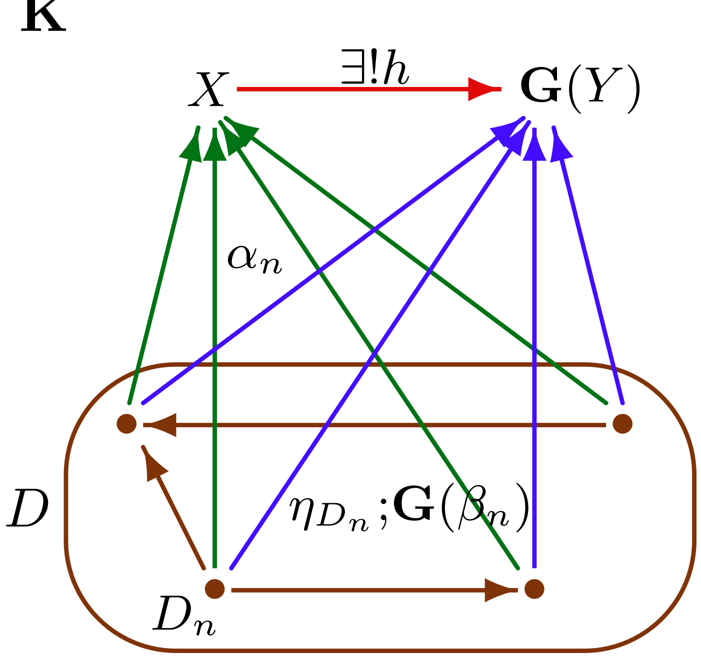

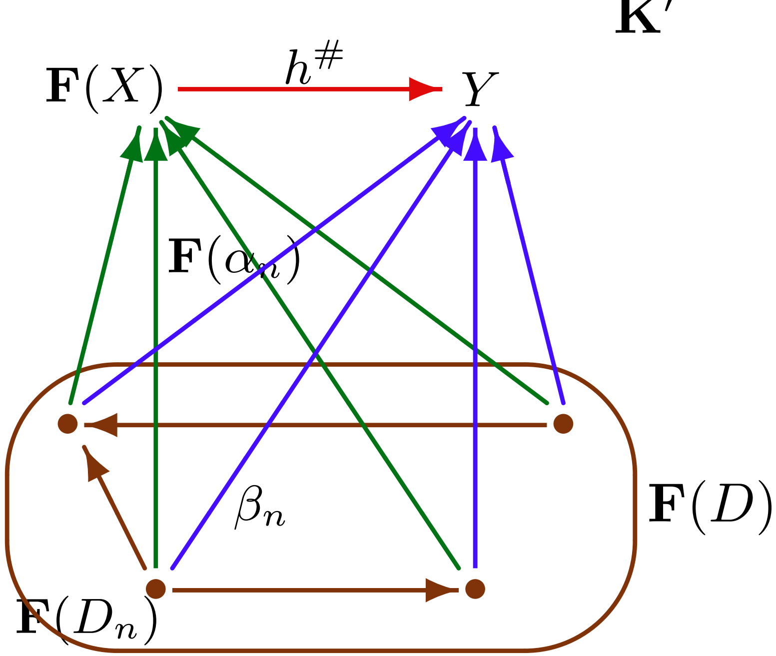

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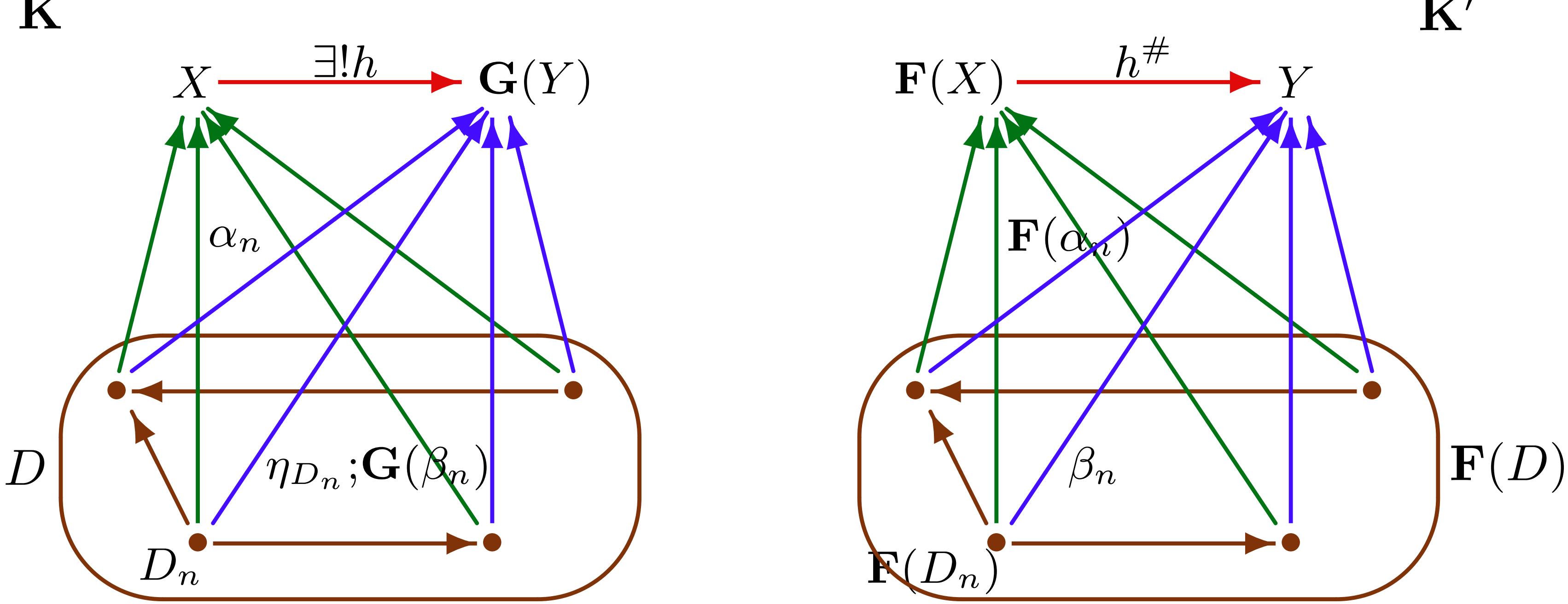

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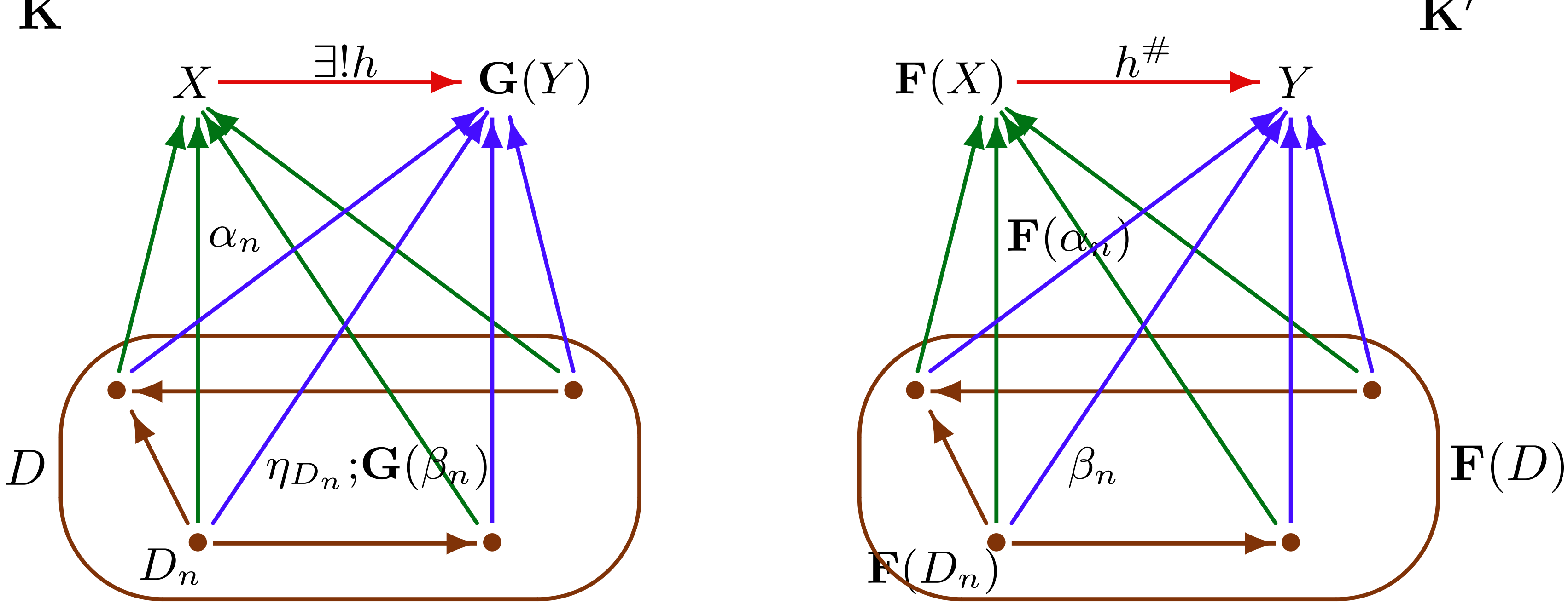

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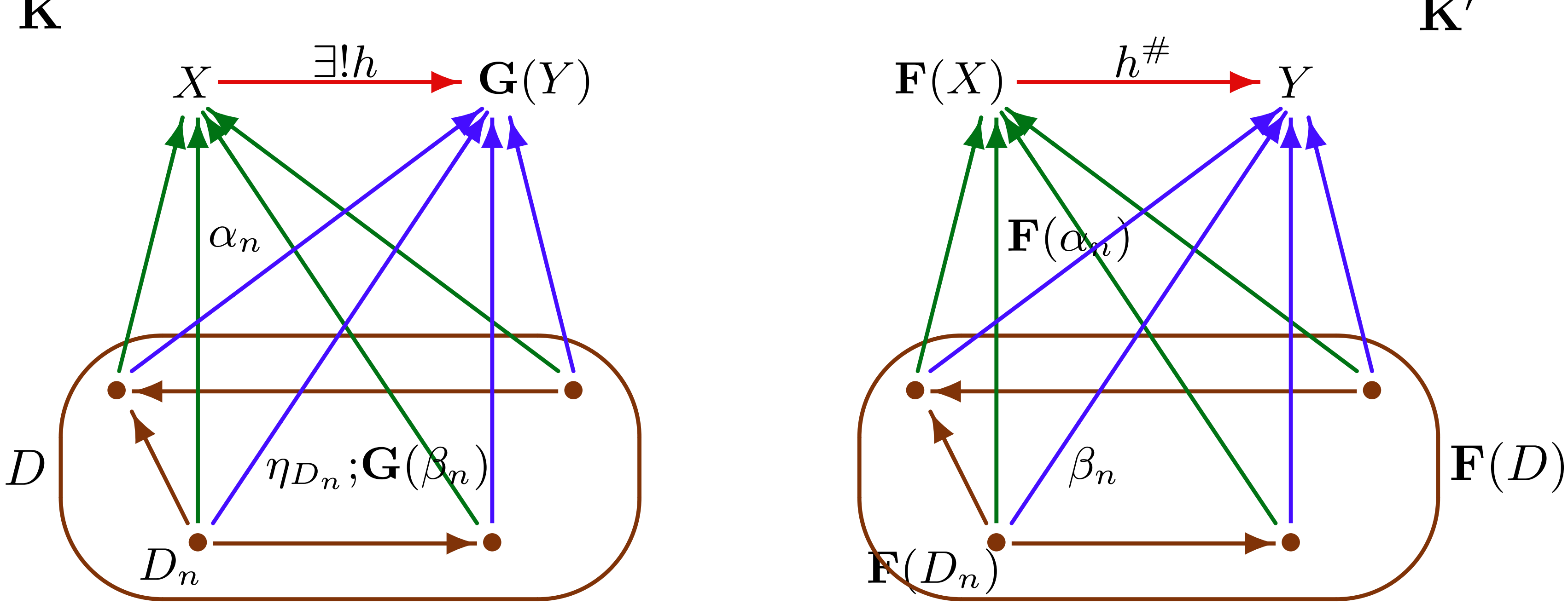
since: $\eta_D;\mathbf{G}(\mathbf{F}(\alpha);h^\#) = \eta_D;\mathbf{G}(\mathbf{F}(\alpha));\mathbf{G}(h^\#) = \alpha;\eta_X;\mathbf{G}(h^\#) = \alpha;h = \eta_D;\mathbf{G}(\beta)$.



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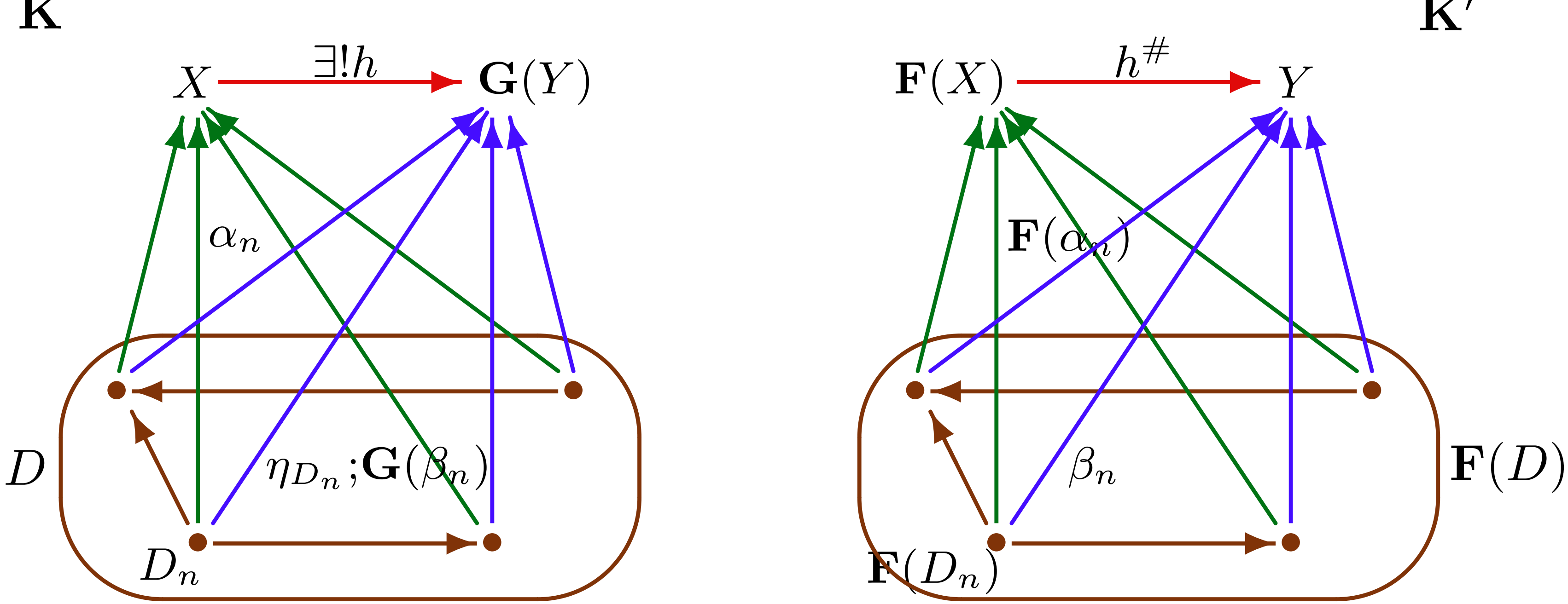

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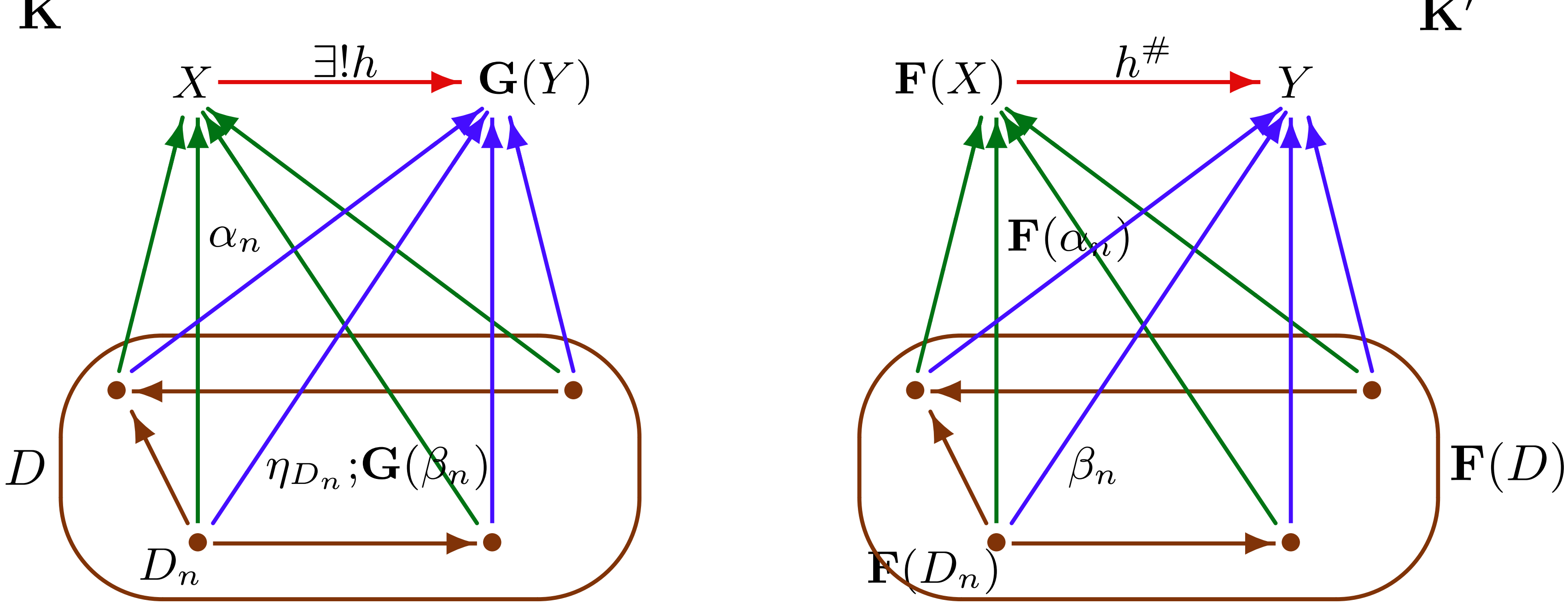

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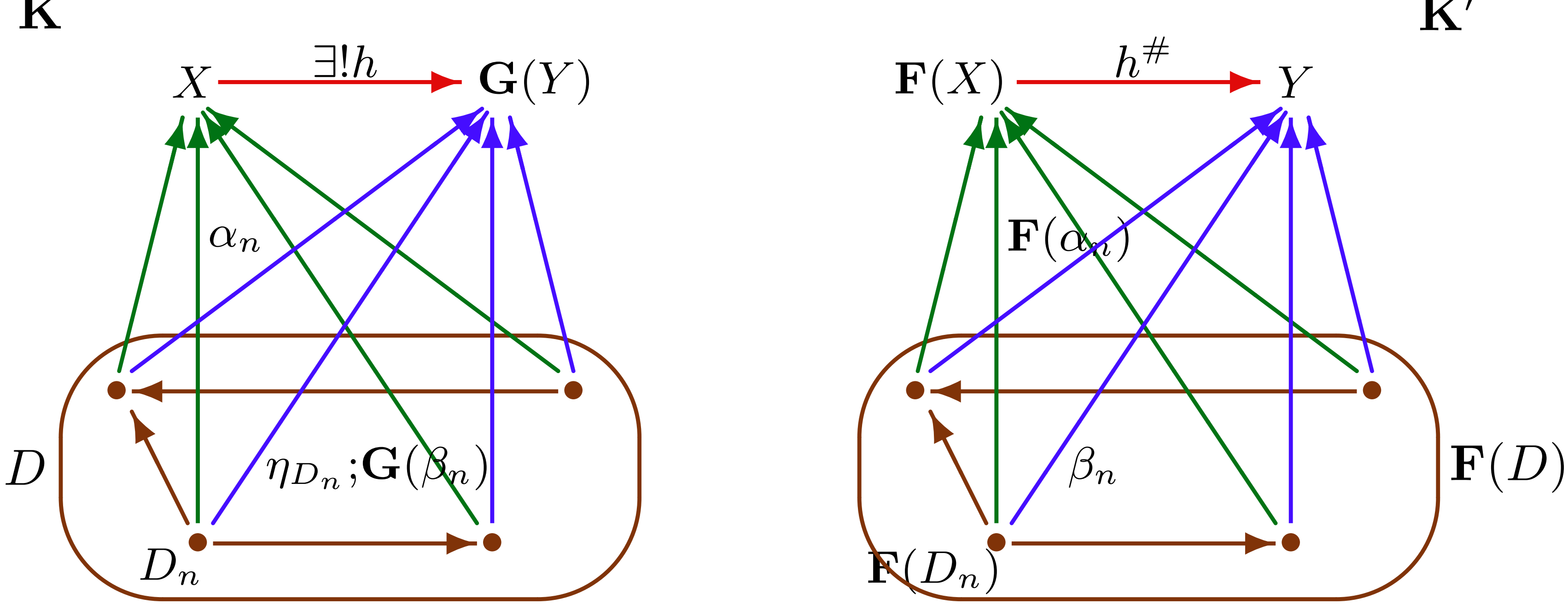

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Left adjoints and colimits

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Theorem: $\mathbf{F}$ *is cocontinuous (preserves colimits).*

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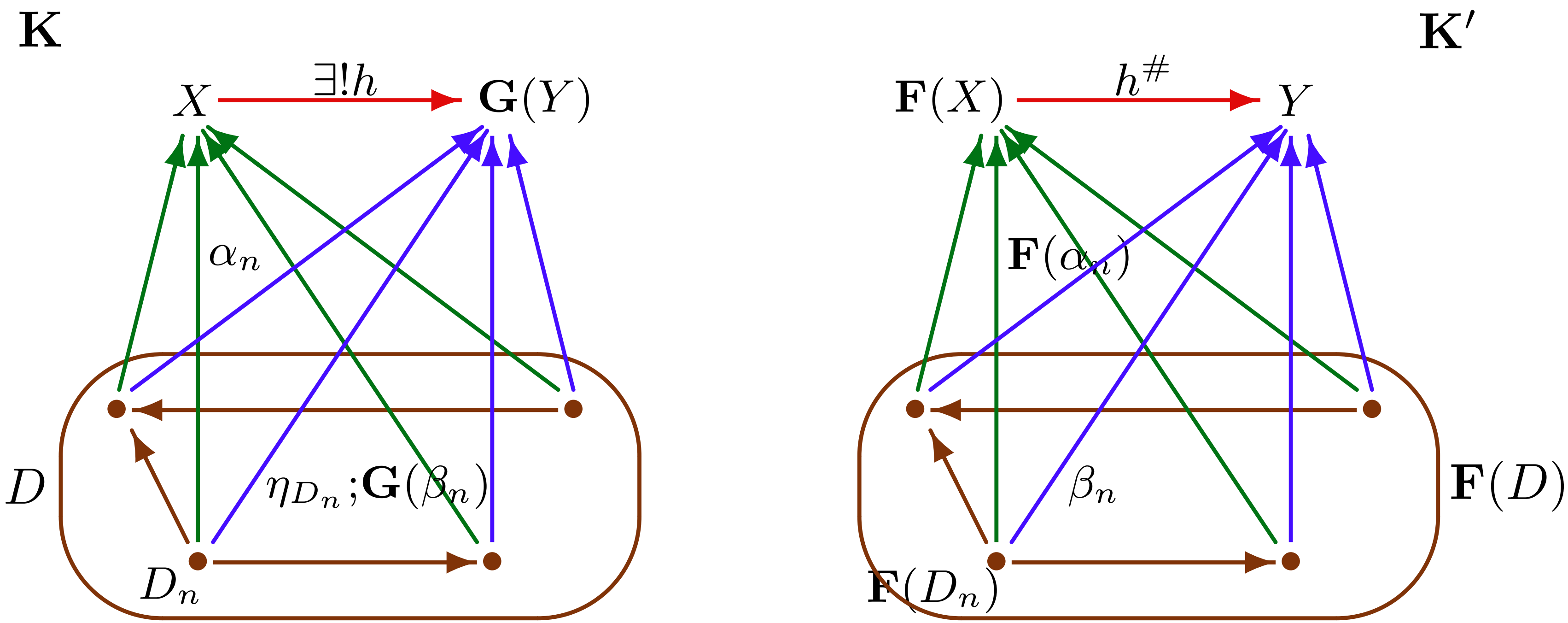

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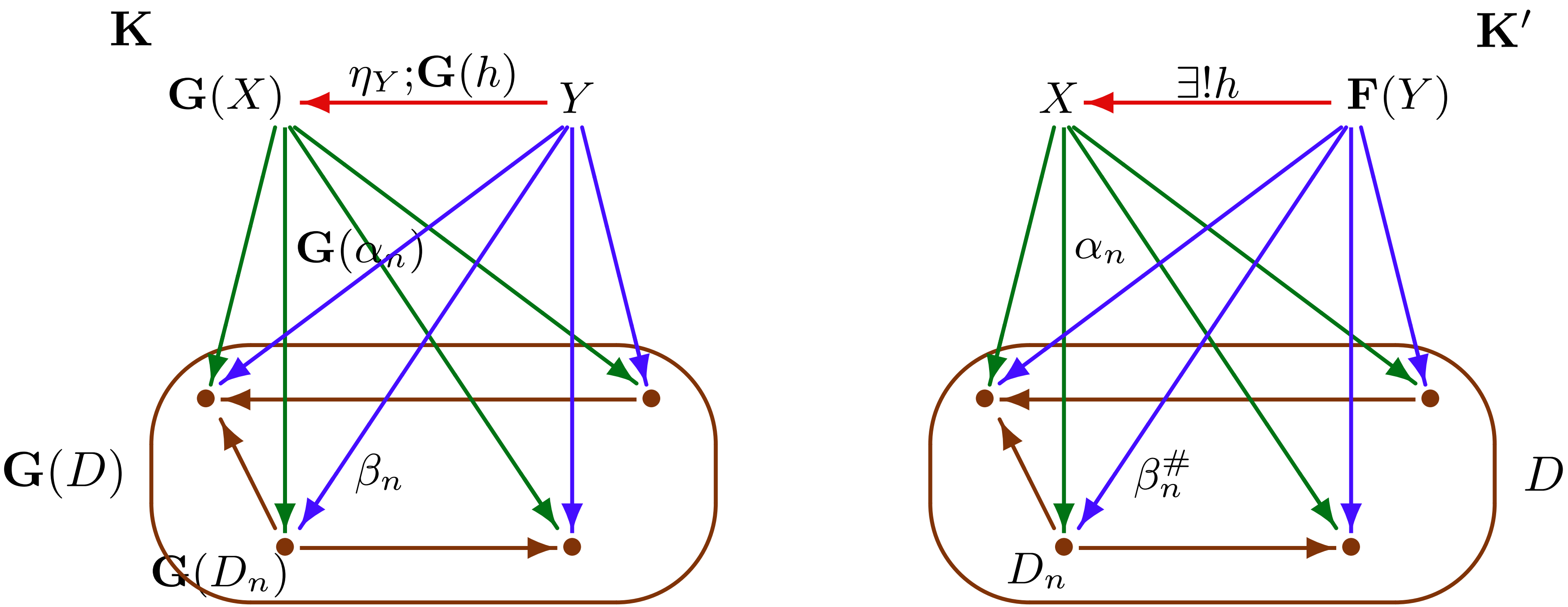

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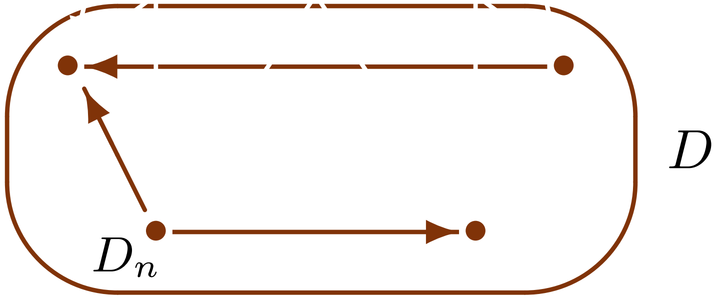

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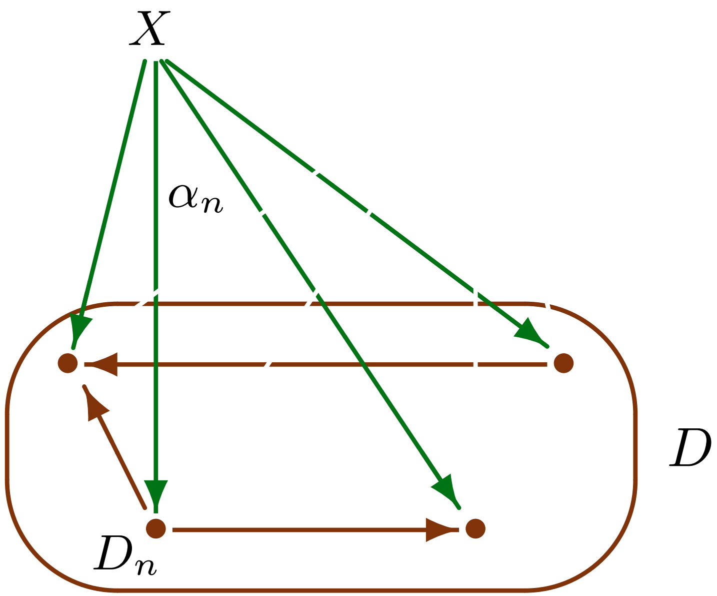

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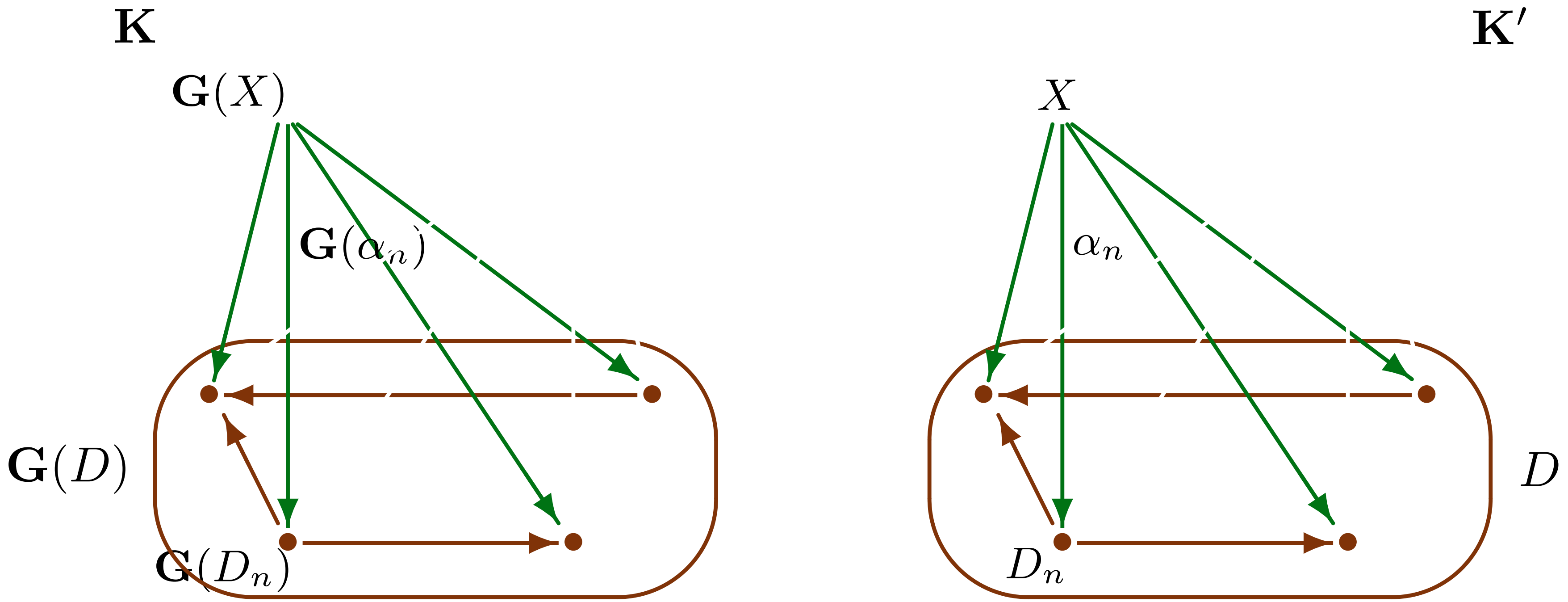

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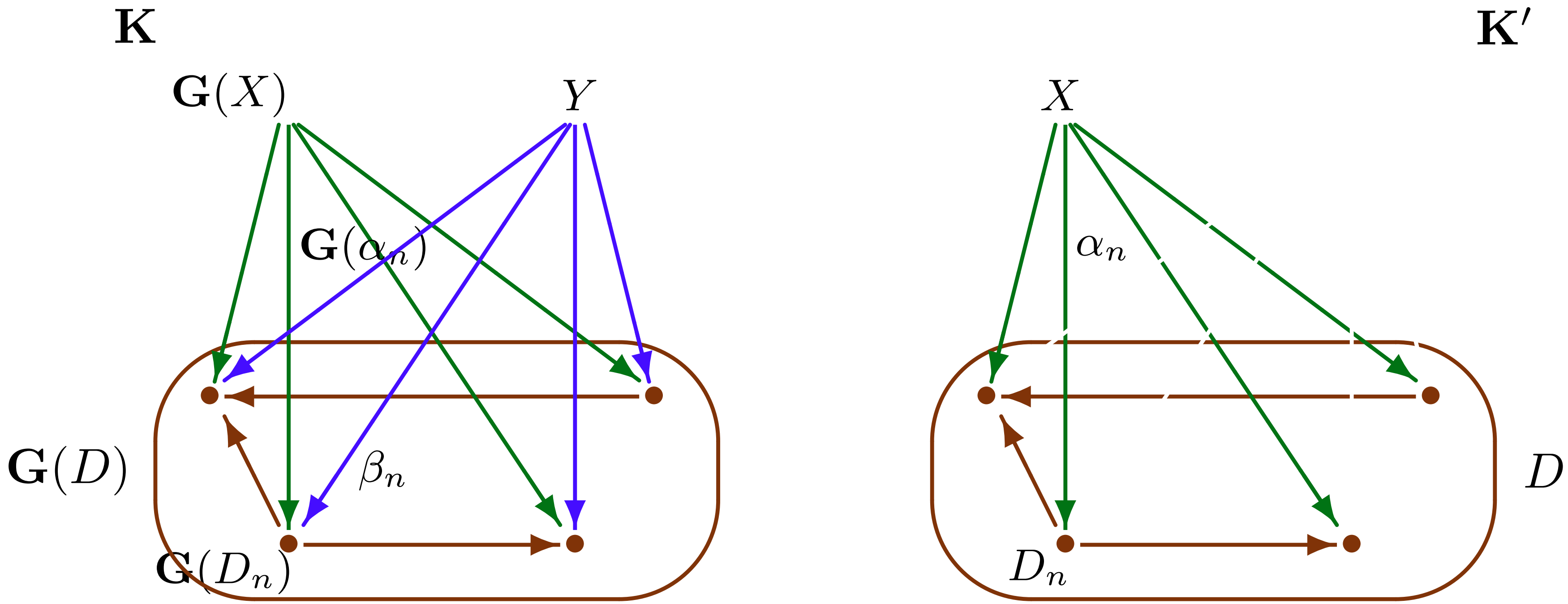

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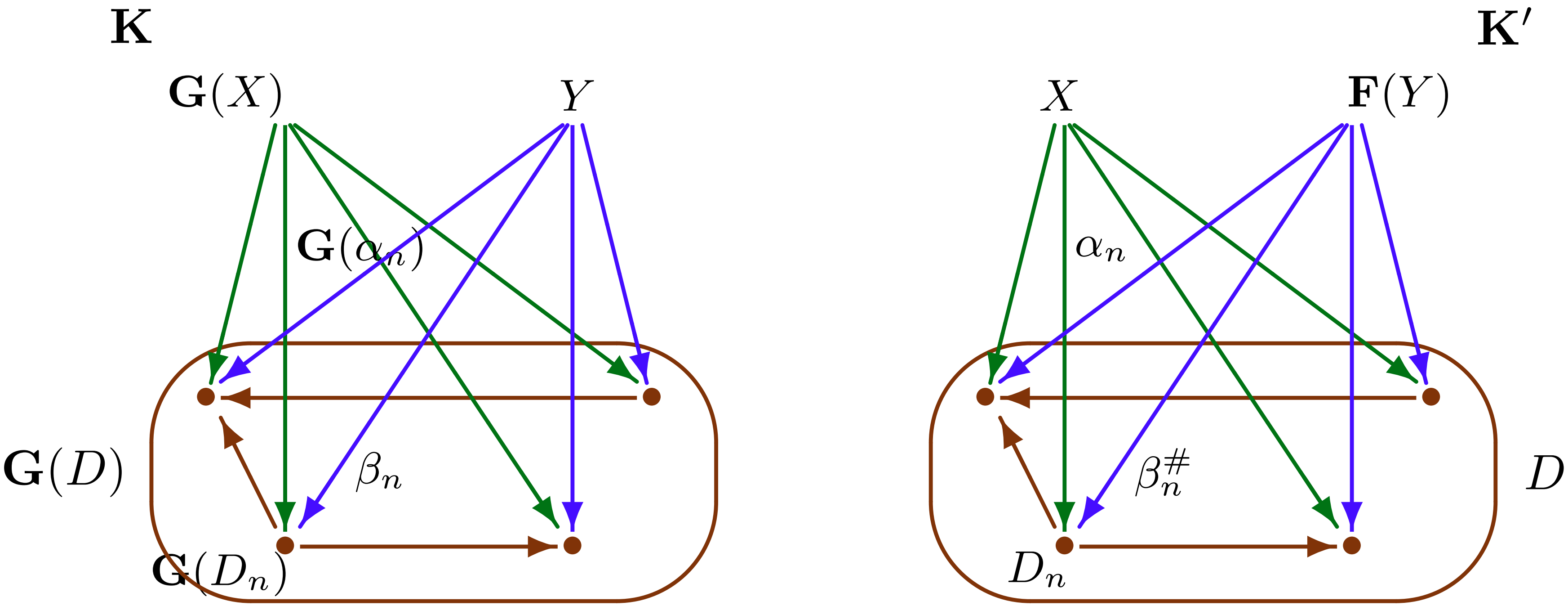

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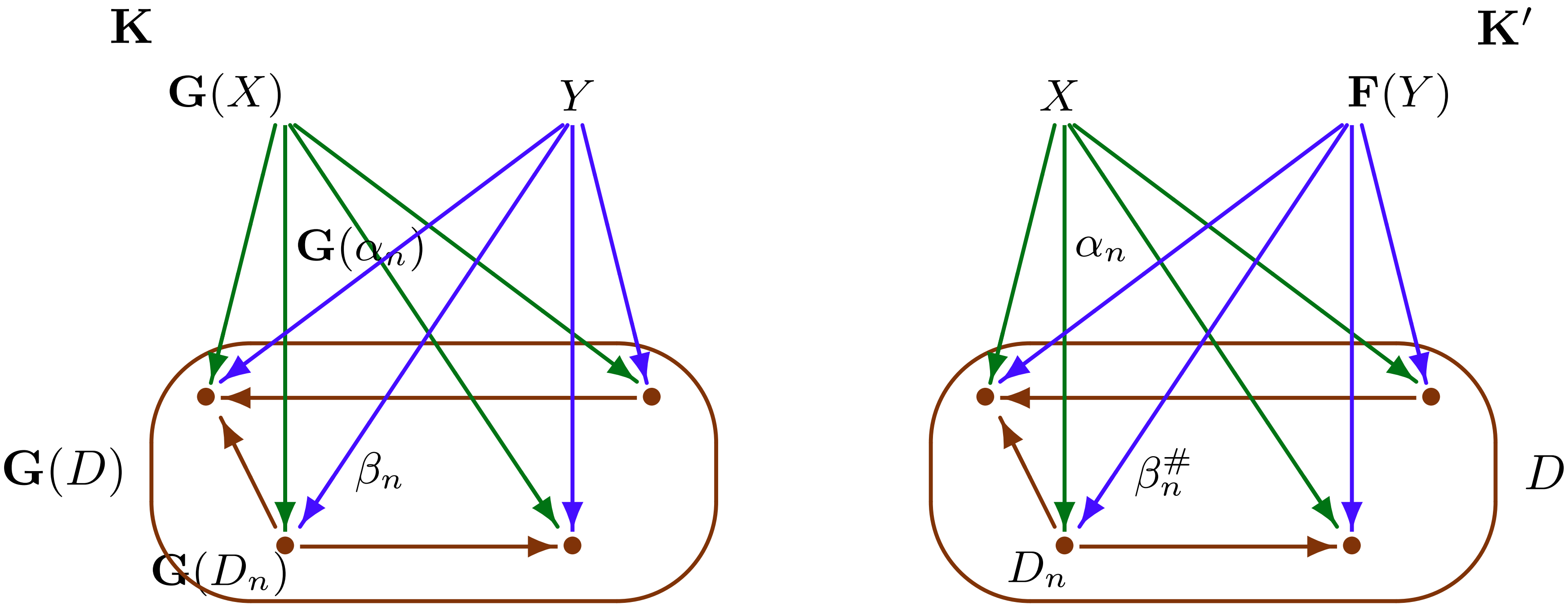

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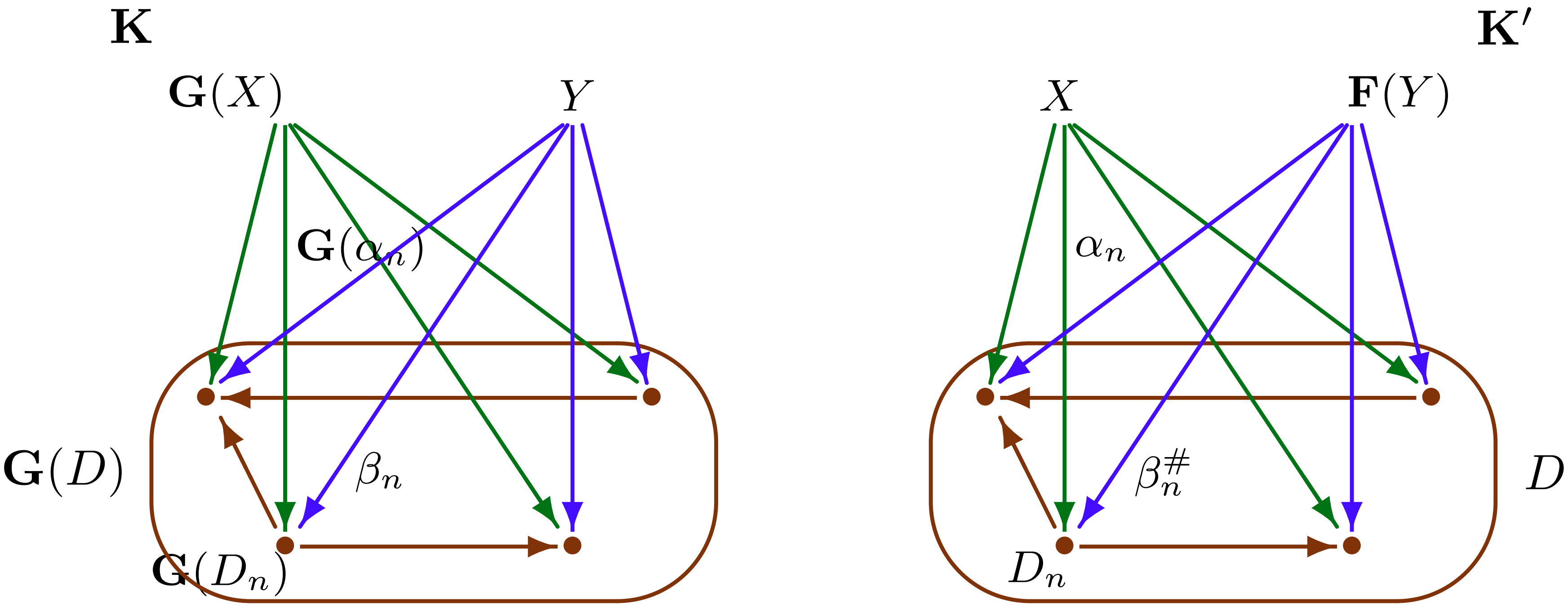

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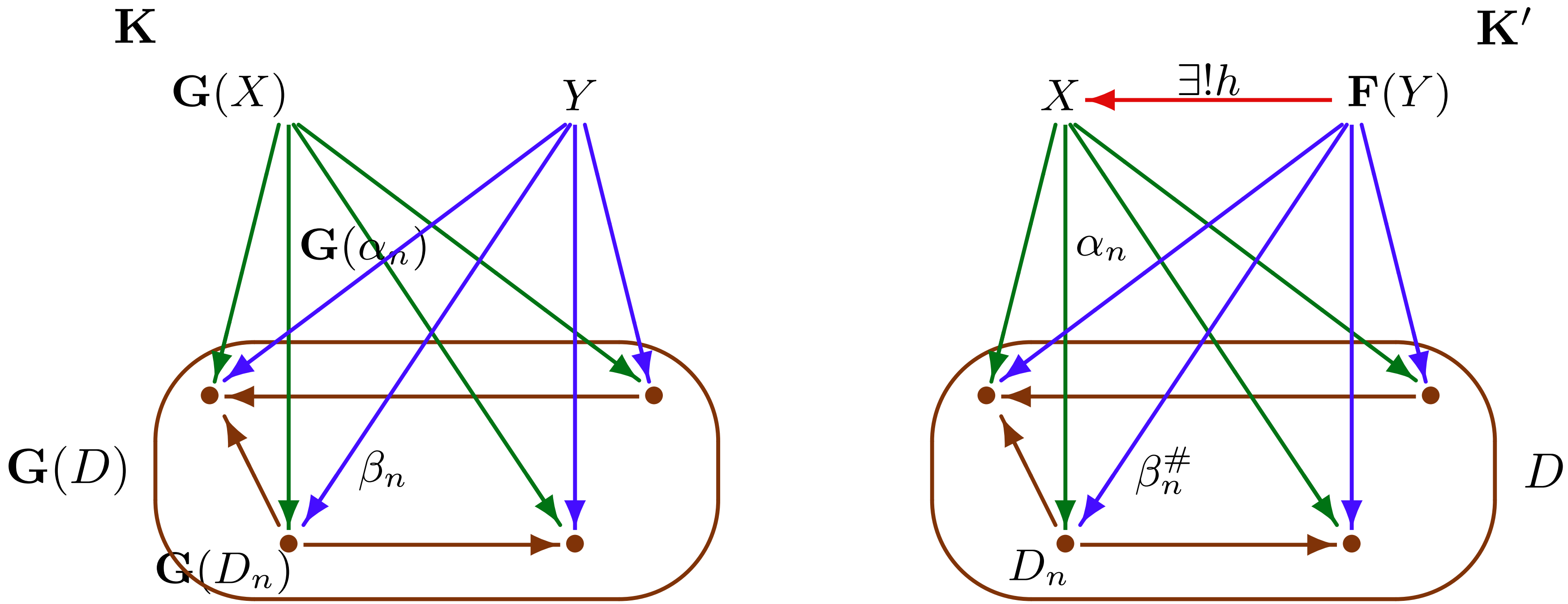

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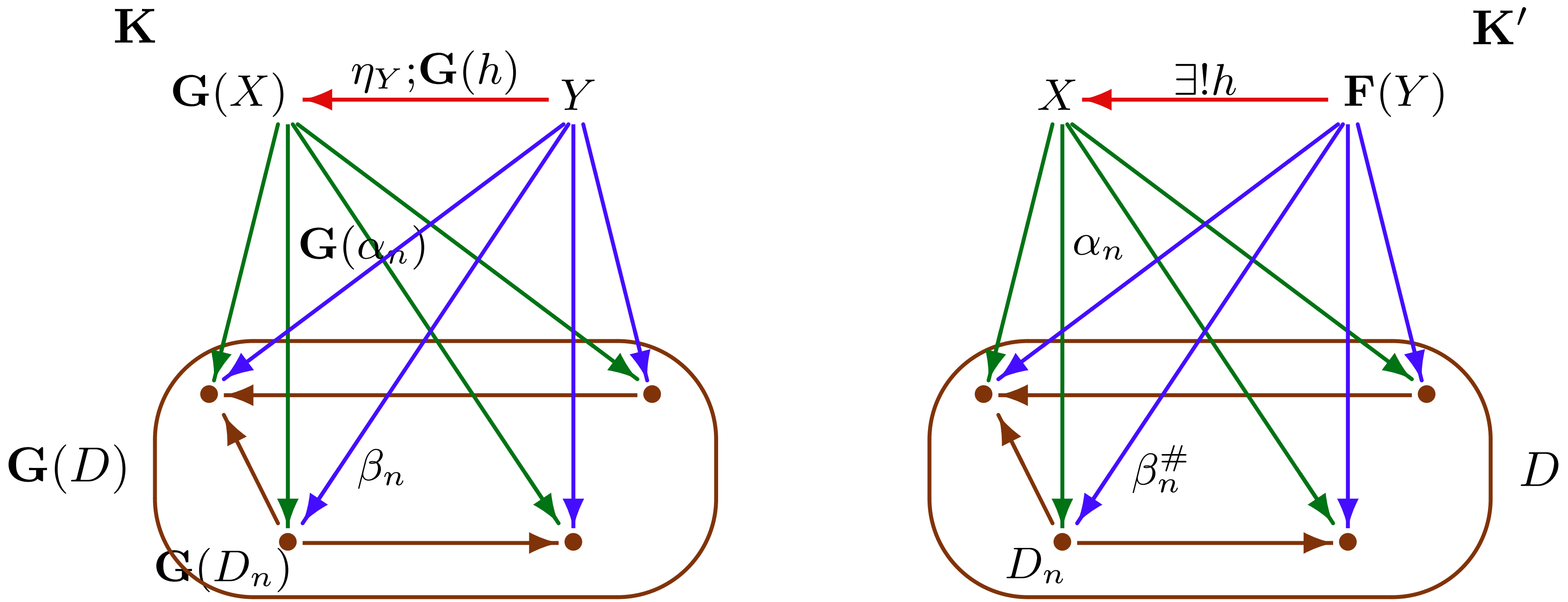

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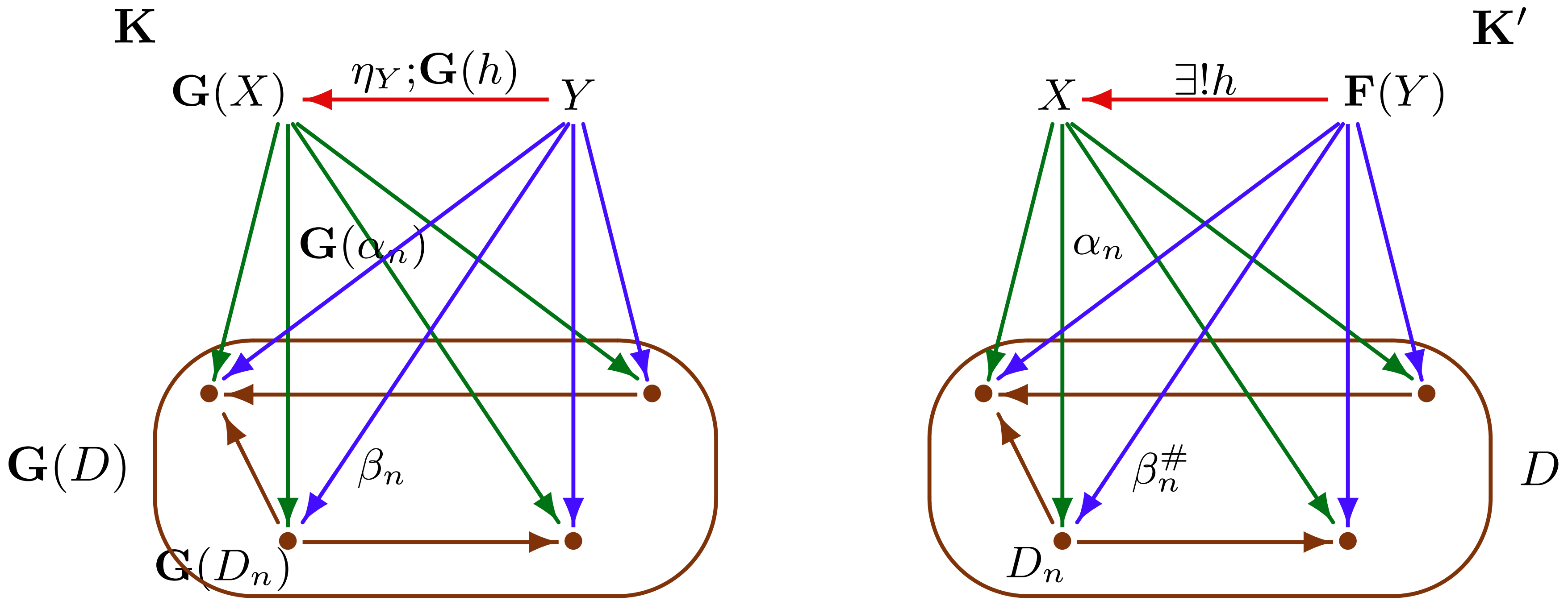

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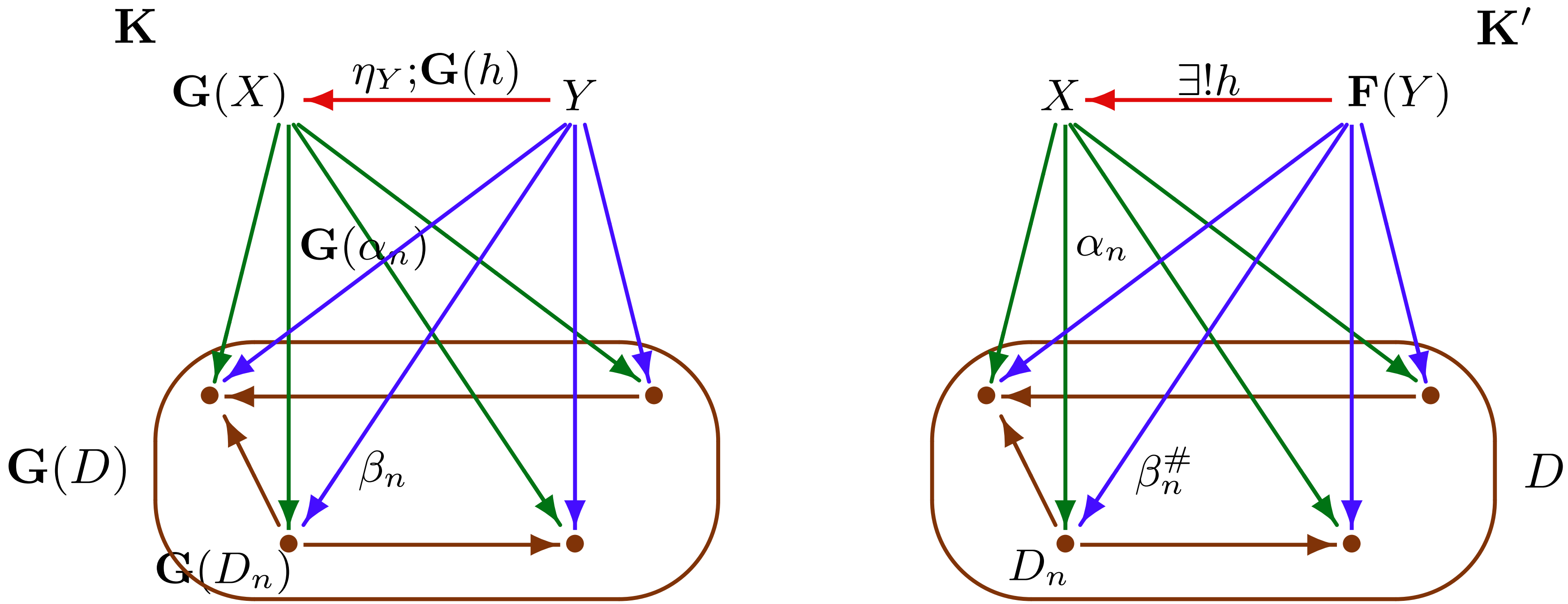

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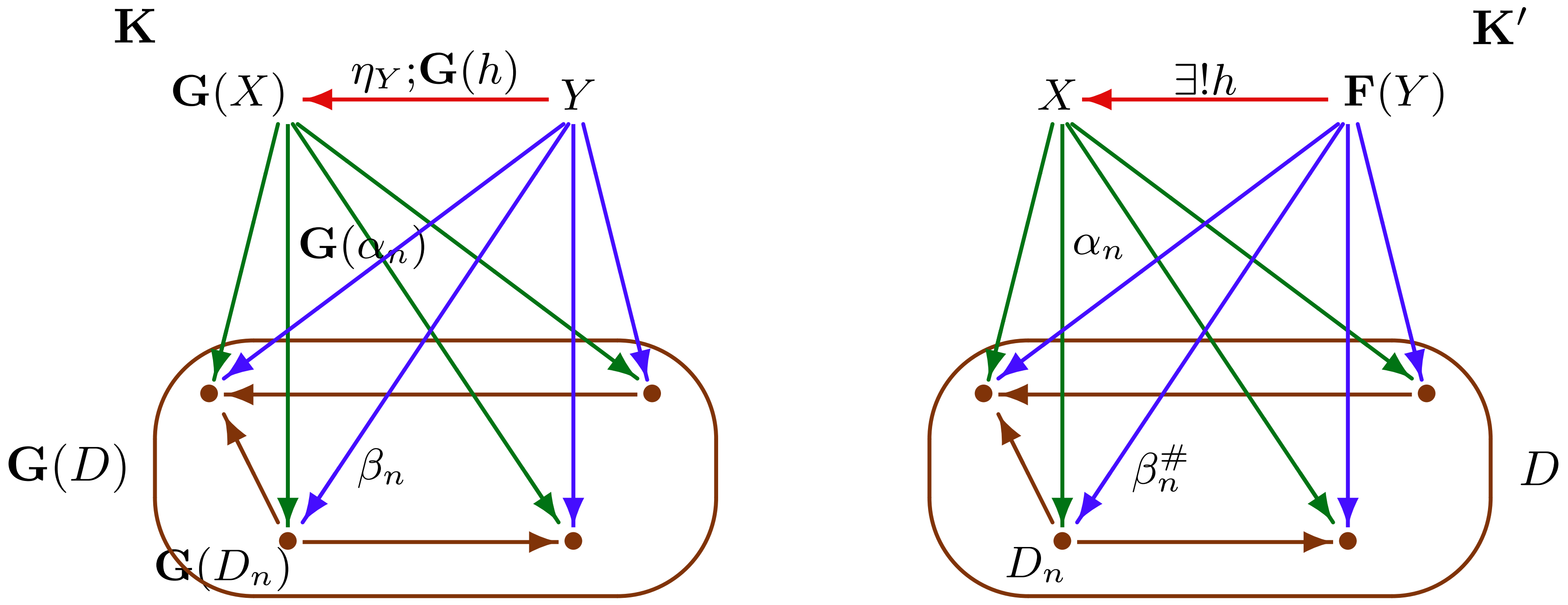

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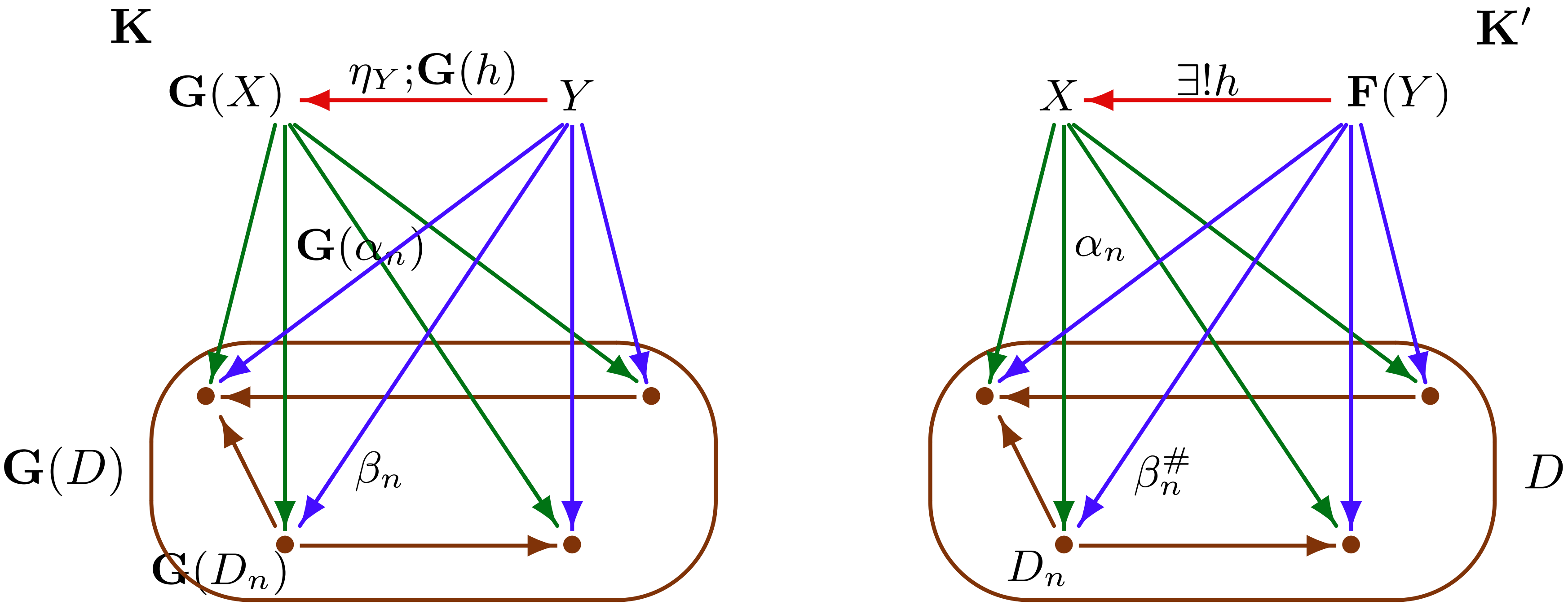

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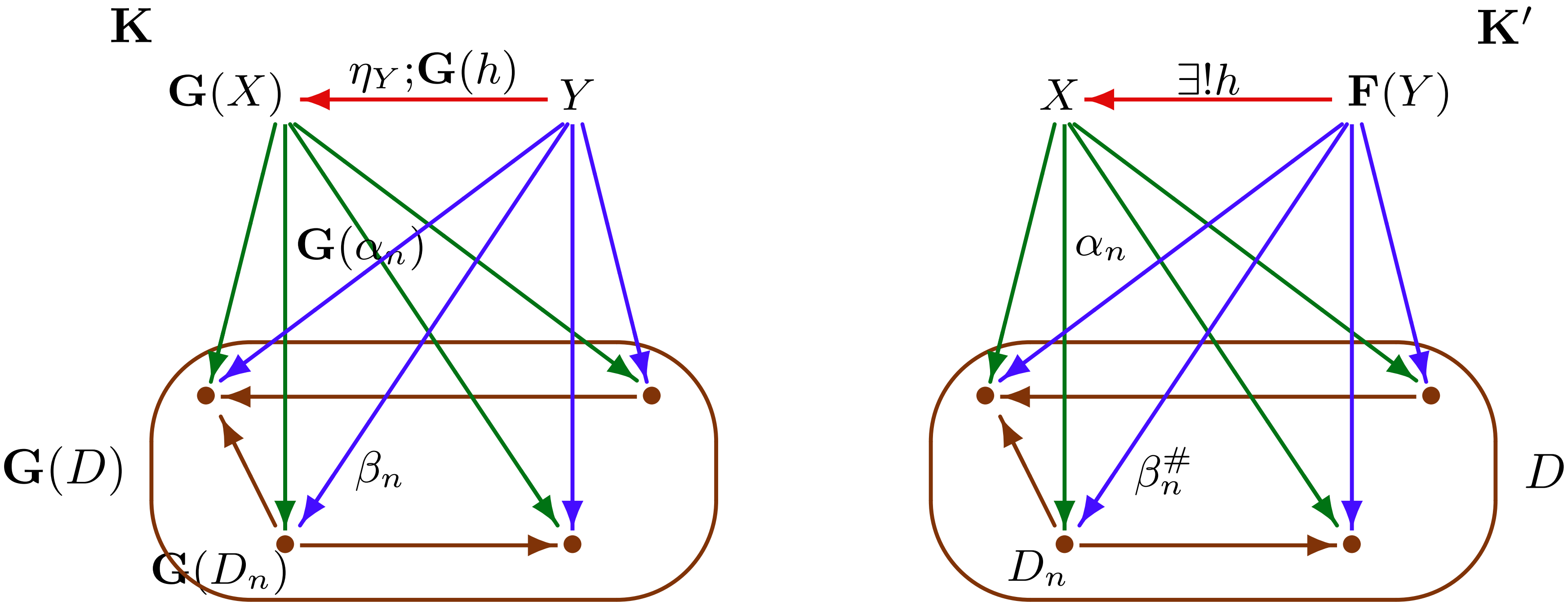

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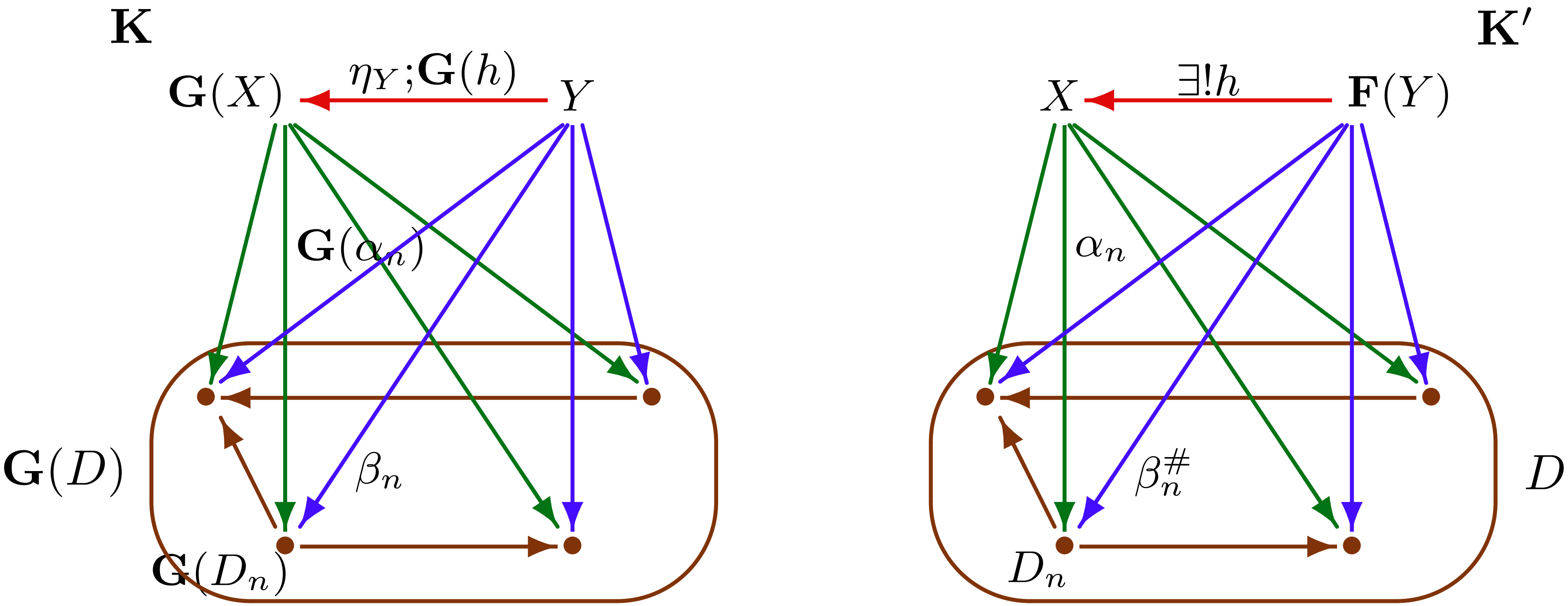

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Left adjoints and limits

Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ with unit $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.

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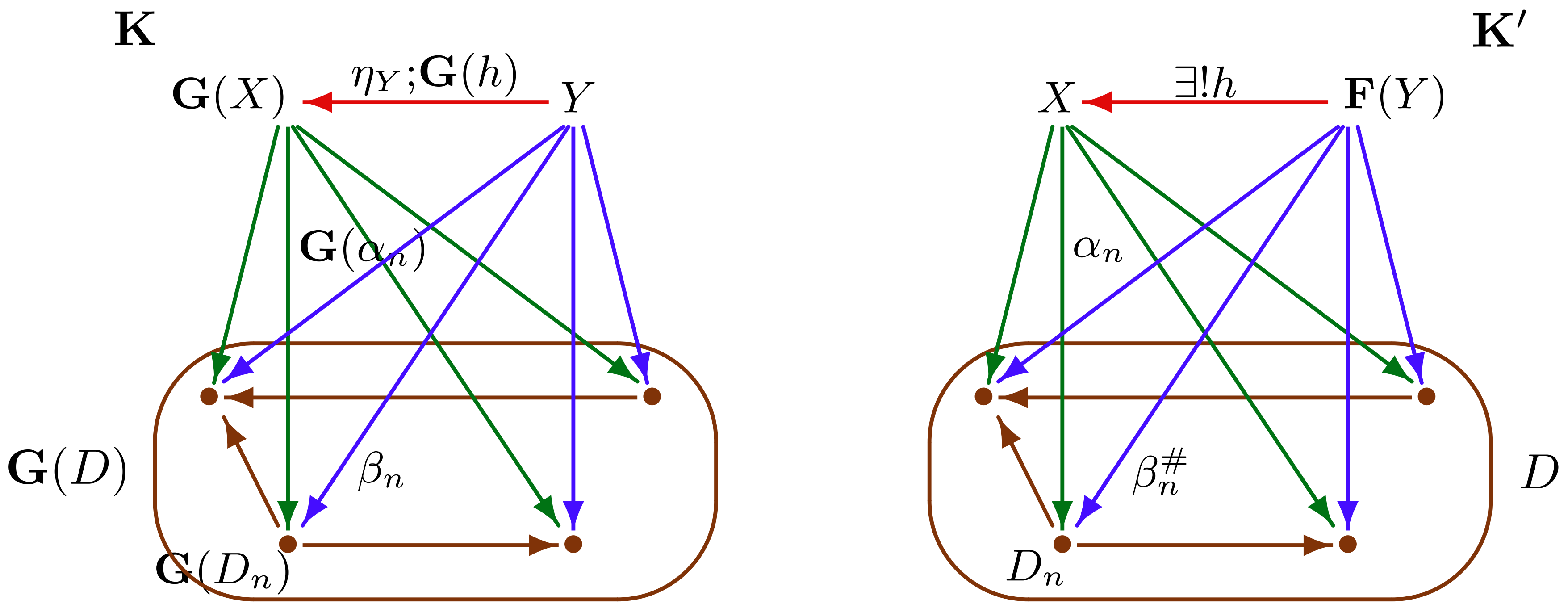

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"$\Leftarrow$": It is enough to show that for each $A \in |\mathbf{K}|$ the comma category $(\mathbf{C}_A, \mathbf{G})$ has an initial object.

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2. *for each $A \in |\mathbf{K}|$ there exists a set $\{f_i\colon A \to \mathbf{G}(X_i) \mid i \in \mathcal{I}\}$ (of objects $X_i \in |\mathbf{K}'|$ with morphisms $f_i\colon A \to \mathbf{G}(X_i)$, $i \in \mathcal{I}$) such that for each $B \in |\mathbf{K}'|$ and $h\colon A \to \mathbf{G}(B)$, for some $f\colon X_i \to B$, $i \in \mathcal{I}$, we have $h = f_i;\mathbf{G}(f)$.*

Proof:

"$\Rightarrow$": Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}$ with unit $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$. Then 1. follows by the previous fact, and for 2. just put $\mathcal{I} = \{*\}$, $X_* = \mathbf{F}(A)$, and $f_* = \eta_A\colon A \to \mathbf{G}(\mathbf{F}(A))$

"$\Leftarrow$": It is enough to show that for each $A \in |\mathbf{K}|$ the comma category $(\mathbf{C}_A, \mathbf{G})$ has an initial object. Under our assumptions, $(\mathbf{C}_A, \mathbf{G})$ is complete. The rest follows by the next fact.

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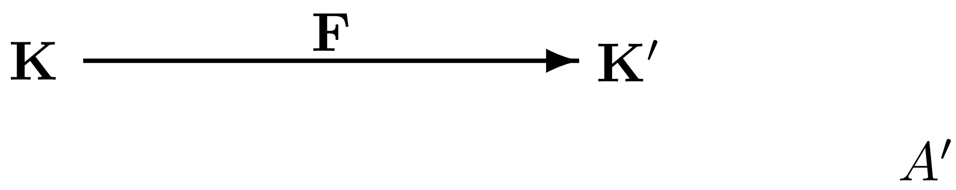

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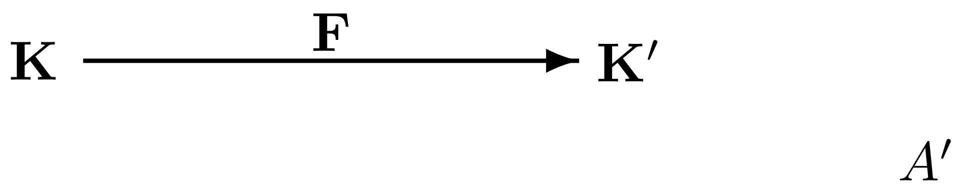

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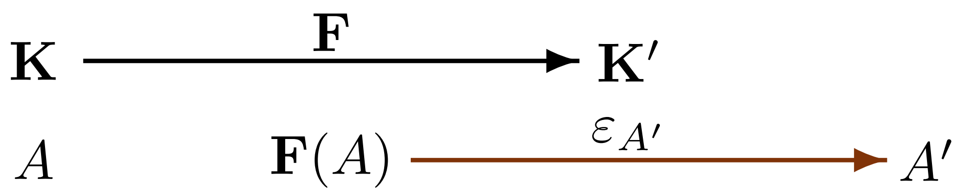

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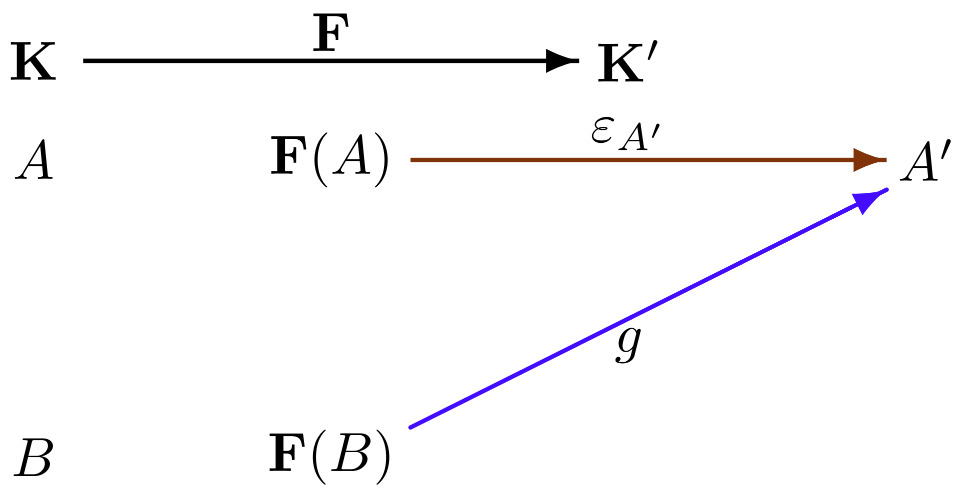

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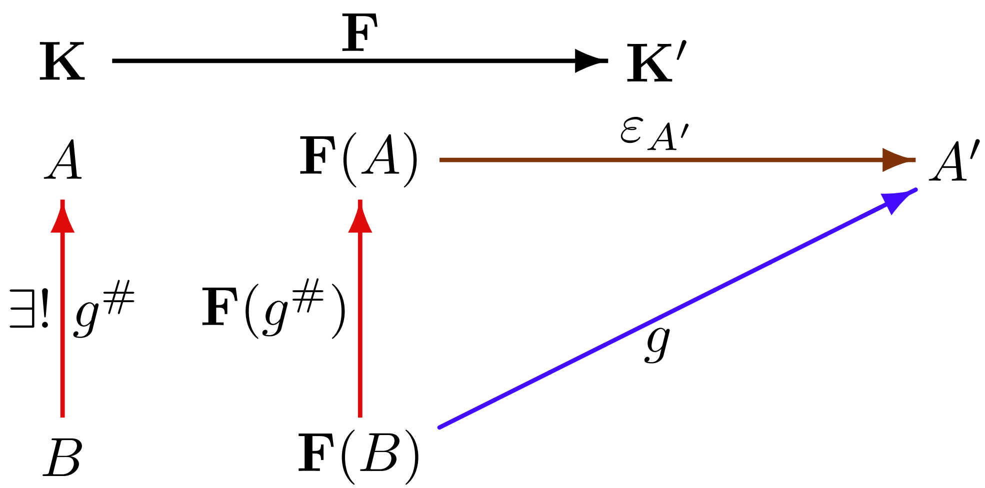

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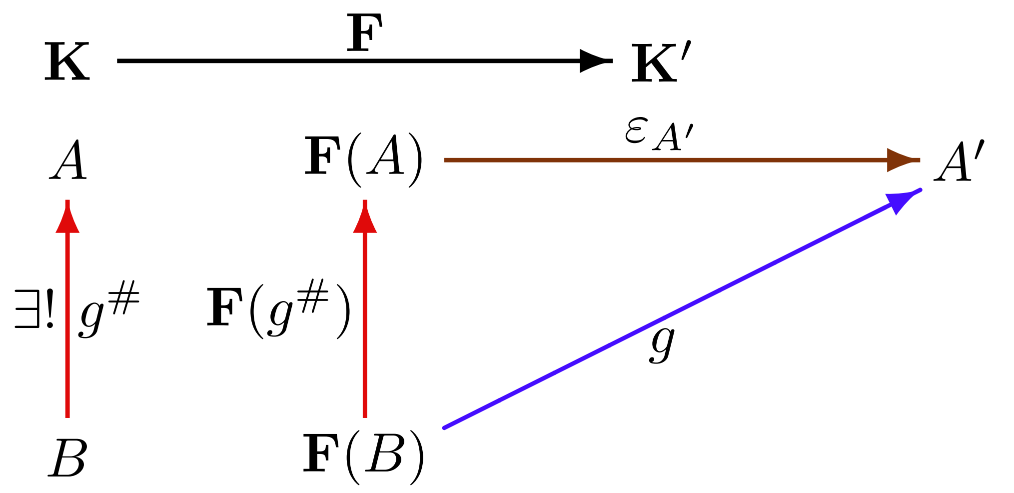
Paradigmatic example:

Function spaces, coming soon



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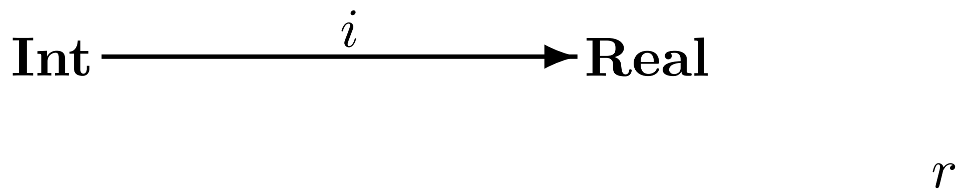

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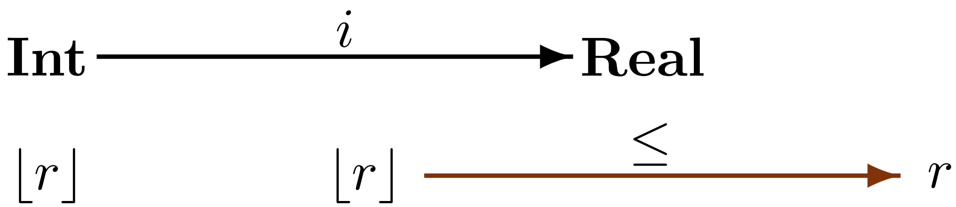

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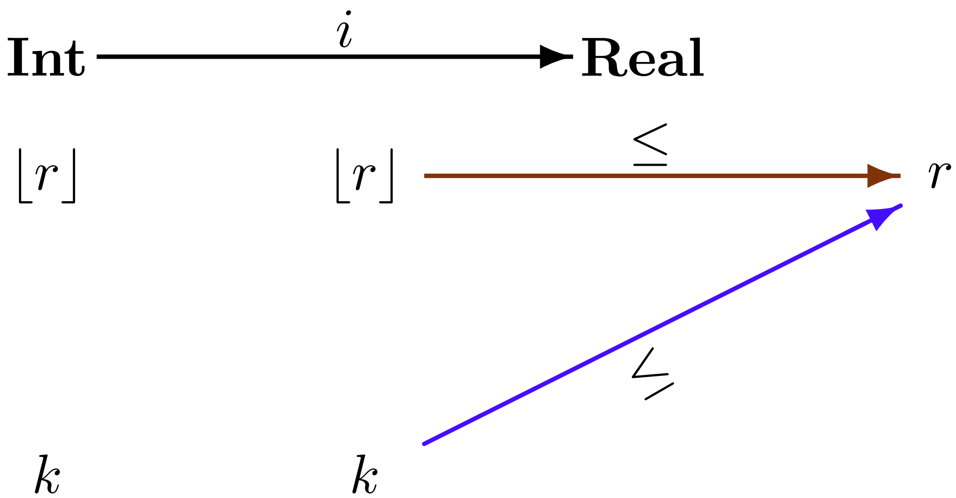

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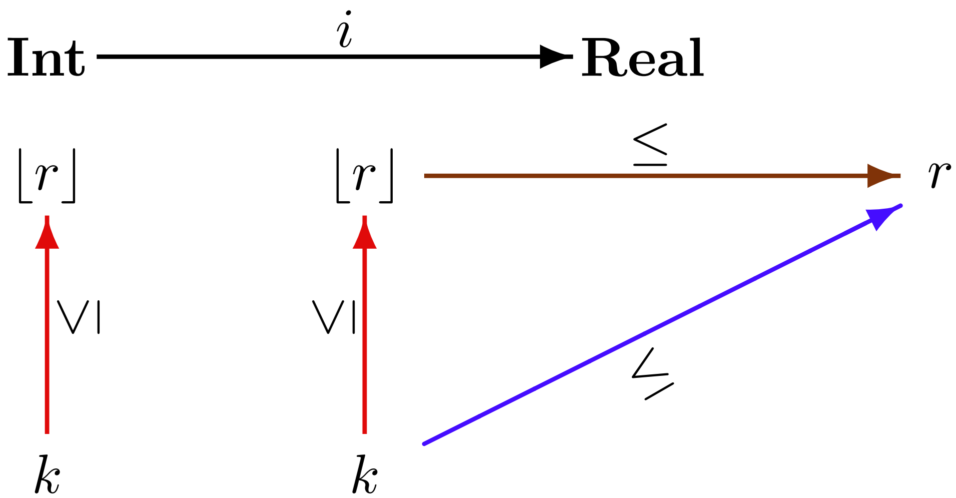

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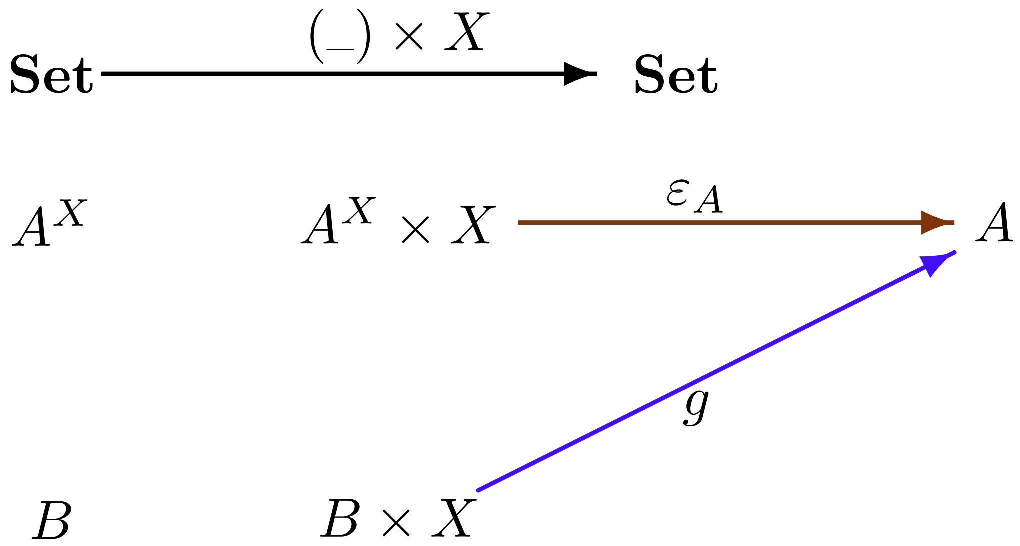
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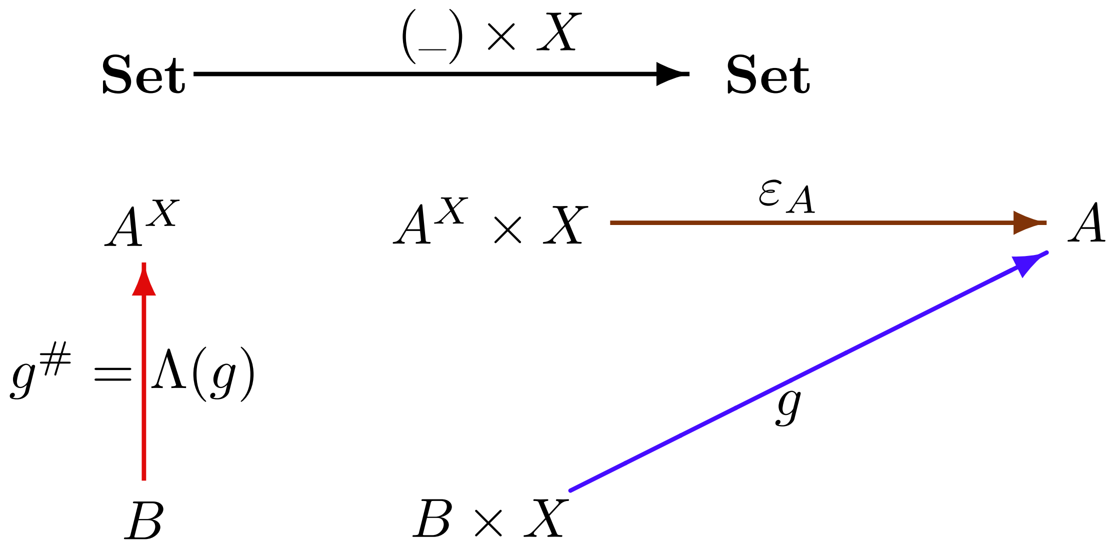

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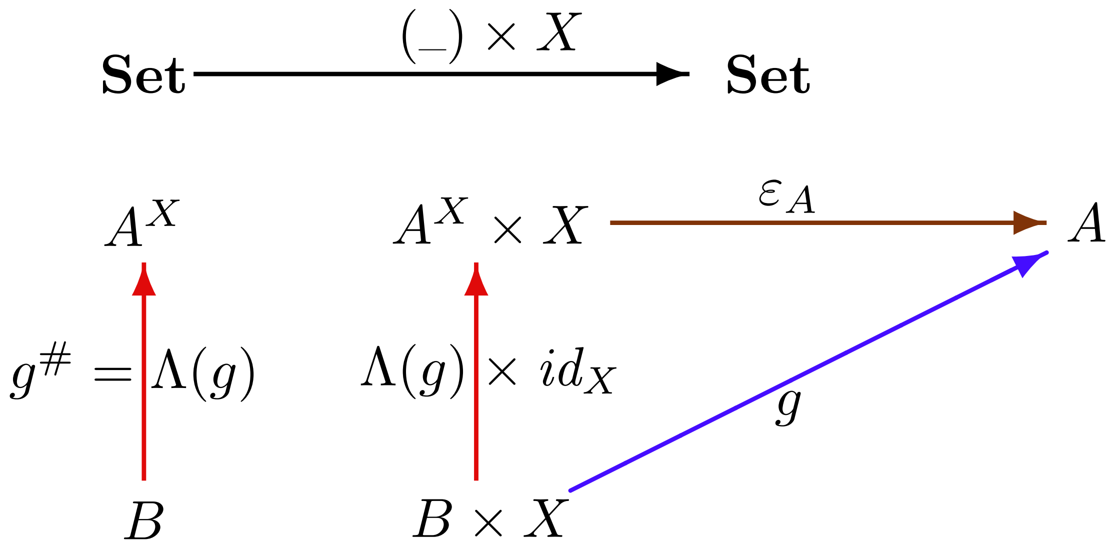

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Examples

- Consider inclusion $i\colon \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the floor of r, $\lfloor r \rfloor \in \mathbf{Int}$ is cofree under r w.r.t. i.

  What about cofree objects w.r.t. the inclusion of rationals into reals?

- Fix a set $X \in |\mathbf{Set}|$. Consider functor $\mathbf{F}_X\colon \mathbf{Set} \to \mathbf{Set}$ defined by:
  - for any set $A \in |\mathbf{Set}|$, $\mathbf{F}_X(A) = A \times X$
  - for any function $f\colon A \to B$, $\mathbf{F}_X(f)\colon A \times X \to B \times X$ is a function given by $\mathbf{F}_X(f)(\langle a, x\rangle) = \langle f(a), x\rangle$.

  Then for any set $A \in |\mathbf{Set}|$, the powerset $A^X \in |\mathbf{Set}|$ (i.e., the set of all functions from X to A) is a cofree objects under A w.r.t. $\mathbf{F}_X$. The counit morphism $\varepsilon_A\colon \mathbf{F}_X(A^X) = A^X \times X \to A$ is the evaluation function: $\varepsilon_A(\langle f, x\rangle) = f(x)$.

  A generalisation to deal with exponential objects will (not) be discussed later

Facts

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Spell this out for terminal objects, products, equalisers, and pullbacks

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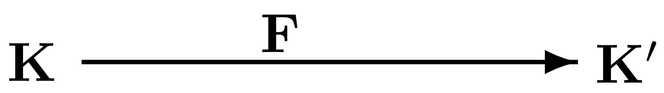

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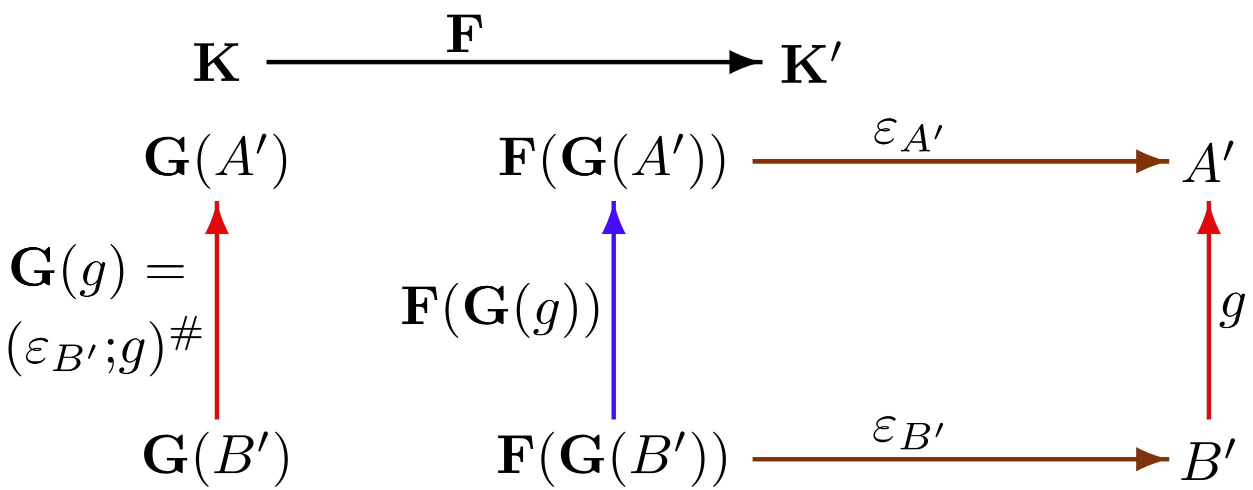

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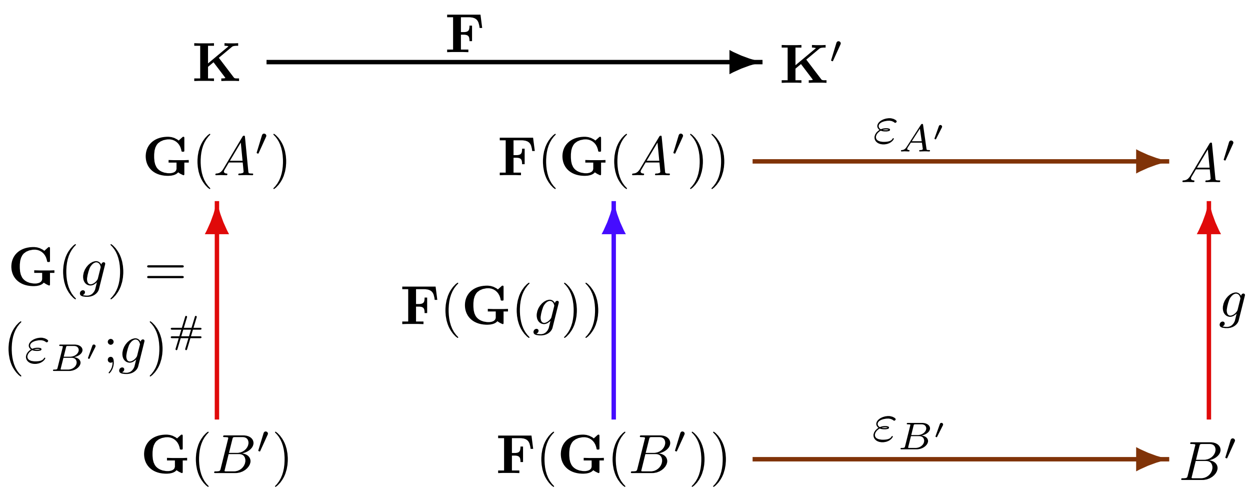
form a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$. Moreover, $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ is a natural transformation.



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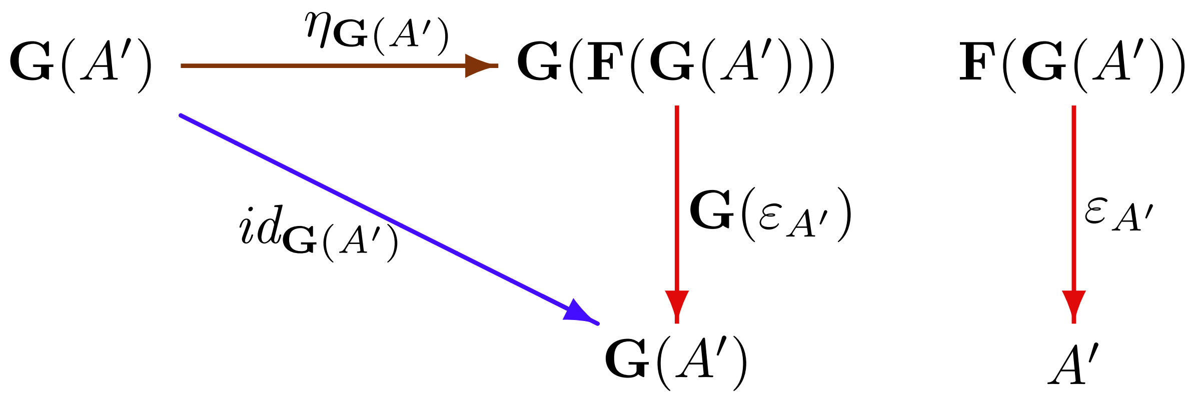

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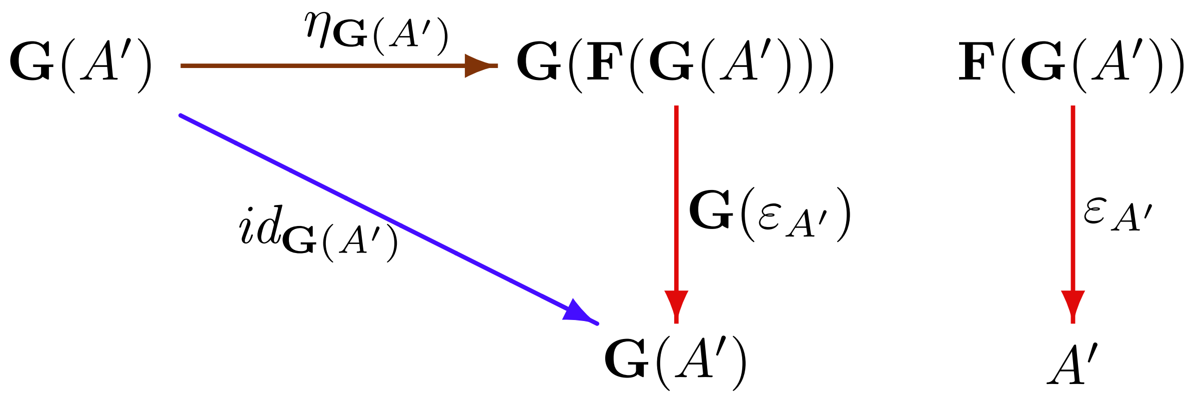

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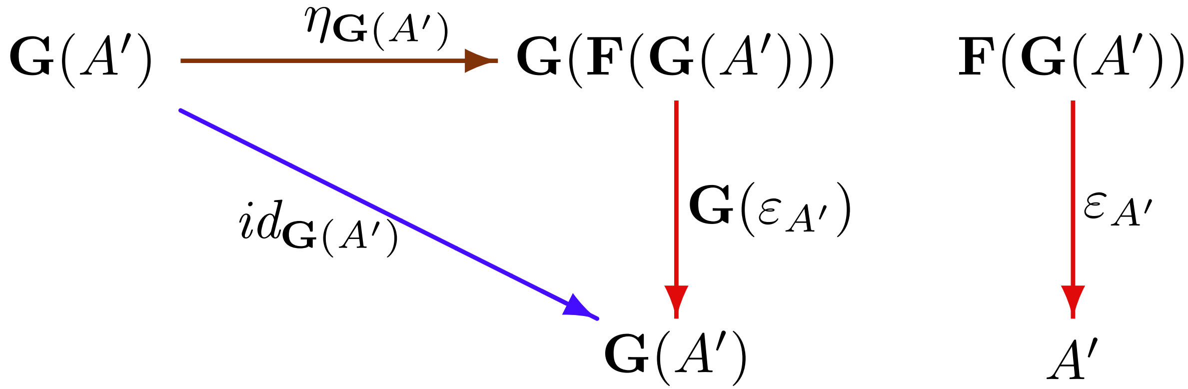

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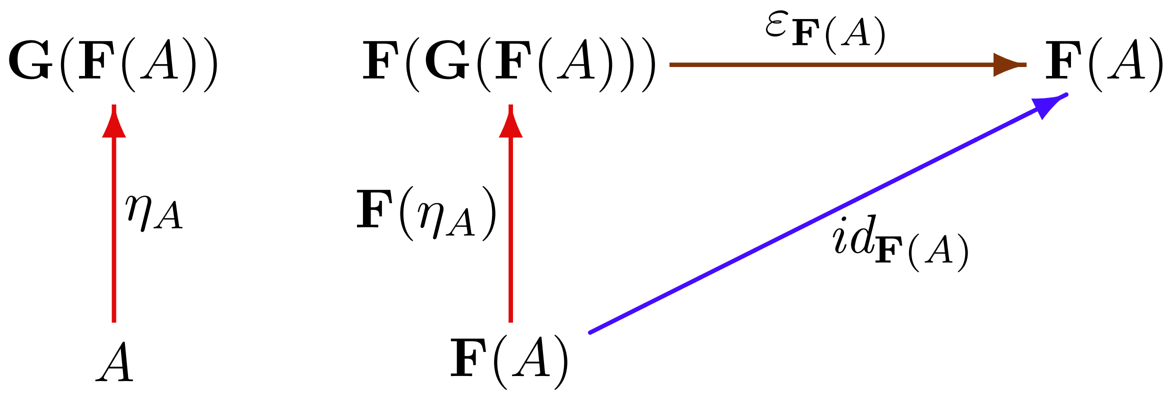

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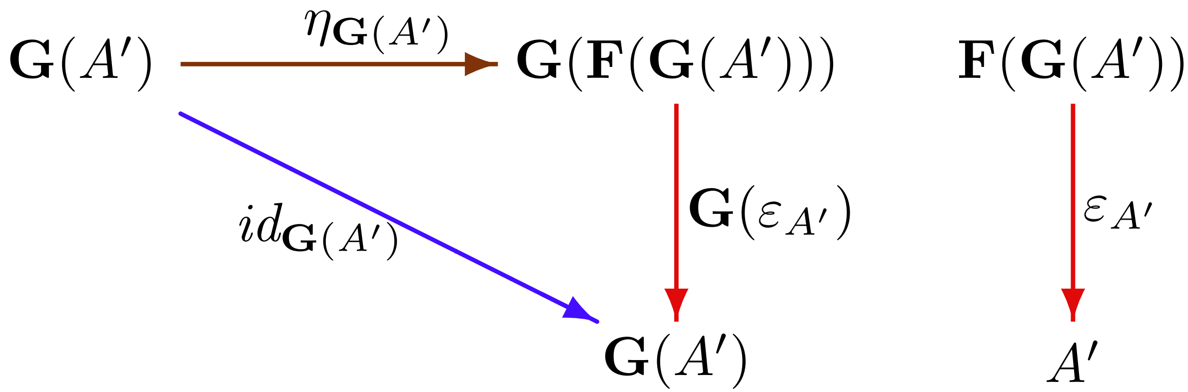

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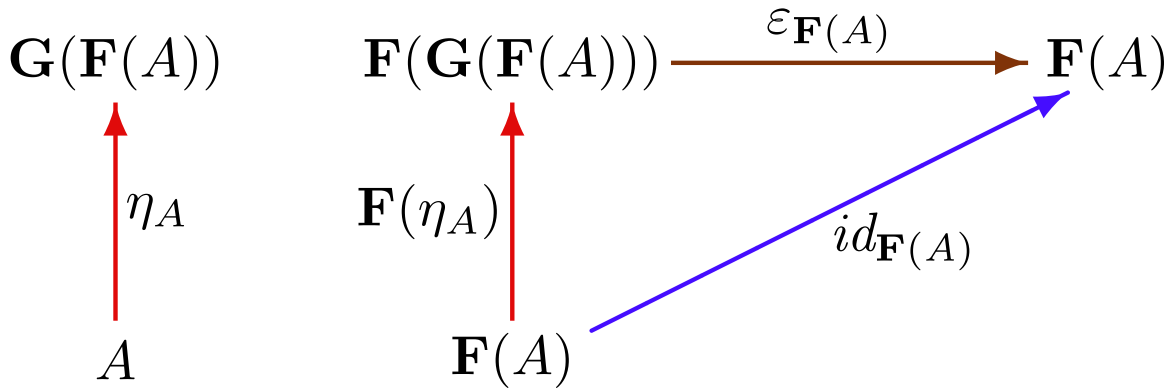

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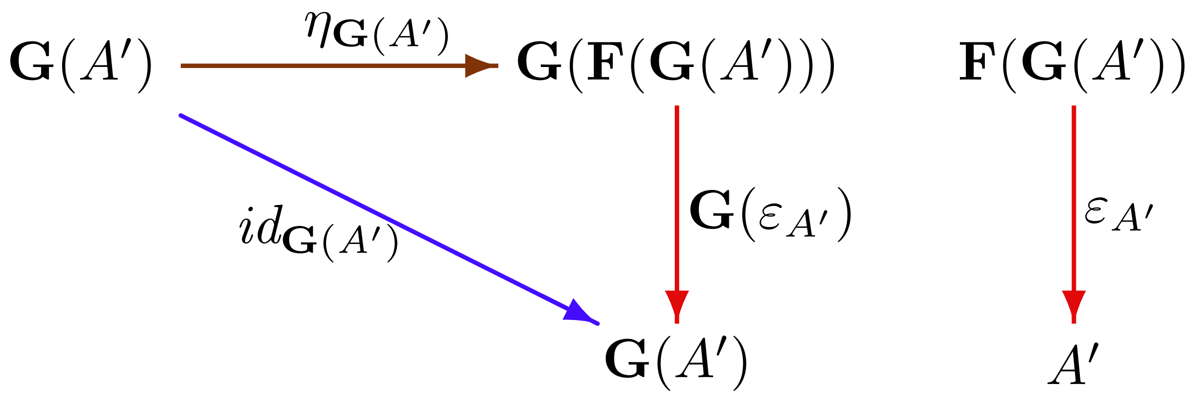

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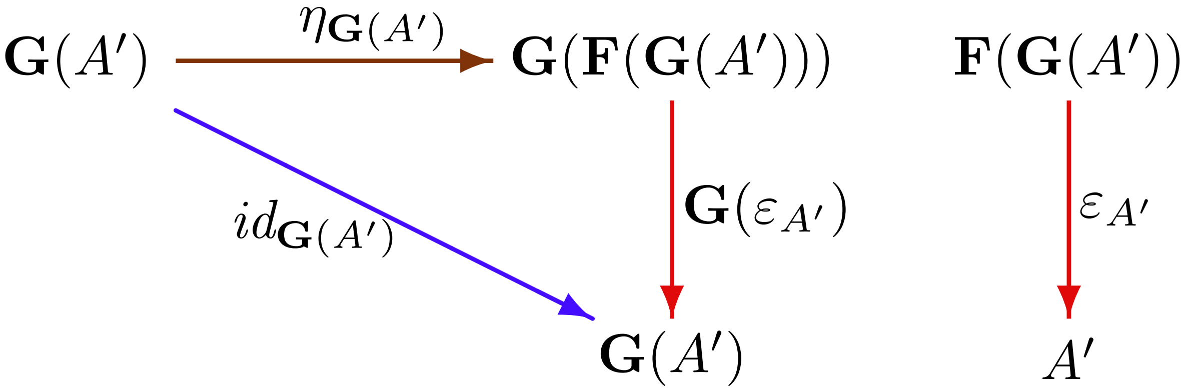

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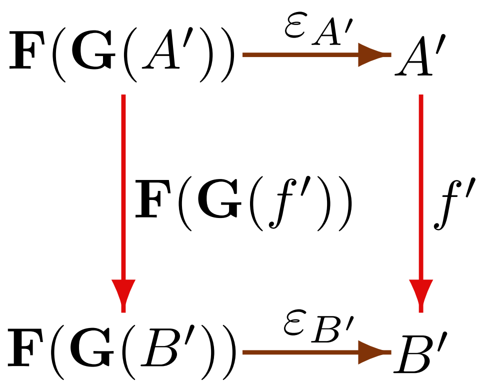

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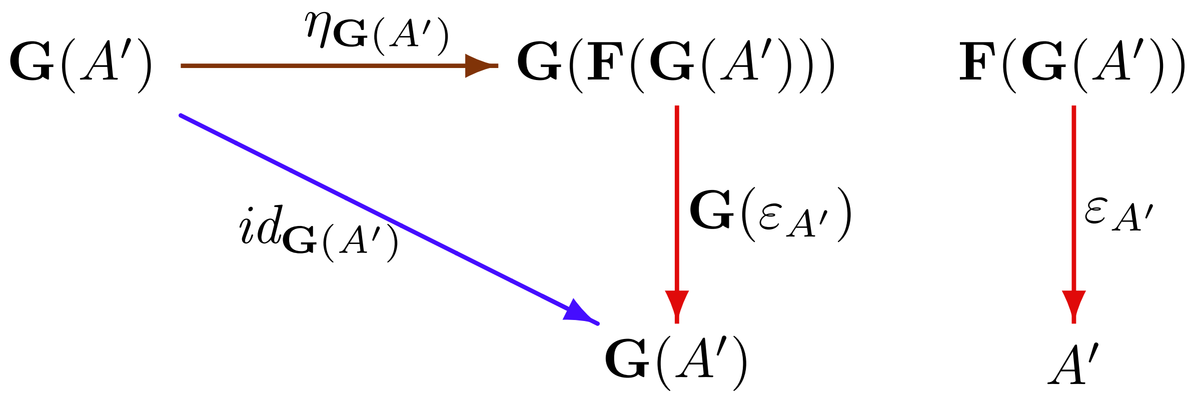

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This holds since $\eta_{\mathbf{G}(A')};\mathbf{G}(\varepsilon_{A'};f') = (\eta_{\mathbf{G}(A')};\mathbf{G}(\varepsilon_{A'}));\mathbf{G}(f') = \mathbf{G}(f')$

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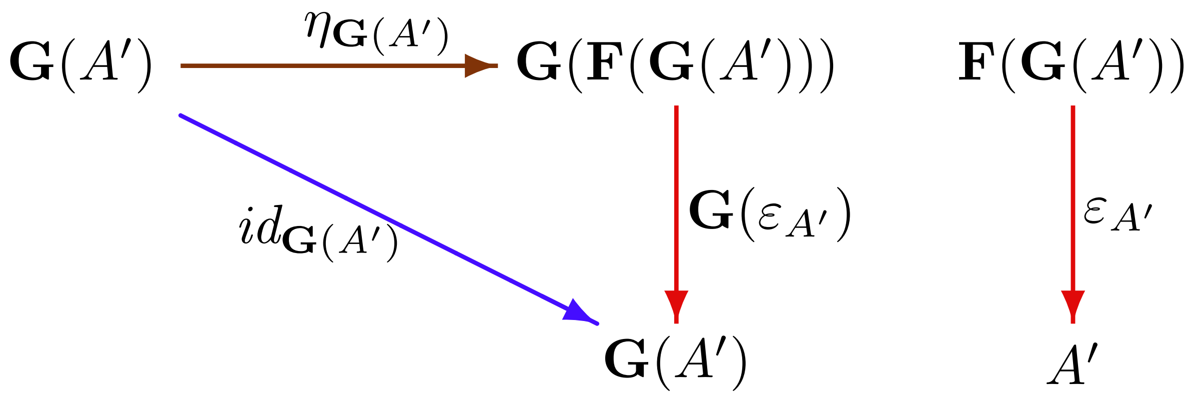

$\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *is indeed natural, i.e. for* $f'\colon A' \to B'$, $\varepsilon_{A'};f' = \mathbf{F}(\mathbf{G}(f'));\varepsilon_{B'}$.
This holds since $\eta_{\mathbf{G}(A')};\mathbf{G}(\varepsilon_{A'};f') = (\eta_{\mathbf{G}(A')};\mathbf{G}(\varepsilon_{A'}));\mathbf{G}(f') = \mathbf{G}(f')$ *and* $\eta_{\mathbf{G}(A')};\mathbf{G}(\mathbf{F}(\mathbf{G}(f'));\varepsilon_{B'}) = (\eta_{\mathbf{G}(A')};\mathbf{G}(\mathbf{F}(\mathbf{G}(f'))));\mathbf{G}(\varepsilon_{B'}) = (\mathbf{G}(f');\eta_{\mathbf{G}(B')});\mathbf{G}(\varepsilon_{B'}) = \mathbf{G}(f')$.

Proof (idea):
Put $\varepsilon_{A'} = (id_{\mathbf{G}(A')})^{\#}$.

From left adjoints to adjunctions

Theorem: *Let* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *be left adjoint to* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *with unit* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$. *Then there is a natural transformation* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *such that:*

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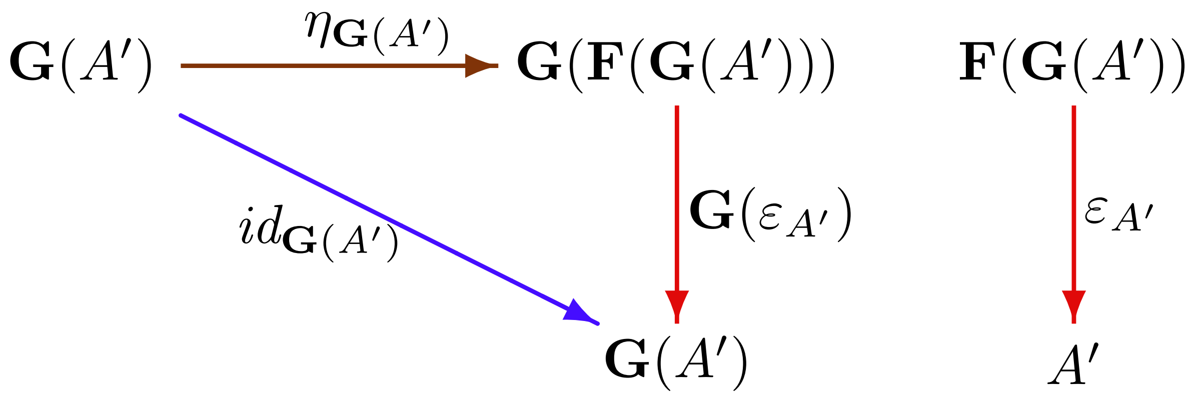

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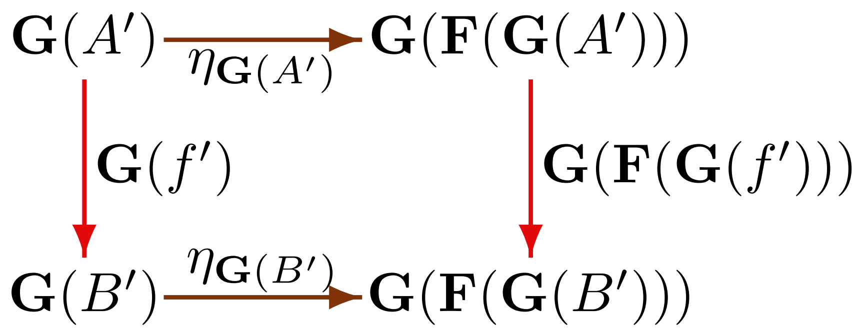

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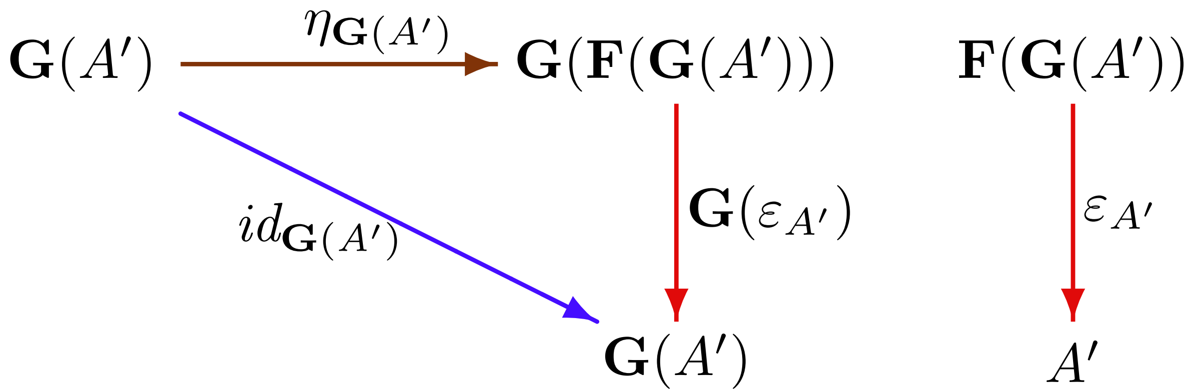
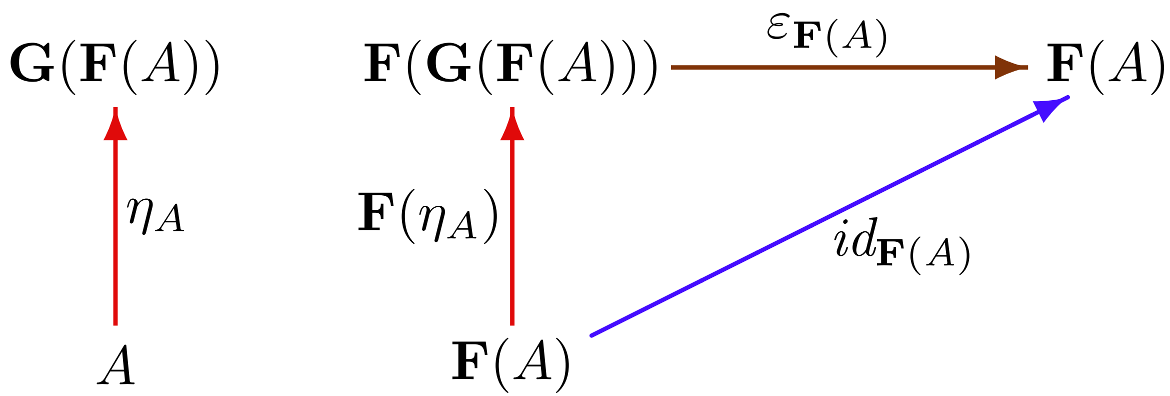


- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$



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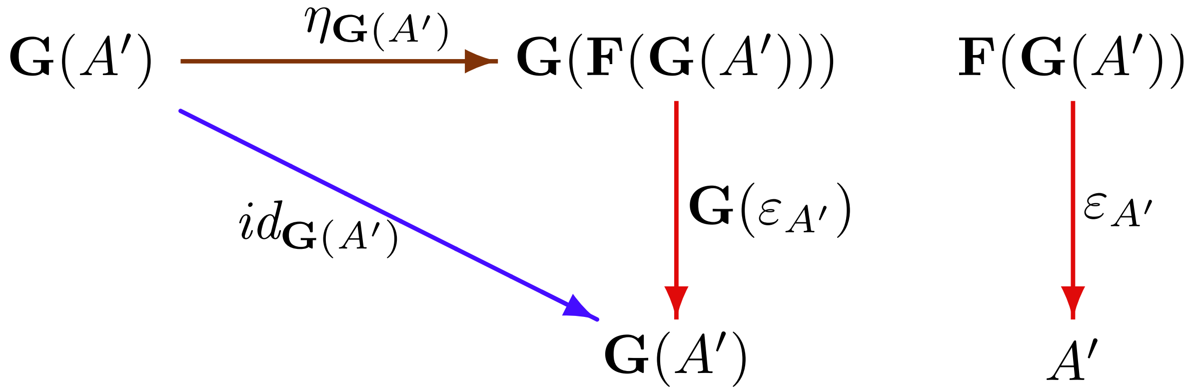

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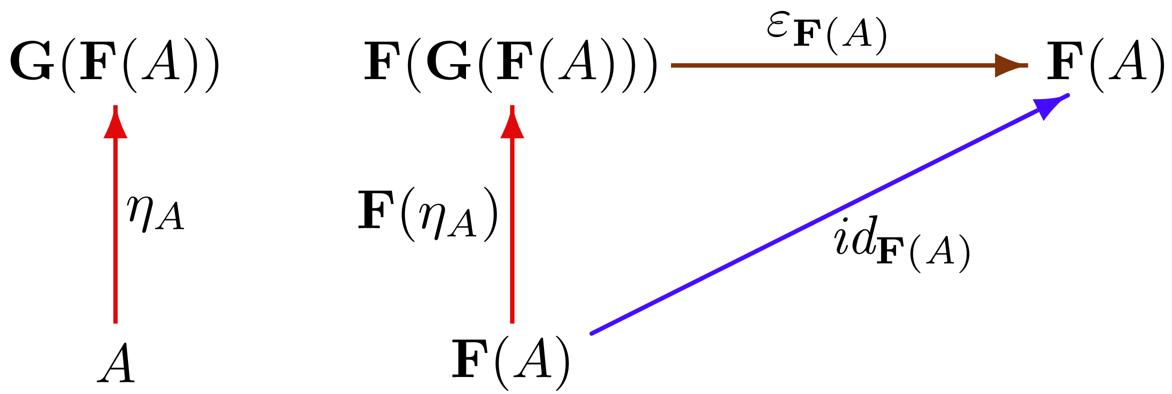

This holds since:

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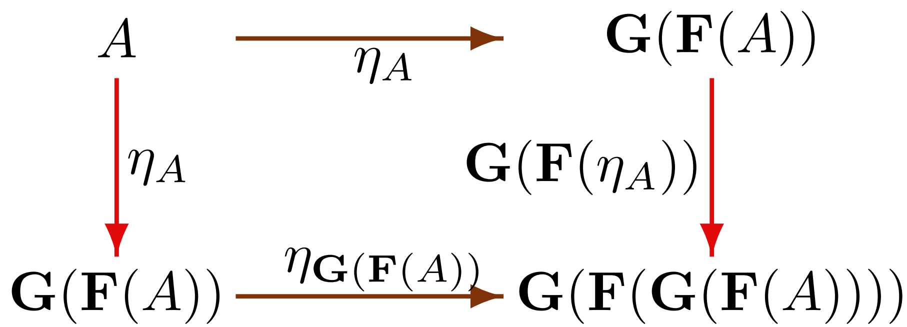

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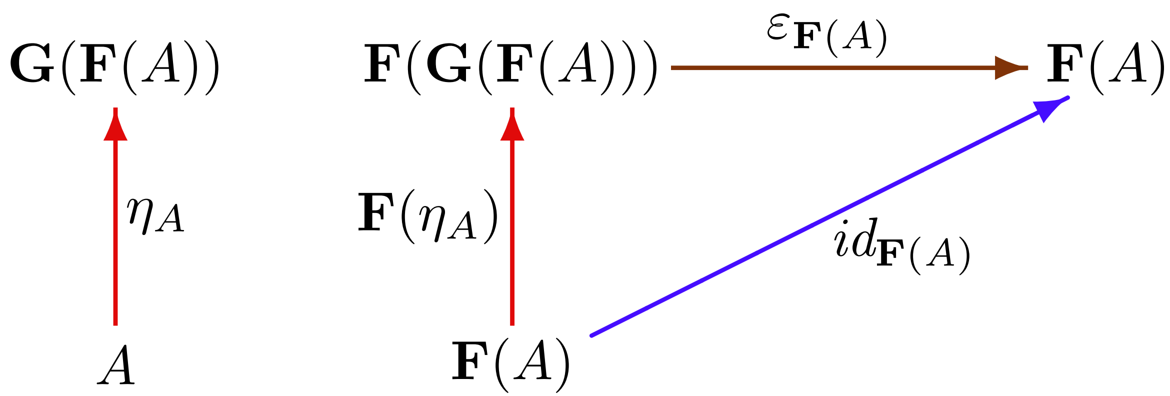

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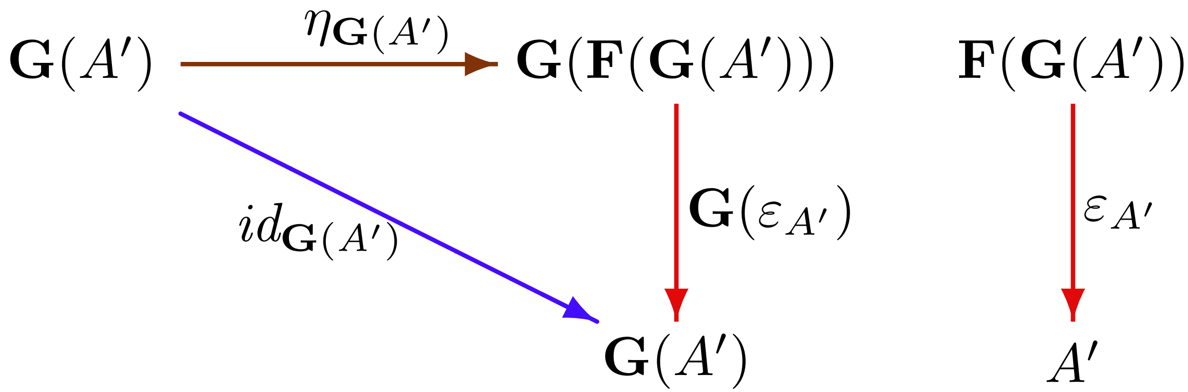

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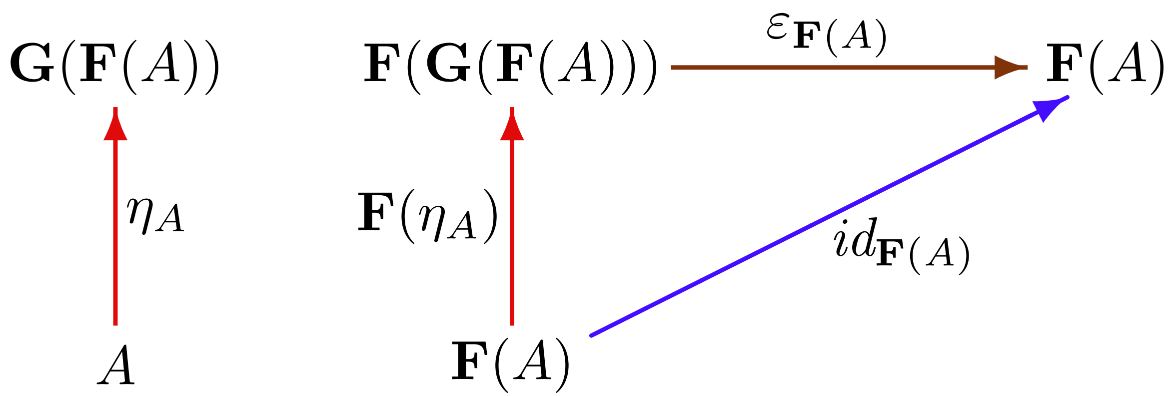

Proof (idea):
Put $\varepsilon_{A'} = (id_{\mathbf{G}(A')})^{\#}$.

From right adjoints to adjunctions

Theorem: *Let* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$. *Then there is a natural transformation* $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *such that:*

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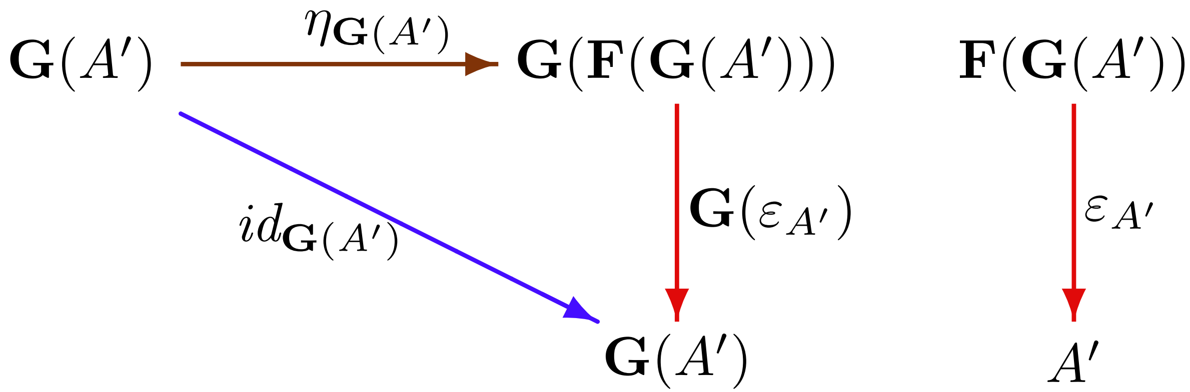

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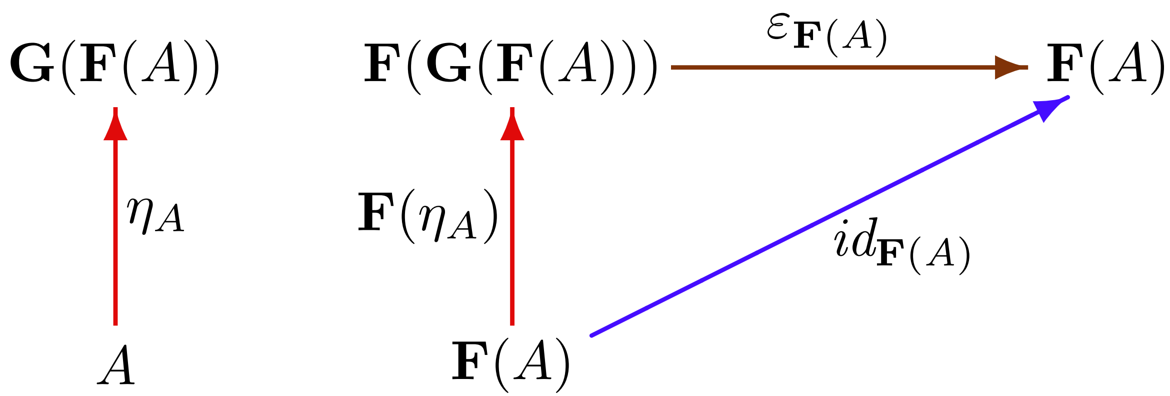

Proof (idea):

Put $\eta_A = (id_{\mathbf{F}(A)})^{\#}$.

From adjunctions to left and right adjoints

Theorem: *Consider two functors* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *with natural transformations* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *such that:*

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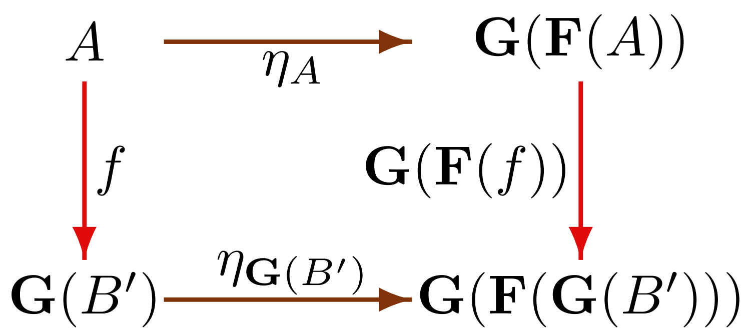

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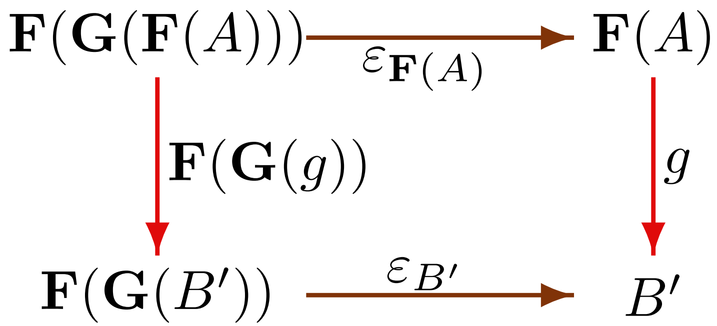

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The proof that $\mathbf{G}$ is right adjoint to $\mathbf{F}$ with counit ε is similar.

Adjunctions

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Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\boxed{\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle}$$

where $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *natural transformations such that:*

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Equivalently, such an adjunction may be given by:

- Functor $\mathbf{G}: \mathbf{K}' \to \mathbf{K}$ and for each $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$.

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Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle$$

where $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *natural transformations such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Equivalently, such an adjunction may be given by:

- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and for each $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$.
- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and its left adjoint.

Adjunctions

Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

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where $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *natural transformations such that:*

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- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and its left adjoint.
- Functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and for each $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$.

Adjunctions

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- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and its left adjoint.
- Functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and for each $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$.
- Functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and its right adjoint.

Adjunctions

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- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

Notation:

$$\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle \colon \mathbf{K} \to \mathbf{K}'$$

$$\mathbf{F} \dashv \mathbf{G}$$

Adjunctions

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$$\boxed{\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle}$$

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- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Exercises

- Yet another way to present adjunctions between locally small categories:
  - a natural isomorphism $(\_)^{\#}\colon \mathbf{Hom}_{\mathbf{K}}(\_, \mathbf{G}(\_)) \to \mathbf{Hom}_{\mathbf{K}'}(\mathbf{F}(\_), \_)$
    ($\colon \mathbf{K}^{op} \times \mathbf{K}' \to \mathbf{Set}$)

Adjunctions

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- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Exercises

- Adjunctions compose: given adjunctions $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle\colon \mathbf{K} \to \mathbf{K}'$ and $\langle \mathbf{F}', \mathbf{G}', \eta', \varepsilon' \rangle\colon \mathbf{K}' \to \mathbf{K}''$, define their composition

$$\langle \mathbf{F};\mathbf{F}', \mathbf{G}';\mathbf{G}, \_, \_ \rangle\colon \mathbf{K} \to \mathbf{K}''$$

Adjunctions

Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle$$

where $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *natural transformations such that:*

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- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Exercises

- Adjunctions compose: given adjunctions $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle \colon \mathbf{K} \to \mathbf{K}'$ and $\langle \mathbf{F}', \mathbf{G}', \eta', \varepsilon' \rangle \colon \mathbf{K}' \to \mathbf{K}''$, define their composition

$$\langle \mathbf{F};\mathbf{F}', \mathbf{G}';\mathbf{G}, \eta;(\mathbf{F}\cdot\eta'\cdot\mathbf{G}), (\mathbf{G}'\cdot\varepsilon\cdot\mathbf{F}');\varepsilon' \rangle \colon \mathbf{K} \to \mathbf{K}''$$