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- $n = 0$ yields $f \colon \to s$, often written $f \colon s$ — *constants* allowed

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    BTW: *constants*: $f_A : \{\langle\rangle\} \rightarrow |A|_s$, i.e. $f_A \in |A|_s$, for $f : s$

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Can $A \in \mathbf{Alg}(\Sigma)$ have empty carriers?

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- for $f : X \to Y$, $g : Y \to Z$, $f;g = \langle f_s;g_s : X_s \to Z_s \rangle_{s \in S} : X \to Z$

  BTW: $(f;g)(x) = g(f(x))$, where by abuse of notation for $x \in X_s$, $f(x) = f_s(x)$

Subalgebras

Definition: *For* $A, A_{sub} \in \mathbf{Alg}(\Sigma)$, A_{sub} *is a* Σ*-subalgebra of* A, *written* $A_{sub} \subseteq A$, *if*

- $|A_{sub}| \subseteq |A|$, *and*
- *for* $f: s_1 \times \ldots \times s_n \to s$, *and* $a_1 \in |A_{sub}|_{s_1}, \ldots, a_n \in |A_{sub}|_{s_n}$,

$$f_{A_{sub}}(a_1, \ldots, a_n) = f_A(a_1, \ldots, a_n)$$

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Then $|\langle A \rangle_X| = \bigcup_{i \geq 0} X_i$ contains X (clearly) and is closed under the operations.
Moreover, if a subset of $|A|$ contains X and is closed under the operations then it contains each X_i, $i \geq 0$, and hence so defined $|\langle A \rangle_X|$ as well.

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Then $|\langle A \rangle_X| = \bigcap \{ |A_{sub}| \mid X \subseteq |A_{sub}|, A_{sub} \subseteq A \}$ is closed under the operations and contains X. Moreover, it is contained in every subalgebra of A that contains X.

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Proof (idea):

- generate the generated subalgebra from X by closing it under operations in A; or
- the intersection of any family of subalgebras of A is a subalgebra of A.

Homomorphisms

- for $A, B \in \mathbf{Alg}(\Sigma)$, a Σ-*homomorphism* $h\colon A \to B$ is a function $h\colon |A| \to |B|$ that preserves the operations:
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$,
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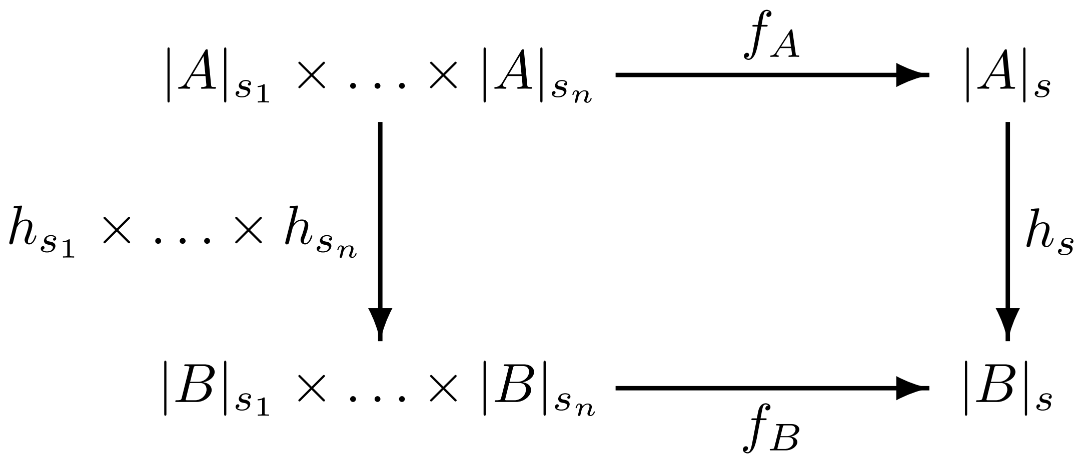

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Proof: Check that:

- $h^{-1}(|B_{sub}|)$ is closed under the operations (in A) – easy!
- $h(|A_{sub}|)$ is closed under the operations (in B) – just a tiny bit more difficult...

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Proof:

- $h(\langle A\rangle_X) \supseteq \langle B\rangle_{h(X)}$, since $h(\langle A\rangle_X)$ is a subalgebra of B and contains $h(X)$;
- $\langle A\rangle_X \subseteq h^{-1}(\langle B\rangle_{h(X)})$, since $h^{-1}(\langle B\rangle_{h(X)})$ is a subalgebra of A and contains X. Hence $h(\langle A\rangle_X) \subseteq h(h^{-1}(\langle B\rangle_{h(X)})) \subseteq \langle B\rangle_{h(X)}$.

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Proof: Check that $\{a \in |A| \mid h_1(a) = h_2(a)\}$ is closed under the operations in A.

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Theorem: *Identity function on the carrier of $A \in \mathbf{Alg}(\Sigma)$ is a homomorphism $id_A\colon A \to A$. Composition of homomorphisms $h\colon A \to B$ and $g\colon B \to C$ is a homomorphism $h;g\colon A \to C$.*

Isomorphisms

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$$A \xrightarrow{\quad i \quad} B$$

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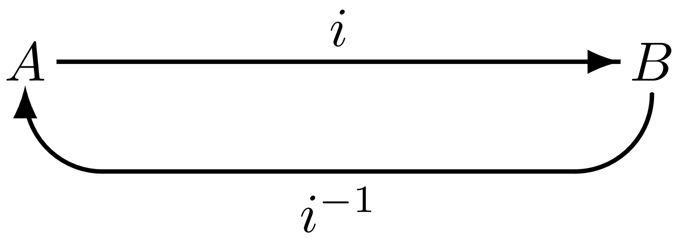

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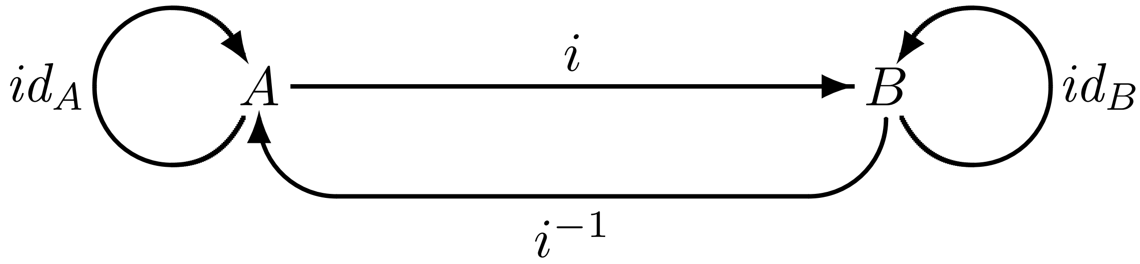

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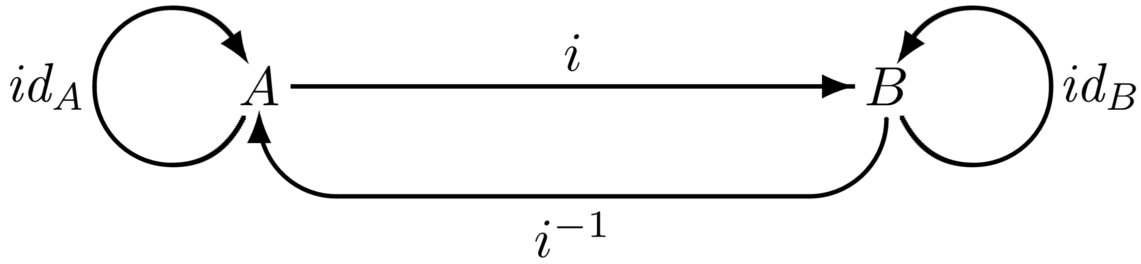
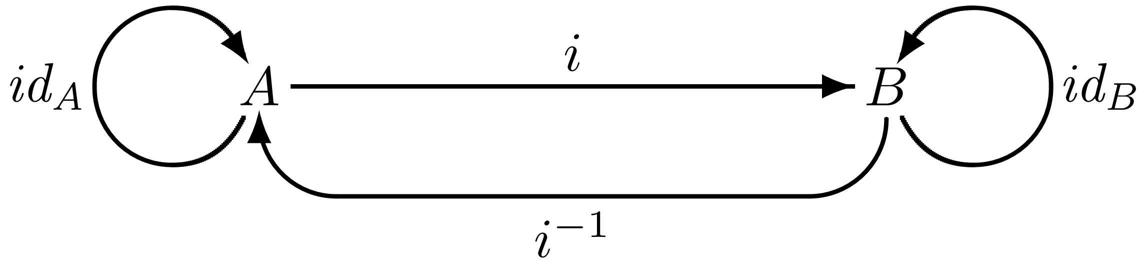


- Σ-algebras are *isomorphic* if there exists an isomorphism between them.

Isomorphisms

- for $A, B \in \mathbf{Alg}(\Sigma)$, a Σ-*isomorphism* is any Σ-homomorphism $i\colon A \to B$ that has an *inverse*, i.e., a Σ-homomorphism $i^{-1}\colon B \to A$ such that $i;i^{-1} = id_A$ and $i^{-1};i = id_B$.



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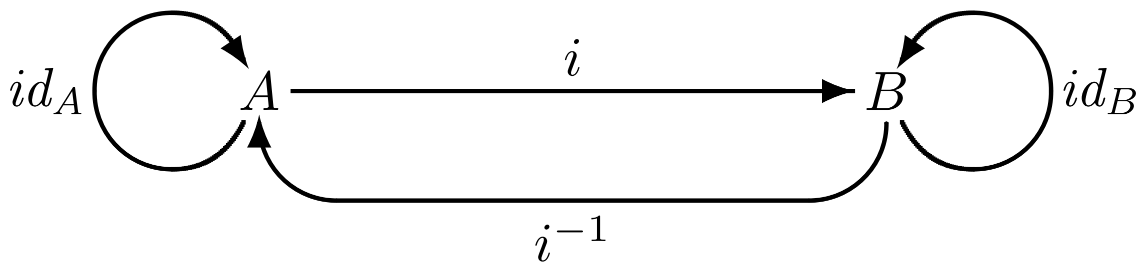

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Proof ("$\Longleftarrow$"): For $f\colon s_1 \times \ldots \times s_n \to s$ and $b_1 \in |B|_{s_1}, \ldots, b_n \in |B|_{s_n}$,

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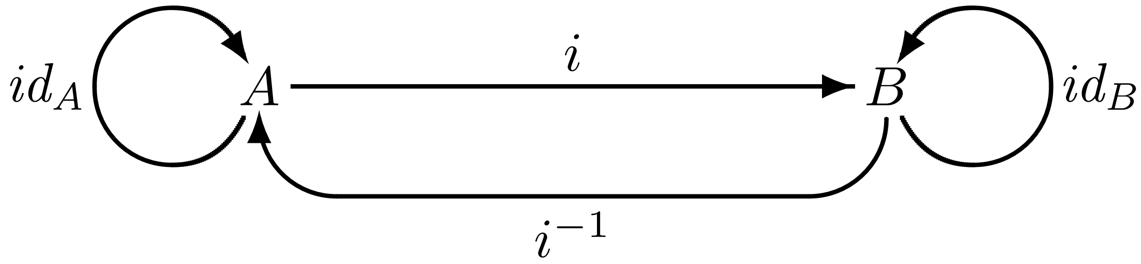

- Σ-algebras are *isomorphic* if there exists an isomorphism between them.

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Theorem: *Identities are isomorphisms, and any composition of isomorphisms is an isomorphism.*

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- for $A \in \mathbf{Alg}(\Sigma)$, a Σ-*congruence on* A is an equivalence $\equiv \subseteq |A| \times |A|$ that is closed under the operations:
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equivalence

$$\approx \subseteq X \times X$$

- reflexivity: $x \approx x$
- symmetry: if $x \approx y$ then $y \approx x$
- transitivity: if $x \approx y$ and $y \approx z$ then $x \approx z$

Then:

- equivalence class: $[x]_\approx = \{y \in X \mid y \approx x\}$
- quotient set: $X/{\approx} = \{[x]_\approx \mid x \in X\}$

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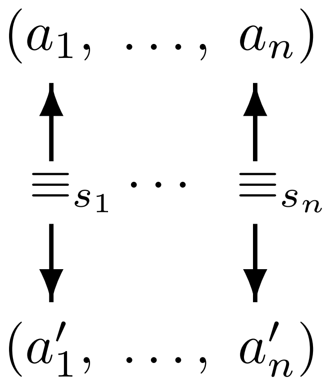

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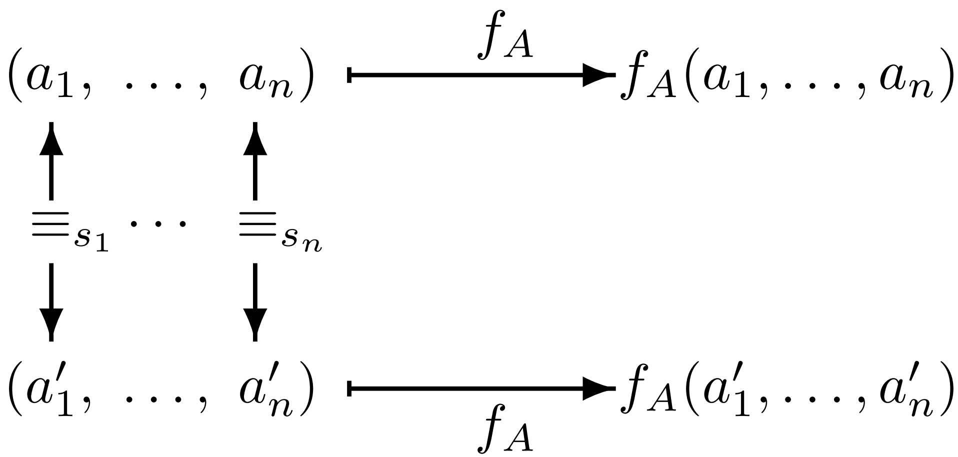

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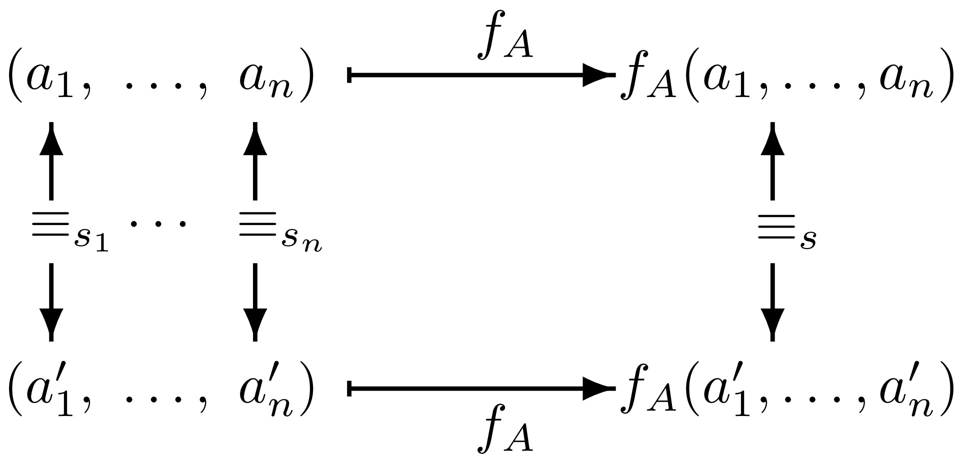

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Proof (idea):

- generate the least congruence from R by closing it under reflexivity, symmetry, transitivity and the operations in A; or
- the intersection of any family of congruences on A is a congruence on A.

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if $a_1\ K(h)_{s_1}\ a'_1, \ldots, a_n\ K(h)_{s_n}\ a'_n$ then $f_A(a_1, \ldots, a_n)\ K(h)_s\ f_A(a'_1, \ldots, a'_n)$, since

$$h_s(f_A(a_1, \ldots, a_n)) = f_B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) = f_B(h_{s_1}(a'_1), \ldots, h_{s_n}(a'_n)) = h_s(f_A(a'_1, \ldots, a'_n)).$$

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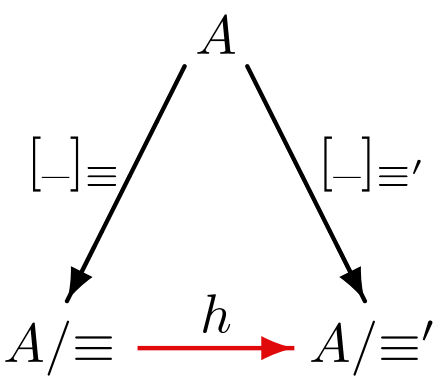
Proof (idea): Define $h([a]_\equiv) = [a]_{\equiv'}$:



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Proof (idea): Check that $i : A/K(h) \to B$ defined by $i([a]_{K(h)}) = h(a)$ is injective and is "onto" $h(A)$.

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BTW:

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- projections $\pi_k \colon \prod_{i \in \mathcal{I}} X_i \to X_k$, $\pi_k(p) = p(k)$.

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– projections $\pi_k \colon \prod_{i \in \mathcal{I}} X_i \to X_k$, $\pi_k(p) = p(k)$.

Products

- for $A_i \in \mathbf{Alg}(\Sigma)$, $i \in \mathcal{I}$, the *product of* $\langle A_i \rangle_{i \in \mathcal{I}}$, $\prod_{i \in \mathcal{I}} A_i$ is built in the natural way on the Cartesian product of the carriers of A_i, $i \in \mathcal{I}$:
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Theorem: *For any family $\langle A_i \rangle_{i \in \mathcal{I}}$ of Σ-algebras, projections $\pi_i(a) = a(i)$, where $i \in \mathcal{I}$ and $a \in \prod_{i \in \mathcal{I}} |A_i|$, are Σ-homomorphisms $\pi_i : \prod_{i \in \mathcal{I}} A_i \to A_i$.*

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> Define the product of the empty family of Σ-algebras.
> When the projection π_i is an isomorphism?

$\mathcal{H} \quad \mathcal{S} \quad \mathcal{P}$

Define $\mathcal{H}, \mathcal{S}, \mathcal{P} \colon 2^{\mathbf{Alg}(\Sigma)} \to 2^{\mathbf{Alg}(\Sigma)}$

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Corollary: $\mathcal{HSP} \colon 2^{\mathbf{Alg}(\Sigma)} \to 2^{\mathbf{Alg}(\Sigma)}$ *is a closure operator on* $2^{\mathbf{Alg}(\Sigma)}$

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No other order of $\mathcal{H}$, $\mathcal{S}$, $\mathcal{P}$ works!

Terms

Consider an S-sorted set X of variables.

- *terms* $t \in |T_\Sigma(X)|$ are built using variables X, constants and operations from Ω in the usual way: $|T_\Sigma(X)|$ is the least set such that
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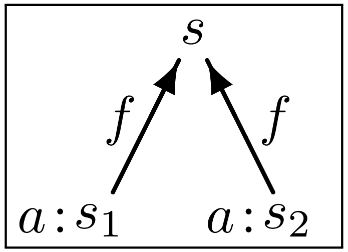

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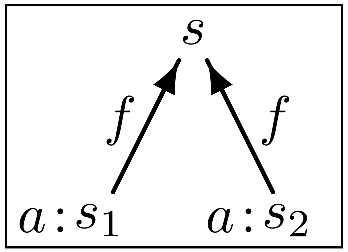
    — better write terms for instance as $f(a{:}s_1){:}s$ and $f(a{:}s_2){:}s$.



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Above and in the following: assuming unambiguous "parsing" of terms!

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  - for $f : s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$, $f(t_1, \ldots, t_n) \in |T_\Sigma(X)|_s$
- for any Σ-algebra A and valuation $v : X \to |A|$, *the value* $t_A[v] \in |A|_s$ *of a term* $t \in |T_\Sigma(X)|_s$ *in* A *under* v is determined inductively:
  - $x_A[v] = v_s(x)$, for $x \in X_s$, $s \in S$
  - $(f(t_1, \ldots, t_n))_A[v] = f_A((t_1)_A[v], \ldots, (t_n)_A[v])$, for $f : s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$

Above and in the following: assuming unambiguous "parsing" of terms!

Term algebras

Consider an S-sorted set X of variables.

- The *term algebra* $T_\Sigma(X)$ has the set of terms as the carrier and operations defined "syntactically":
  - for $f\colon s_1 \times \ldots \times s_n \to s$ and $t_1 \in |T_\Sigma(X)|_{s_1}, \ldots, t_n \in |T_\Sigma(X)|_{s_n}$, $f_{T_\Sigma(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

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- *Ground terms*: terms with no variables.
- *Ground term algebra*:

$$\boxed{T_\Sigma = T_\Sigma(\emptyset)}$$

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Fact: $T_\Sigma(X)$ *is generated by* X*;* T_Σ *is reachable.*

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Theorem: *For any S-sorted set X of variables,*

$$X \xrightarrow{id_{X \hookrightarrow |T_\Sigma(X)|}} |T_\Sigma(X)| \qquad\qquad T_\Sigma(X)$$

$$\mathbf{Set}^S \qquad\qquad\qquad\qquad \mathbf{Alg}(\Sigma)$$

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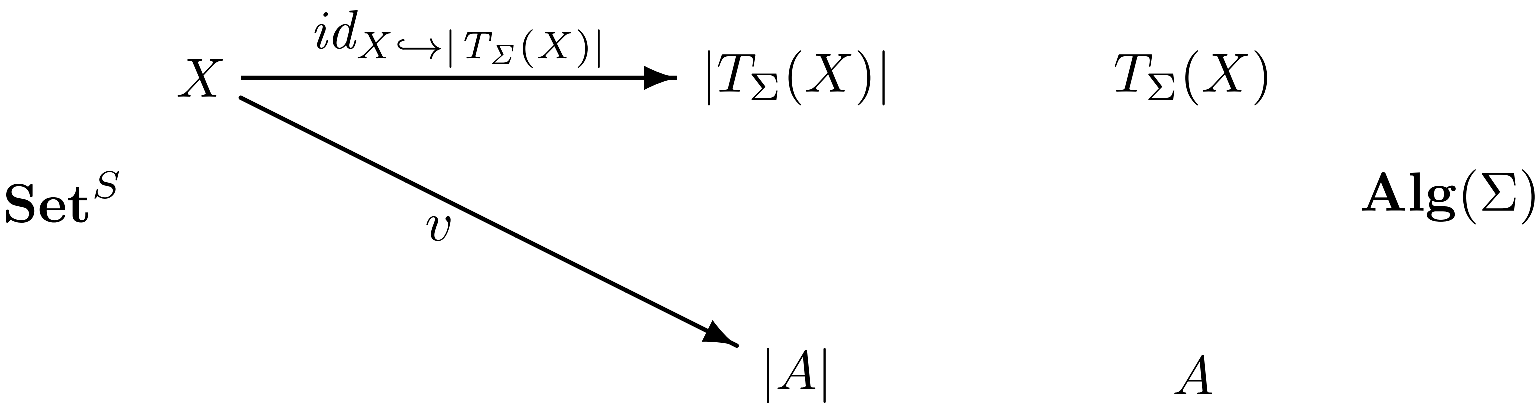

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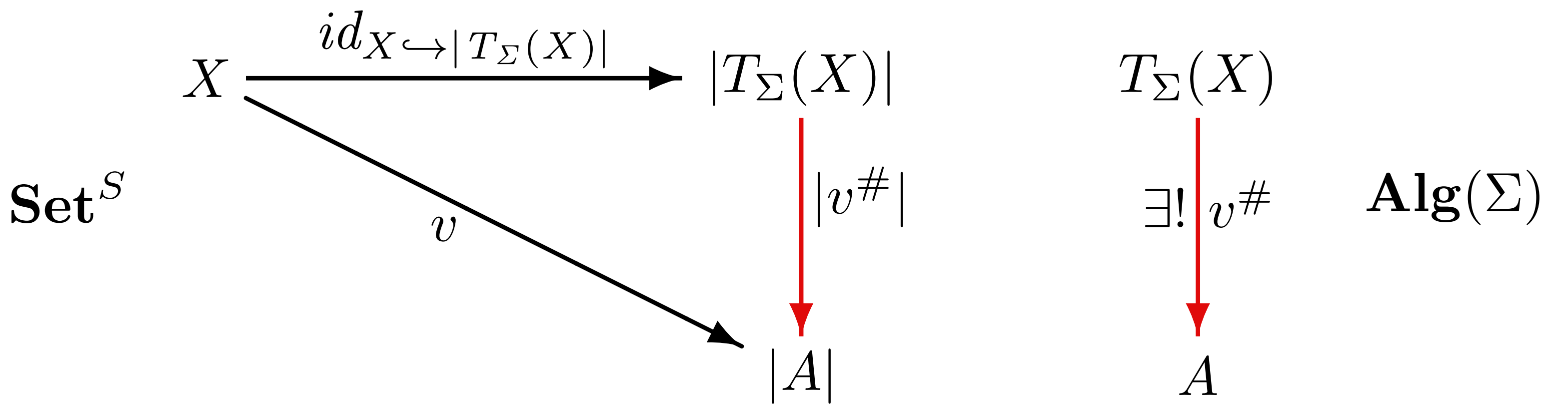

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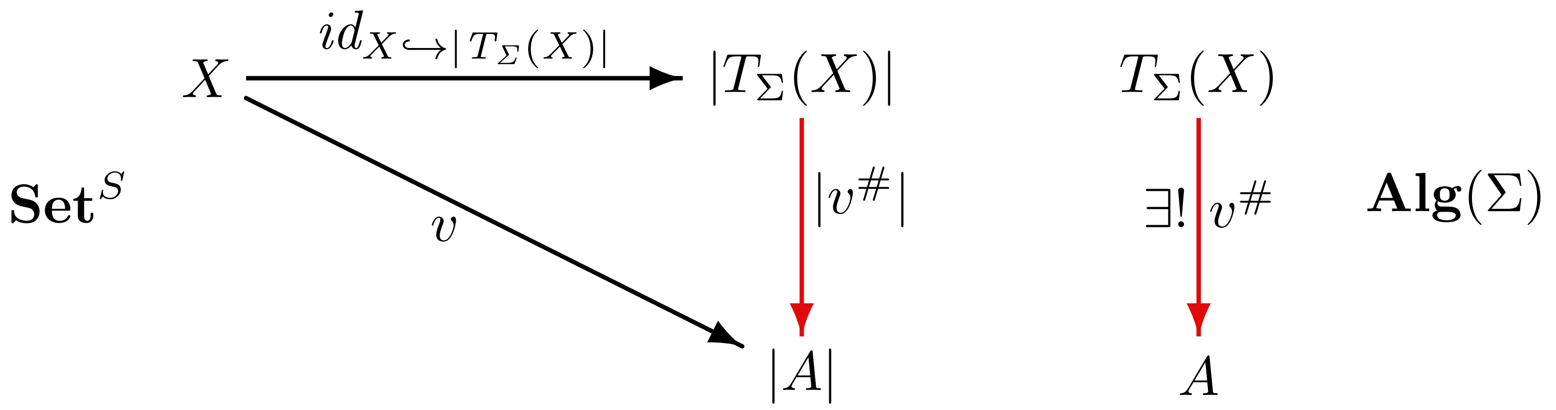

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Notation: *Given* $t \in |T_\Sigma(X)|$, $x_1 \in X_{s_1}$, $t_1 \in |T_\Sigma(X)|_{s_1}$, ..., $x_n \in X_{s_n}$, $t_n \in |T_\Sigma(X)|_{s_n}$, x_1, ..., x_n *mutually distinct:*

t *with* t_1, ..., t_n *simultaneously substituted for* x_1, ..., x_n, *respectively:*

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Alternative:

Generalise!

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Notation: *Given substitution* $\theta \colon X \to |T_\Sigma(X)|$*:*

t *with substitution* θ *carried out:* $t[\theta]$

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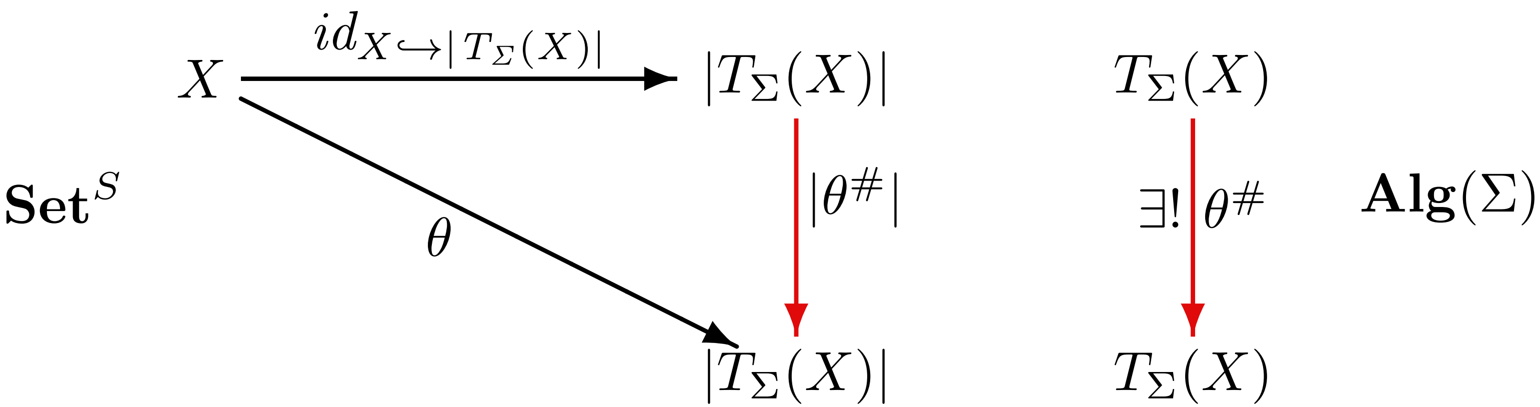

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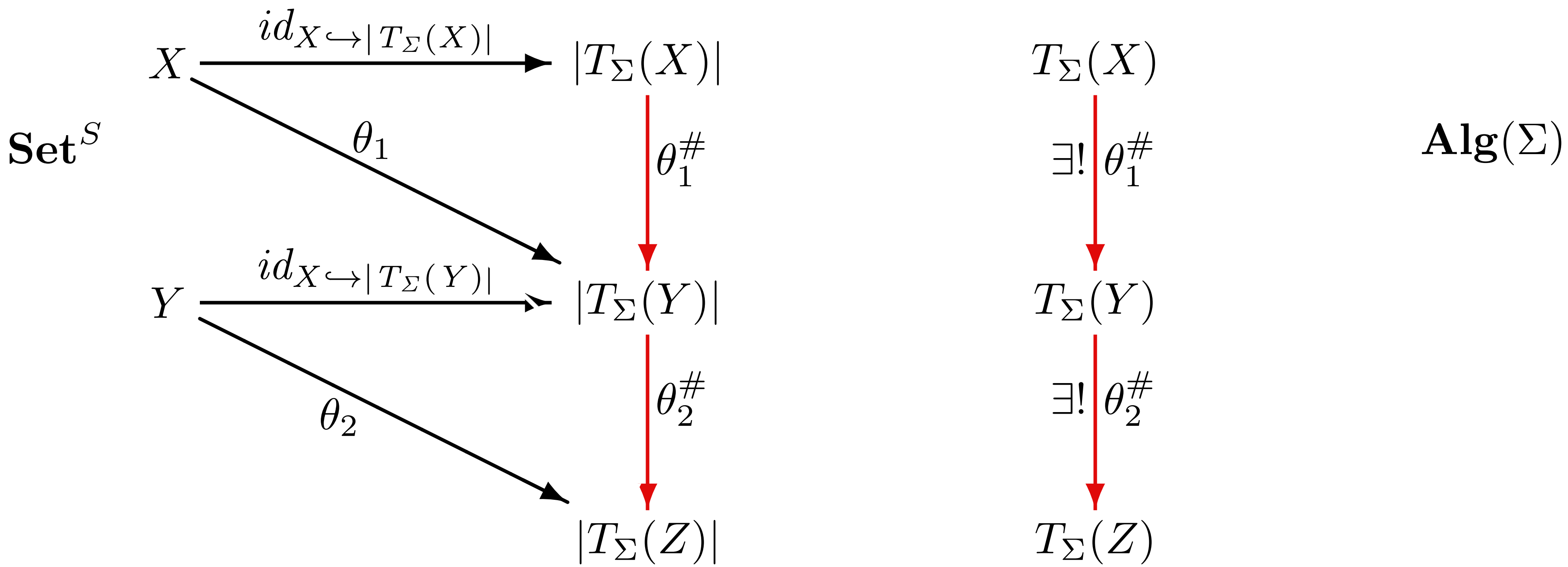
Theorem: *For any S-sorted sets X, Y and Z (of variables) and substitutions* $\theta_1 : X \to |T_\Sigma(Y)|$ *and* $\theta_2 : Y \to |T_\Sigma(Z)|$



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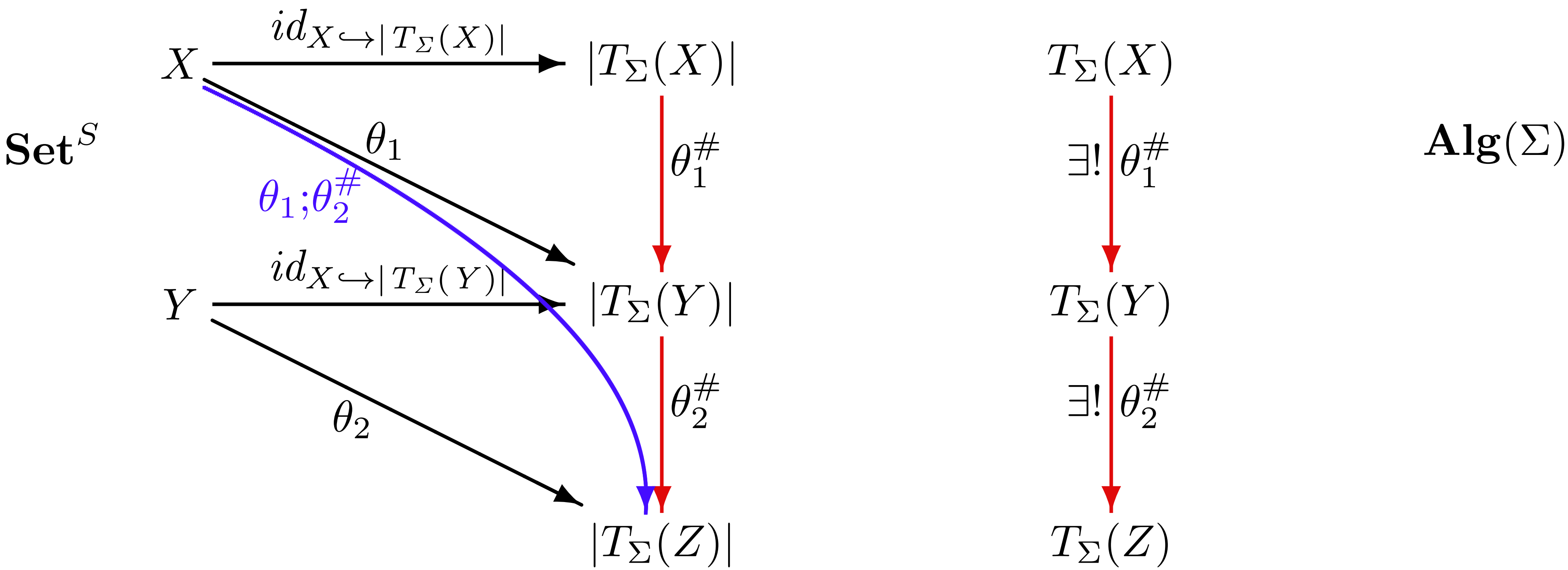

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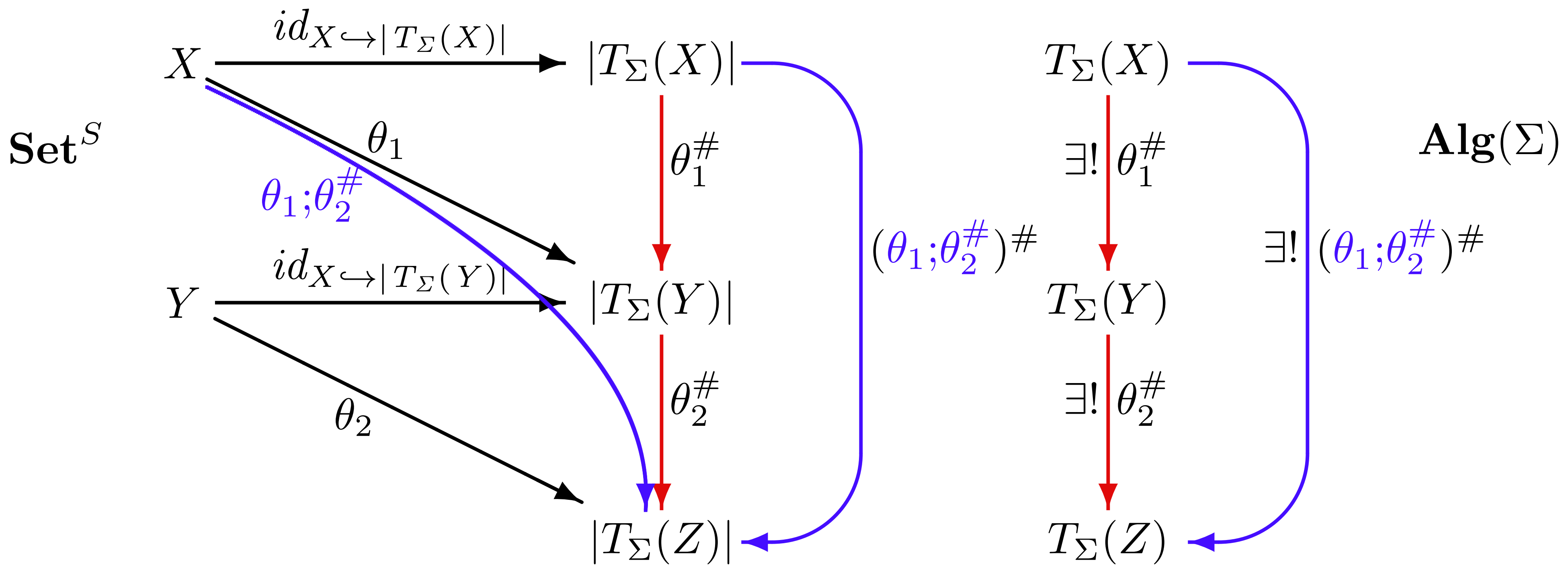

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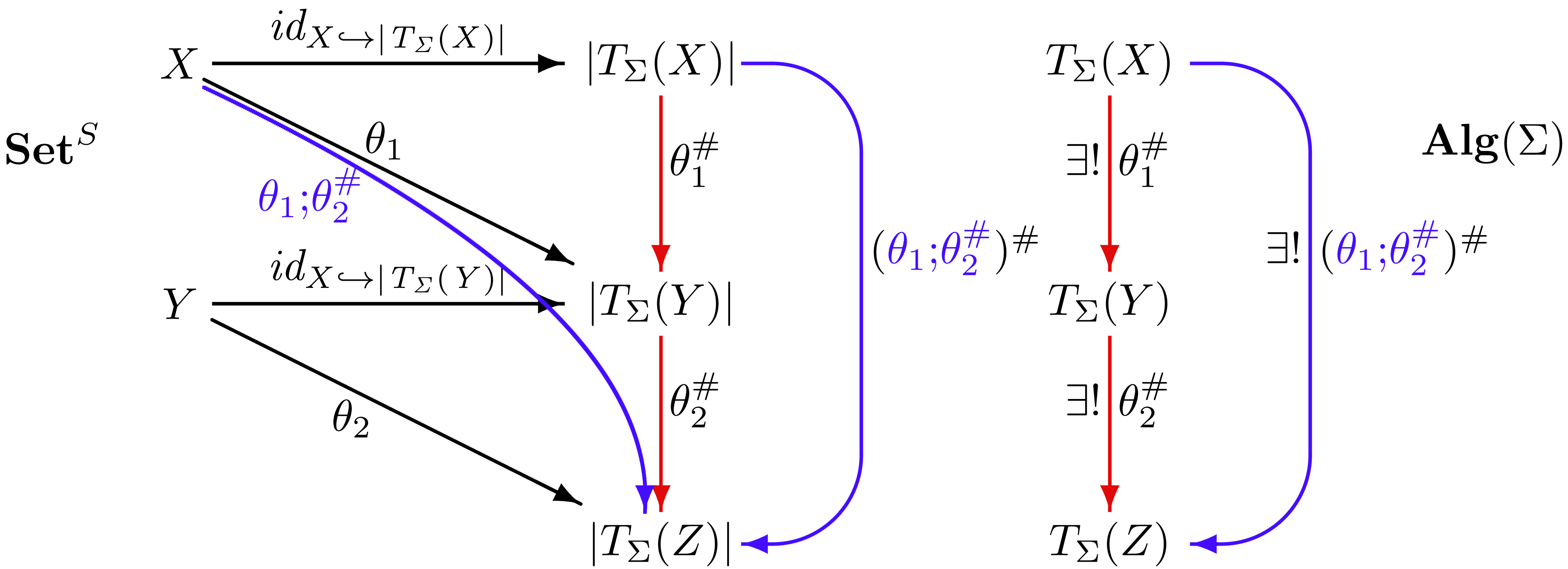

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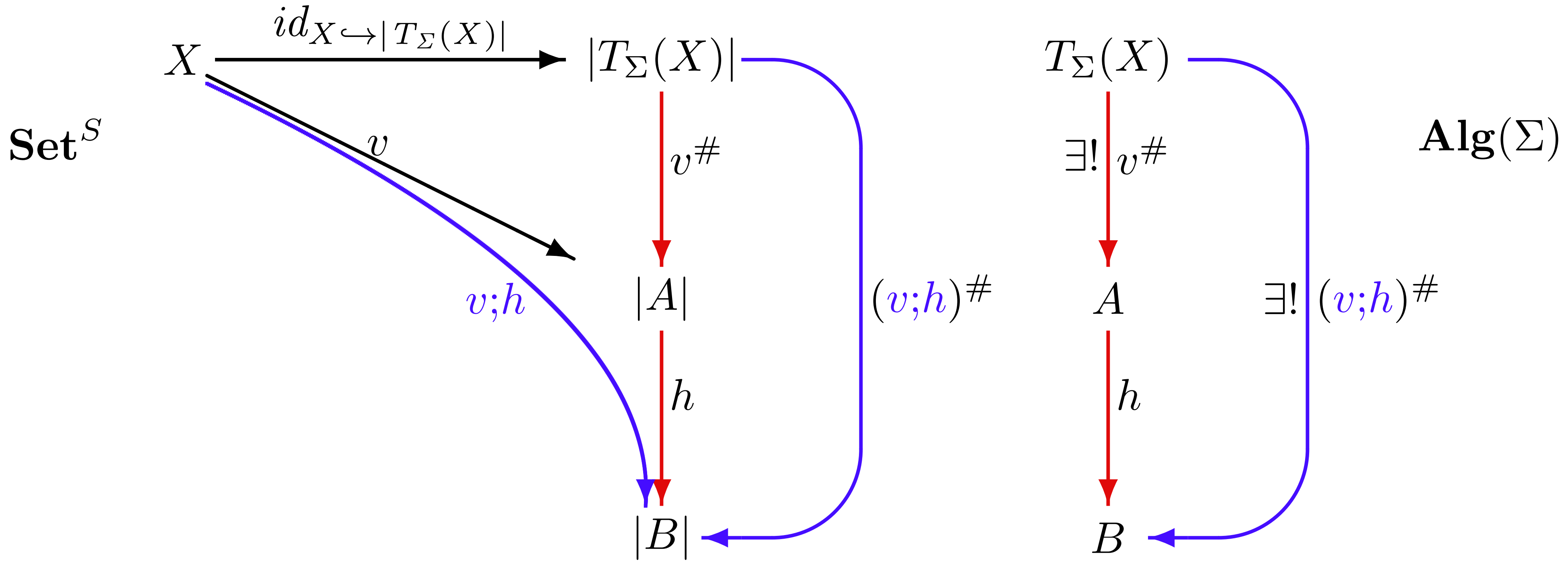
$$\boxed{v^{\#};h = (v;h)^{\#}}$$

In other words, for any term $t \in |T_\Sigma(X)|_s$, $h_s(t_A[v]) = t_B[v;h]$.



Consequences for reachability

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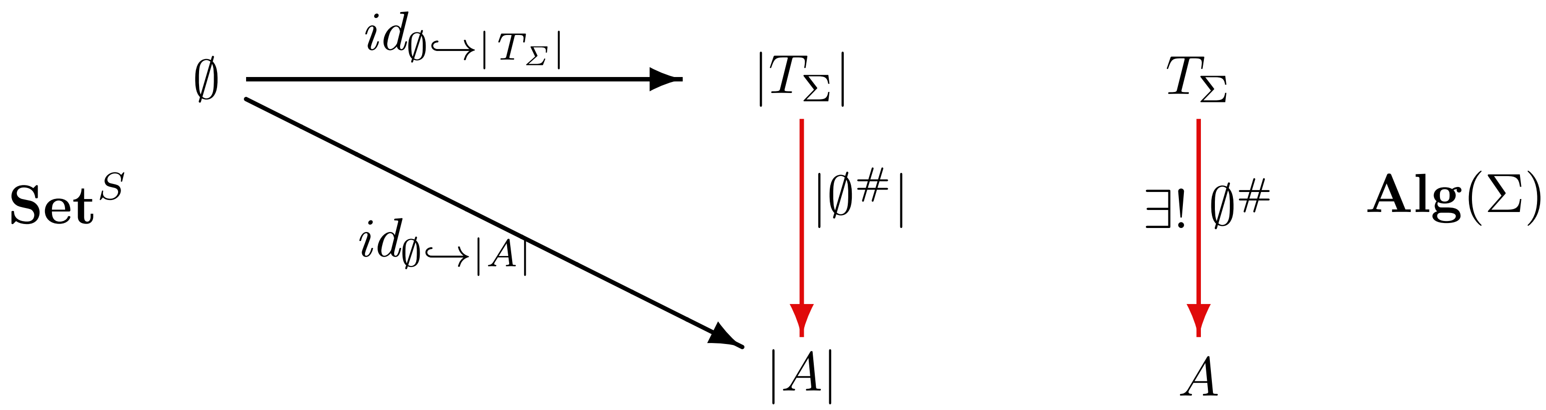

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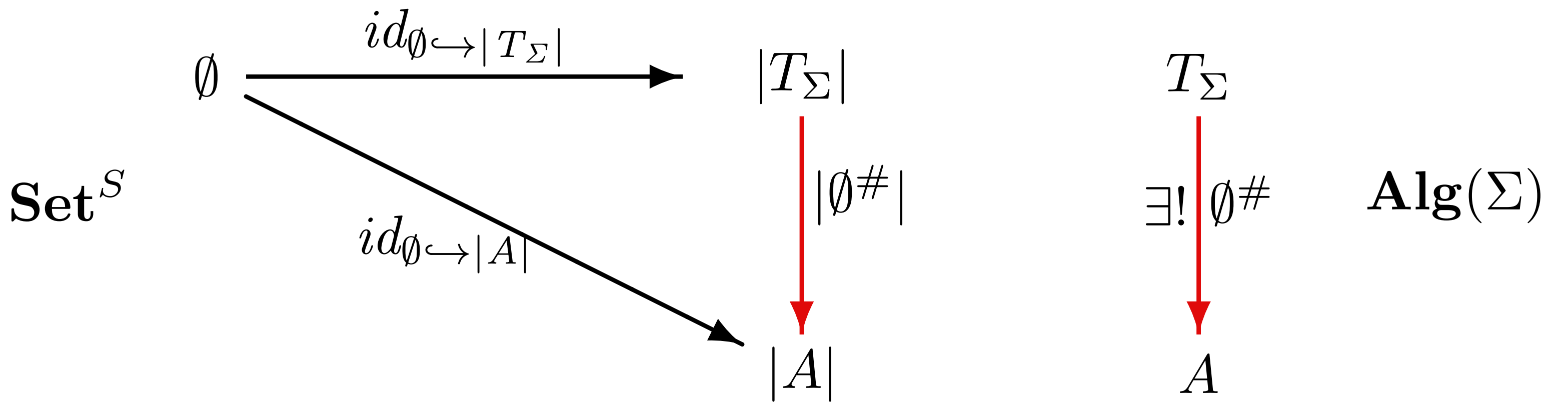

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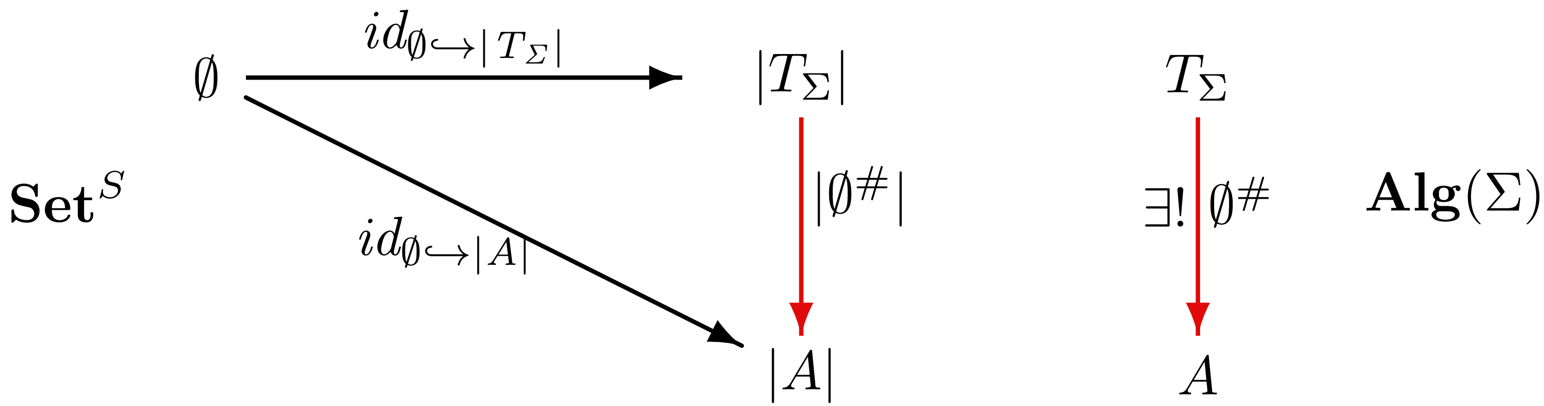

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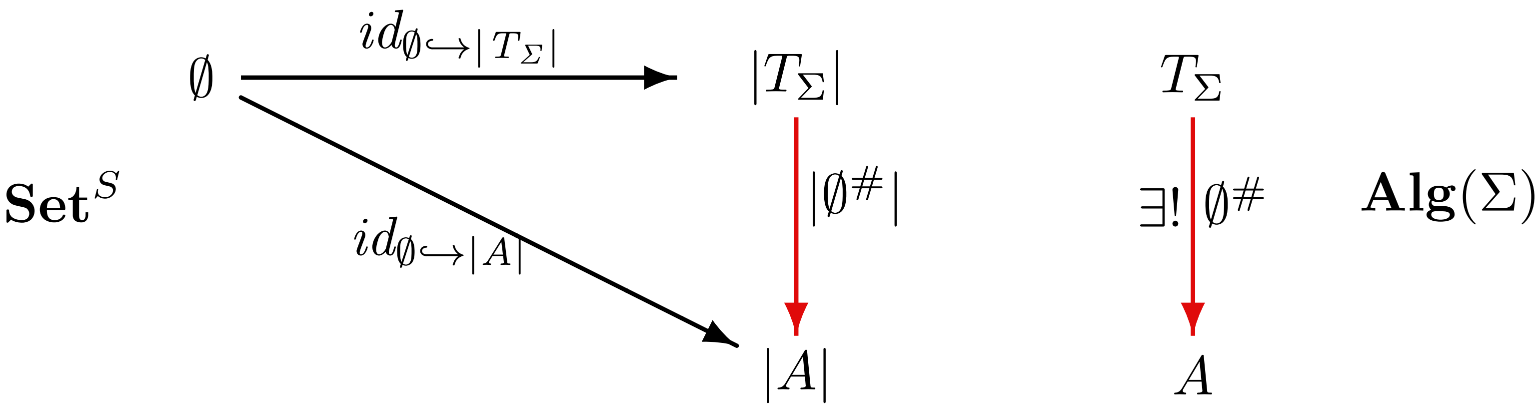

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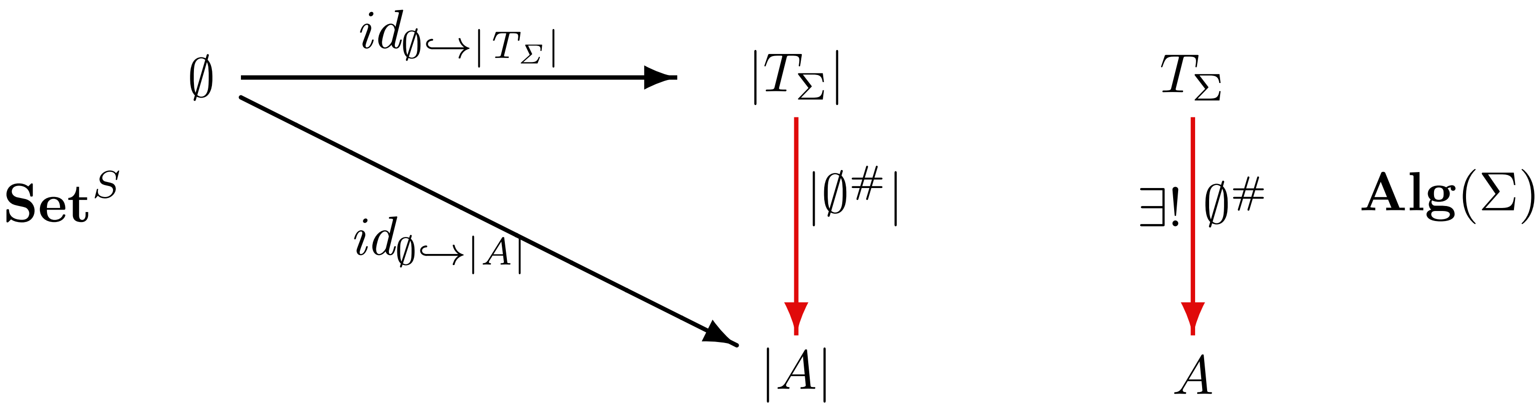

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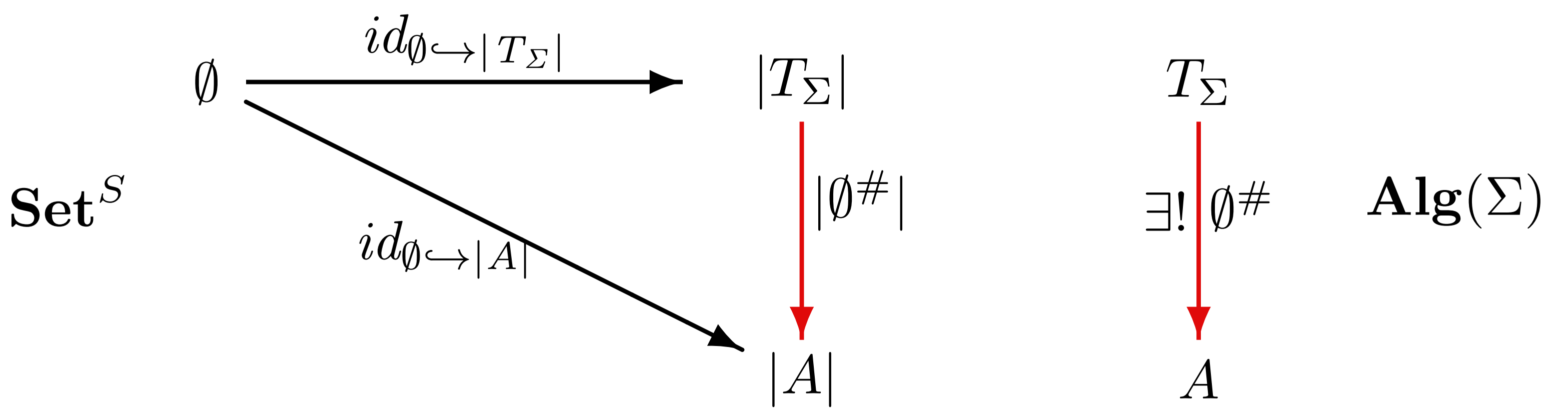

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$$\boxed{\forall X.t = t'}$$

where:

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BTW: $A \models \forall X.t = t'$ holds "trivially" if for some $s \in S$, $X_s \neq \emptyset$ and $|A|_s = \emptyset$.

Semantic entailment

$$\Phi \models_\Sigma \varphi$$

Σ-*equation* φ *is a* *semantic consequence* *of a set of* Σ-*equations* Φ *if* φ *holds in every* Σ-*algebra that satisfies* Φ.

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- Mod and Th form a *Galois connection*: $Mod(\Phi) \supseteq \mathcal{C}$ iff $\Phi \subseteq Th(\mathcal{C})$.
  - $\mathcal{C} \subseteq Mod(Th(\mathcal{C}))$, $\Phi \subseteq Th(Mod(\Phi))$
  - $Mod(Th(Mod(\Phi))) = Mod(\Phi)$, $Th(Mod(Th(\mathcal{C}))) = Th(\mathcal{C})$

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$$\langle \Sigma, \Phi \rangle$$

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"$\Longleftarrow$": *Not so easy, hints later...*

Example

$$\textbf{spec}\ \textsc{NaiveNat} = \textbf{sort}\ Nat$$
$$\textbf{ops}\ 0\colon Nat;$$
$$succ\colon Nat \to Nat;$$
$$\_+\_\colon Nat \times Nat \to Nat$$
$$\textbf{axioms}\ \forall n{:}Nat \bullet n + 0 = n;$$
$$\forall n, m{:}Nat \bullet n + succ(m) = succ(n + m)$$

Now:

$$\textsc{NaiveNat} \not\models \forall n, m{:}Nat \bullet n + m = m + n$$

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  Constraints can be thought of as special (higher-order) formulae.

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Theorem: *Every equational specification $\langle \Sigma, \Phi \rangle$ has an initial model: there exists a Σ-algebra $I \in Mod(\Phi)$ such that for every Σ-algebra $M \in Mod(\Phi)$ there exists a unique Σ-homomorphism from I to M.*

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P

where $P = \prod_{\equiv \in \{\equiv \mid (T_\Sigma/\equiv) \models \Phi\}} T_\Sigma/\equiv$

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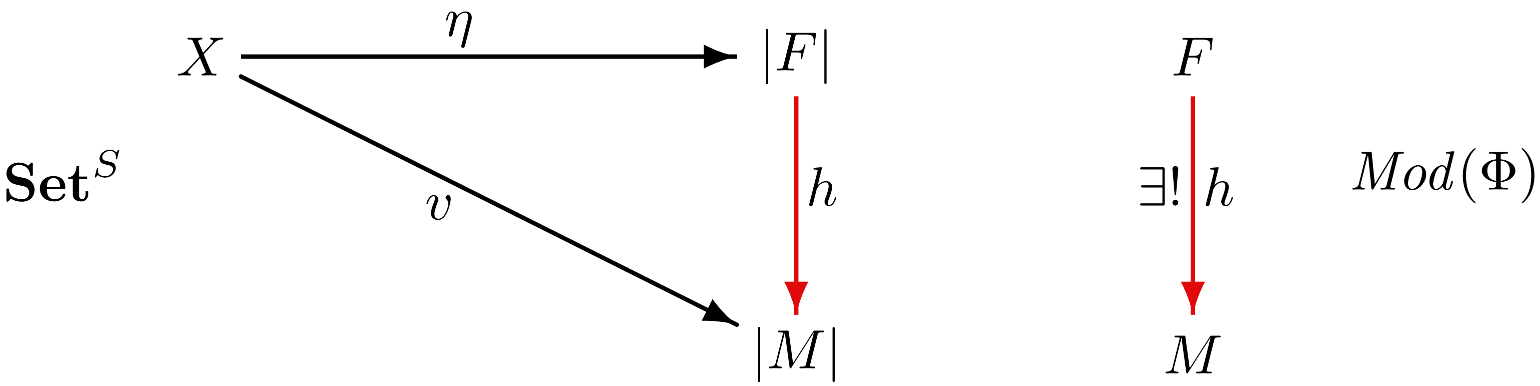

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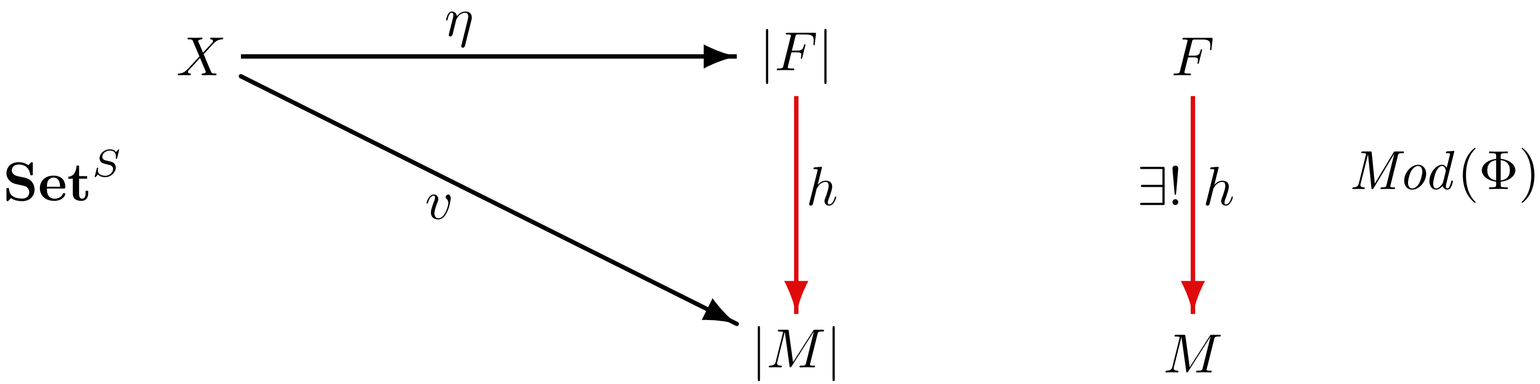

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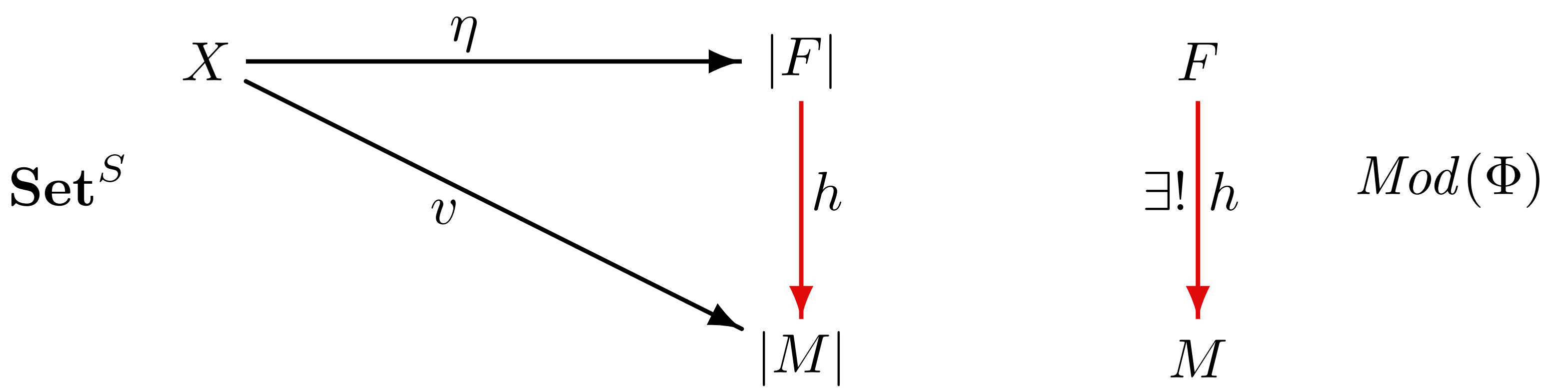

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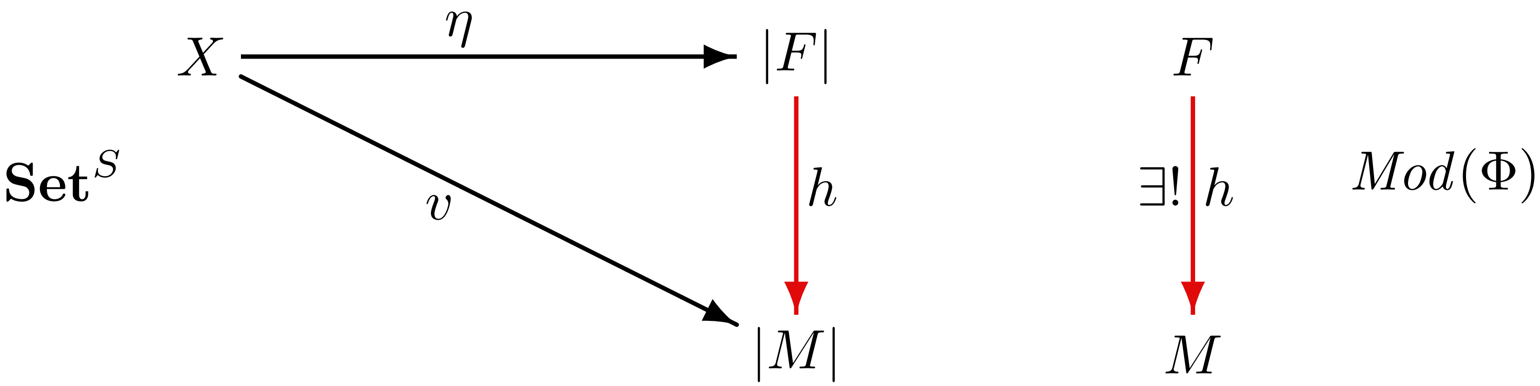

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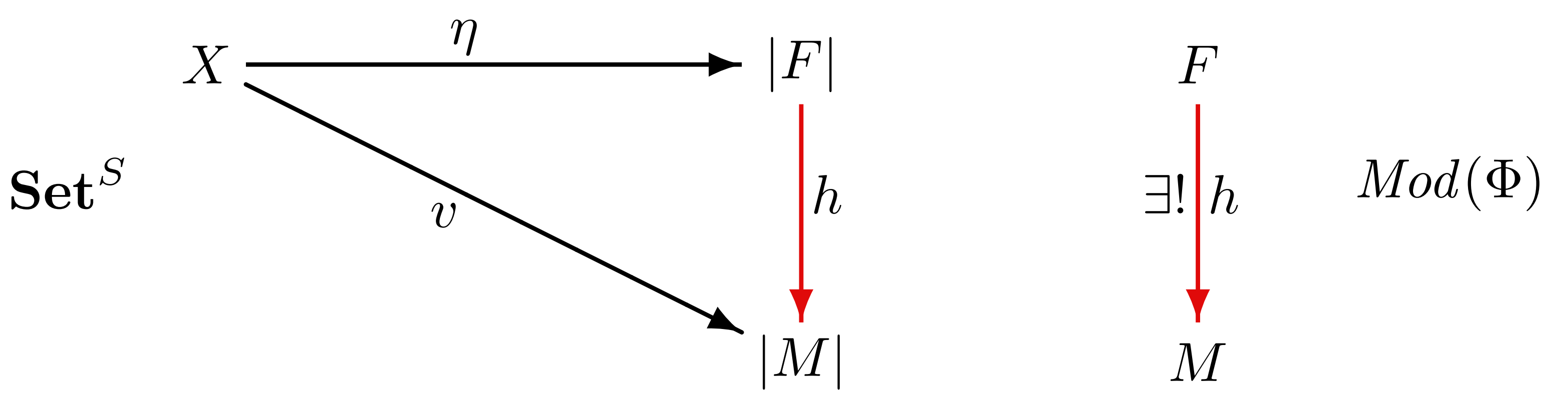

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  - reflexivity, transitivity, symmetry: easy!
  - congruence property: easy as well!

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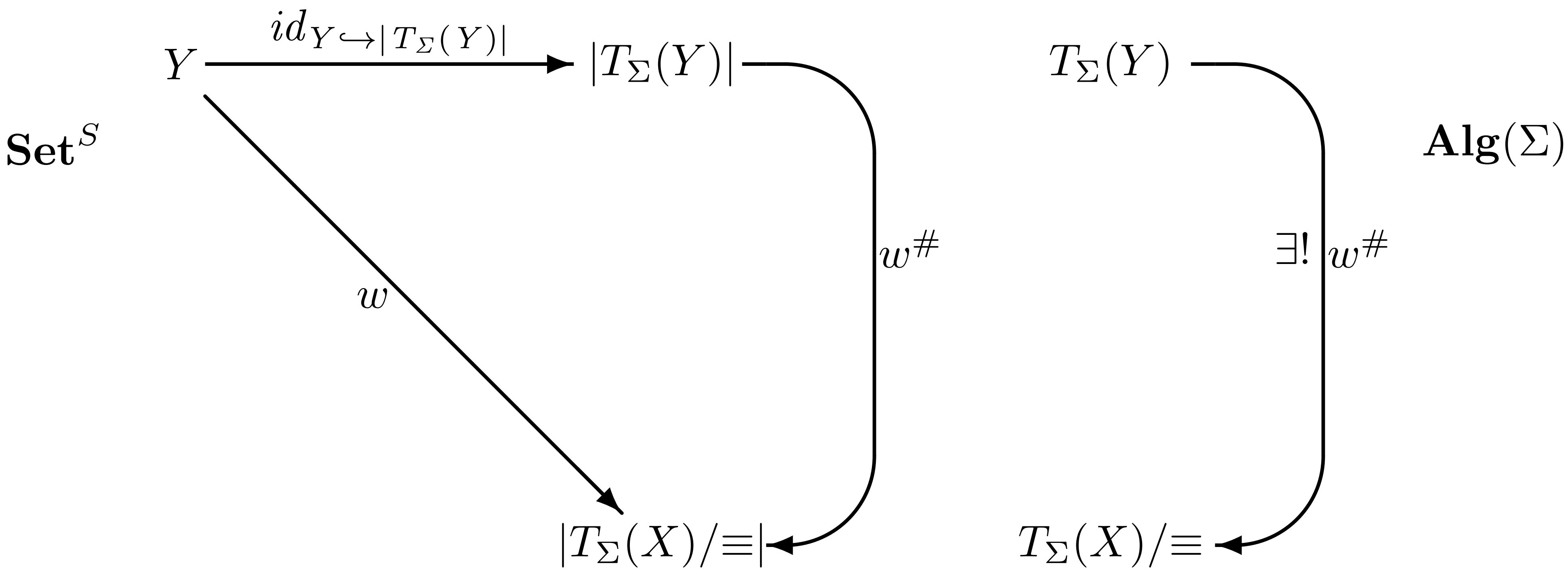

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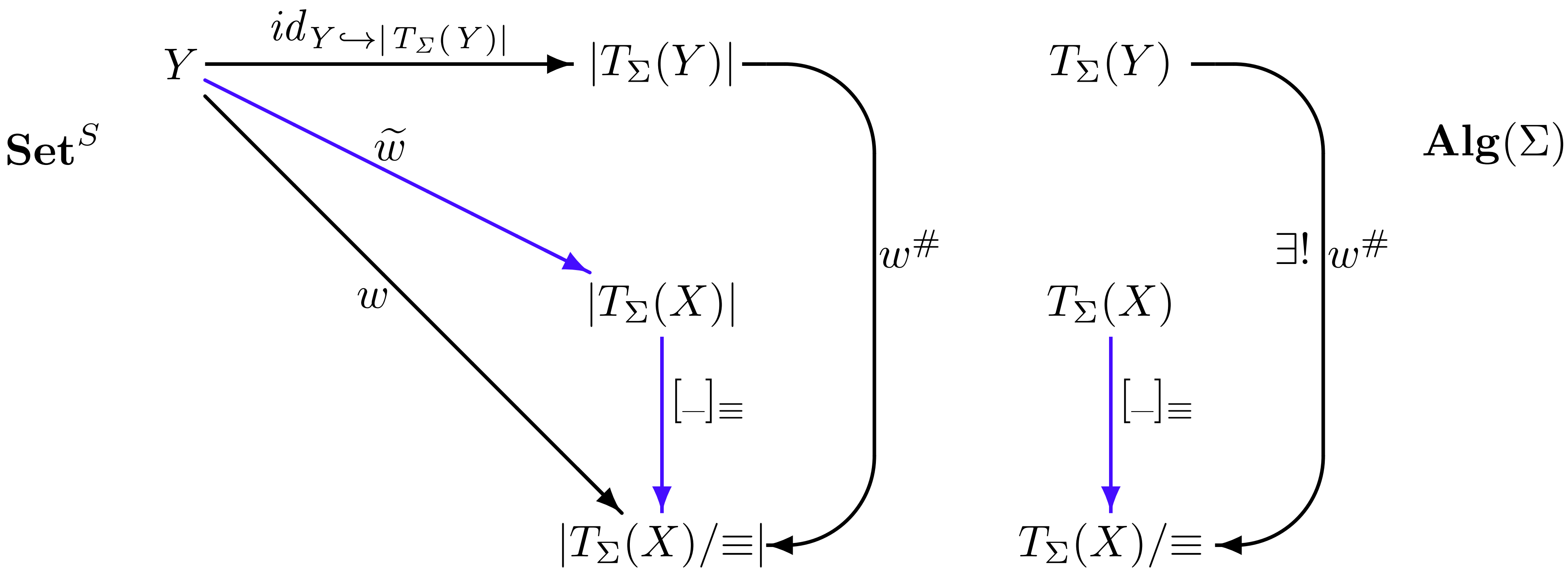

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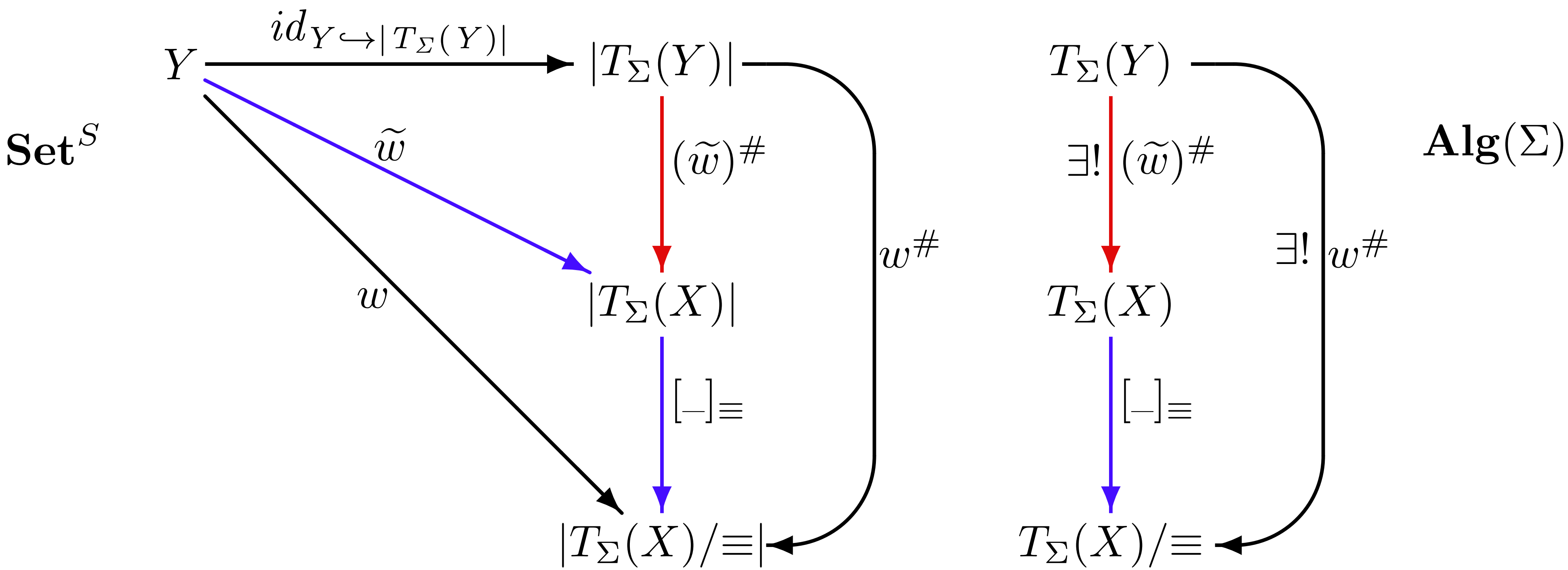

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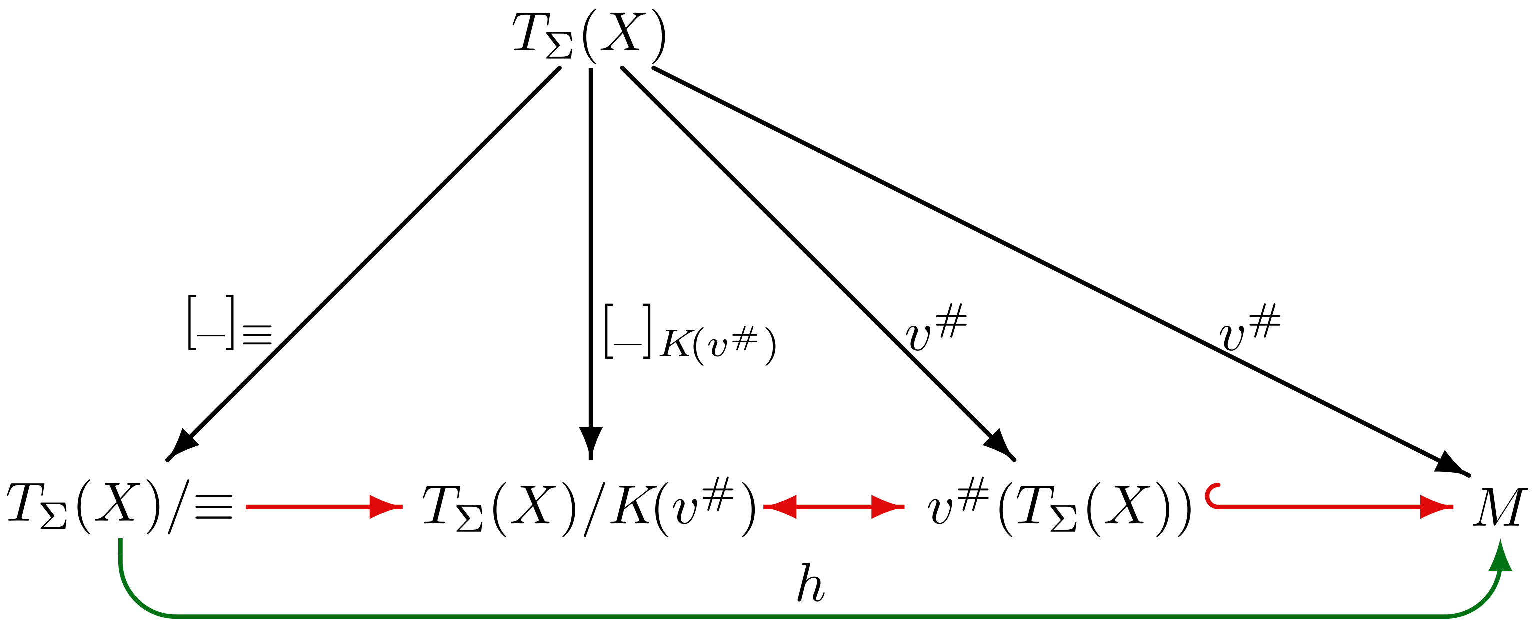

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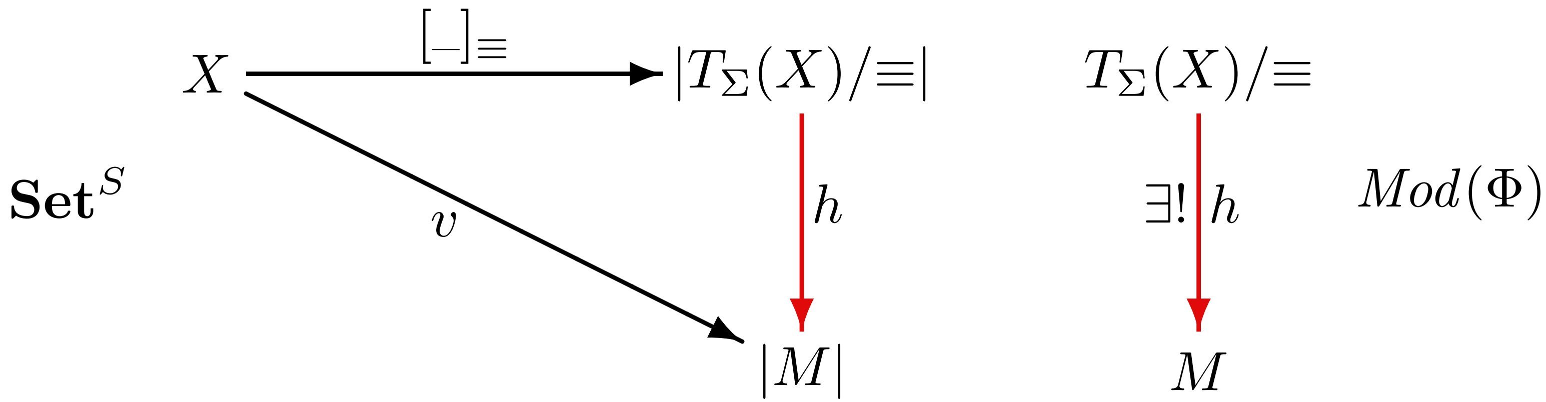

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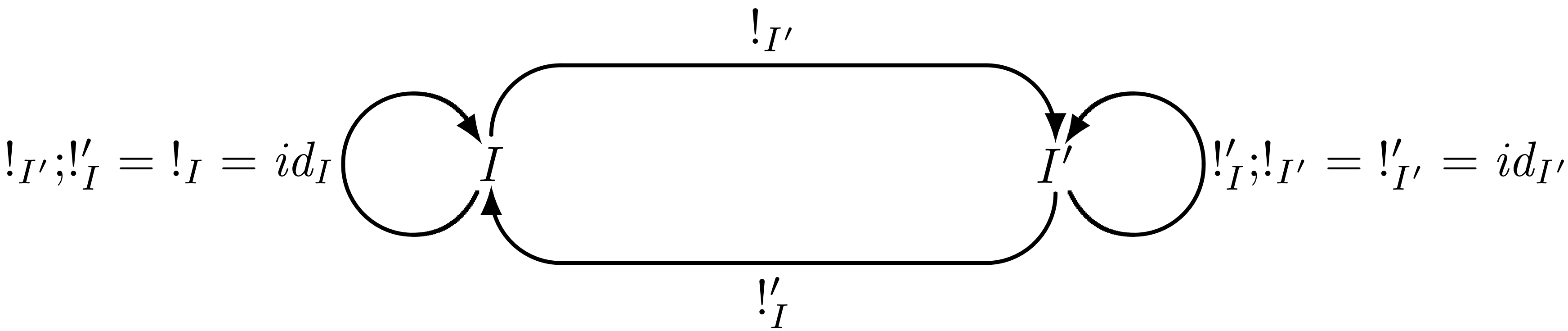
Fact: *Any two initial models of an equational specification are isomorphic.*

BTW: This can be generalised for free models of $\langle \Sigma, \Phi \rangle$ over any (many-sorted) set of data.

Initial models

Theorem: *Every equational specification $\langle\Sigma,\Phi\rangle$ has an initial model: there exists a Σ-algebra $I \in Mod(\Phi)$ such that for every Σ-algebra $M \in Mod(\Phi)$ there exists a unique Σ-homomorphism from I to M.*

Fact: *Any two initial models of an equational specification are isomorphic.*



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Example

$$\textbf{spec } \mathrm{NAT} = \textbf{free } \{ \textbf{ sort } Nat$$
$$\textbf{ops } 0\colon Nat;$$
$$succ\colon Nat \to Nat;$$
$$\_+\_\colon Nat \times Nat \to Nat$$
$$\textbf{axioms } \forall n{:}Nat \bullet n + 0 = n;$$
$$\forall n, m{:}Nat \bullet n + succ(m) = succ(n+m)$$
$$\}$$

Now:

$$\mathrm{NAT} \models \forall n, m{:}Nat \bullet n + m = m + n$$

Example′

spec $\textsc{Nat}'$ = **free type** $Nat ::= 0 \mid succ(Nat)$

op $\_+\_ : Nat \times Nat \to Nat$

axioms $\forall n{:}Nat \bullet n + 0 = n;$

$\forall n, m{:}Nat \bullet n + succ(m) = succ(n + m)$

$$\textsc{Nat} \equiv \textsc{Nat}'$$

Another example

spec STRING =

generated { **sort** *String*

ops $nil : String$;

$a, \ldots, z : String$;

$\_ \frown \_ : String \times String \to String$ }

axioms $\forall s{:}String \bullet s \frown nil = s$;

$\forall s{:}String \bullet nil \frown s = s$;

$\forall s, t, v{:}String \bullet s \frown (t \frown v) = (s \frown t) \frown v$

}

Birkhoff's Theorem

Theorem: *A class of Σ-algebras is equationally definable iff it is closed under subalgebras, products and homomorphic images.*

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- If $\mathcal{C}$ is closed under subalgebras and products then for any set X, there exists an algebra $F_X \in \mathcal{C}$ that is free in $\mathcal{C}$ over X with unit $\eta_X : X \to |F_X|$,

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- For $t, t' \in |T_\Sigma(X)|_s$, if $t_{F_X}[\eta_X] = t'_{F_X}[\eta_X]$ then $\forall X. t = t' \in Th(\mathcal{C})$.

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- For $t, t' \in |T_\Sigma(X)|_s$, if $t_{F_X}[\eta_X] = t'_{F_X}[\eta_X]$ then $\forall X.t = t' \in Th(\mathcal{C})$.
- Let $A \in Mod(Th(\mathcal{C}))$. Then there is a homomorphism $h \colon F_{|A|} \to A$ such that $\eta_{|A|};h = id_{|A|}$.

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Conclude:

$$\boxed{Mod(Th(\mathcal{C})) = \mathcal{C}}$$

Equational calculus

$$\frac{}{\forall X.t = t} \qquad \frac{\forall X.t = t'}{\forall X.t' = t} \qquad \frac{\forall X.t = t' \quad \forall X.t' = t''}{\forall X.t = t''}$$

$$\frac{\forall X.t_1 = t'_1 \quad \dots \quad \forall X.t_n = t'_n}{\forall X.f(t_1 \dots t_n) = f(t'_1 \dots t'_n)} \qquad \frac{\forall X.t = t'}{\forall Y.t[\theta] = t'[\theta]} \text{ for } \theta \colon X \to |T_\Sigma(Y)|$$

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Mind the variables!

$a = b$ does *not* follow from $a = f(x)$ and $f(x) = b$

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In general, $\forall x{:}s.(a{:}s') = (b{:}s') \not\models \forall \emptyset.(a{:}s') = (b{:}s')$.

For instance, over signature Σ with sorts s, s' and constants $a, b\colon s'$ and no other operations, for any algebra $A \in \mathbf{Alg}(\Sigma)$ such that $|A|_s = \emptyset$

$A \models \forall x{:}s.a = b$, even if $a_A \neq b_A$

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Mind the variables!

$a = b$ does *not* follow from $a = f(x)$ and $f(x) = b$ without a "witness" for x

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- *congruence*: clear as well

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- *reflexivity*, *symmetry*, *transitivity*: clear
- *congruence*: clear as well
- *substitution* allows one to:
  - substitute terms for (some) variables, possibly with different variables
  - increase the set of variables
  - remove unused variables, if "witnesses" to substitute for them remain

Proof-theoretic entailment

$$\Phi \vdash_\Sigma \varphi$$

Σ-equation φ is a proof-theoretic consequence of a set of Σ-equations Φ if φ can be derived from Φ by the rules.

How to justify this?

Semantics!

Soundness & completeness

Theorem: *The equational calculus is sound and complete:*

$$\Phi \models \varphi \iff \Phi \vdash \varphi$$

- soundness: "all that can be proved, is true" ($\Phi \models \varphi \Longleftarrow \Phi \vdash \varphi$)
- completeness: "all that is true, can be proved" ($\Phi \models \varphi \Longrightarrow \Phi \vdash \varphi$)

Proof (idea):

- soundness: easy!
- completeness: not so easy!

“Ground” completeness

$$\Phi \models \forall\emptyset.t_1 = t_2 \Longrightarrow \Phi \vdash \forall\emptyset.t_1 = t_2$$

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Proof (idea):

– Define $\approx \subseteq |T_\Sigma| \times |T_\Sigma|$: $t_1 \approx t_2$ iff $\Phi \vdash \forall\emptyset.t_1 = t_2$

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– Show that $\approx$ is a congruence on T_Σ, and $T_\Sigma/{\approx} \models \Phi$

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- Therefore $T_\Sigma/{\equiv}$ and $T_\Sigma/{\approx}$ are isomorphic

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$$\Phi \models \forall\emptyset.t_1 = t_2 \Longrightarrow \Phi \vdash \forall\emptyset.t_1 = t_2$$

Completeness

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Proof (idea):

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Proof (idea): Generalise the previous proof by building a free algebra $T_\Sigma(X)/{\approx}$ in $Mod(\Phi)$ with unit $[\_]_\approx \colon X \to T_\Sigma(X)/{\approx}$, where $\approx \subseteq |T_\Sigma(X)| \times |T_\Sigma(X)|$ is given by $t_1 \approx t_2$ iff $\Phi \vdash \forall X.t_1 = t_2$.

Completeness

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- For each signature Σ and a set of variables X, define a new signature $\Sigma(X)$ that extends Σ by variables from X as constants
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  - easy!

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  - Straightforward induction on the structure of derivation does not go through!

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- Show $\Phi \vdash_\Sigma \forall X.t_1 = t_2$ iff $\Phi \vdash_{\Sigma(X)} \forall\emptyset.t_1 = t_2$
  - Straightforward induction on the structure of derivation does not go through!
  - Induction works for a more general thesis:

    $\Phi \vdash_\Sigma \forall X \cup Y.t_1 = t_2$ iff $\Phi \vdash_{\Sigma(X)} \forall Y.t_1 = t_2$

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- Show $\Phi \models_\Sigma \forall X.t_1 = t_2$ iff $\Phi \models_{\Sigma(X)} \forall \emptyset.t_1 = t_2$
- Show $\Phi \vdash_\Sigma \forall X.t_1 = t_2$ iff $\Phi \vdash_{\Sigma(X)} \forall \emptyset.t_1 = t_2$
- Using ground completeness, conclude: $\Phi \models_\Sigma \forall X.t_1 = t_2$ iff $\Phi \models_{\Sigma(X)} \forall \emptyset.t_1 = t_2$ iff $\Phi \vdash_{\Sigma(X)} \forall \emptyset.t_1 = t_2$ iff $\Phi \vdash_\Sigma \forall X.t_1 = t_2$

Moving between signatures

Let $\Sigma = (S, \Omega)$ and $\Sigma' = (S', \Omega')$

$$\sigma : \Sigma \to \Sigma'$$

- *Signature morphism* maps:
  - sorts to sorts: $\sigma : S \to S'$
  - operation names to operation names, preserving their profiles:
    $\sigma : \Omega_{w,s} \to \Omega'_{\sigma(w),\sigma(s)}$, for $w \in S^*$, $s \in S$

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  - operation names to operation names, preserving their profiles:
    $\sigma \colon \Omega_{w,s} \to \Omega'_{\sigma(w),\sigma(s)}$, for $w \in S^*$, $s \in S$, that is:
    if $f \colon s_1 \times \ldots \times s_n \to s$ then $\sigma(f) \colon \sigma(s_1) \times \ldots \times \sigma(s_n) \to \sigma(s)$,

Let $\sigma\colon \Sigma \to \Sigma'$

Translating syntax

- *translation of variables*: $X \mapsto X'$, where $X'_{s'} = \biguplus_{\sigma(s)=s'} X_s$
- *translation of terms*: $\sigma\colon |T_\Sigma(X)|_s \to |T_{\Sigma'}(X')|_{\sigma(s)}$, for $s \in S$
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- σ*-reduct*: $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$, where for $A' \in \mathbf{Alg}(\Sigma')$
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    - for $f\colon s_1 \times \ldots \times s_n \to s$, $f_{A'|_\sigma}\colon |A'|_\sigma|_{s_1} \times \ldots \times |A'|_\sigma|_{s_n} \to |A'|_\sigma|_s$ since $\sigma(f)_{A'}\colon |A'|_{\sigma(s_1)} \times \ldots \times |A'|_{\sigma(s_n)} \to |A'|_{\sigma(s)}$

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BTW: Given a Σ'-homomorphism $h'\colon A' \to B'$, Σ-homomorphism $h'|_\sigma\colon A'|_\sigma \to B'|_\sigma$ is defined by $(h'|_\sigma)_s = h'_{\sigma(s)}$ for $s \in S$.

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Note the contravariancy!

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Theorem: *For any signature morphism* $\sigma: \Sigma \to \Sigma'$, Σ'*-algebra* A' *and* Σ*-equation* φ*:*

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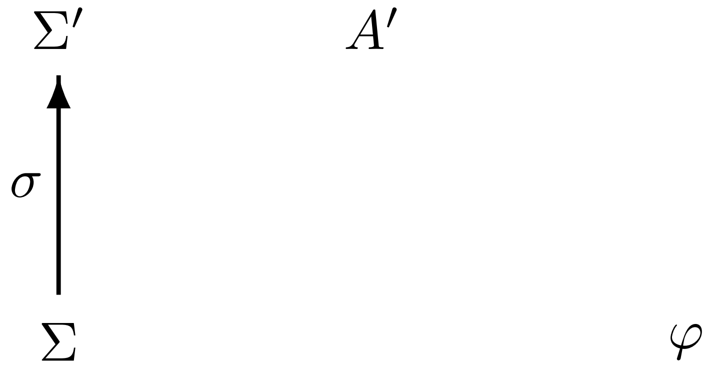

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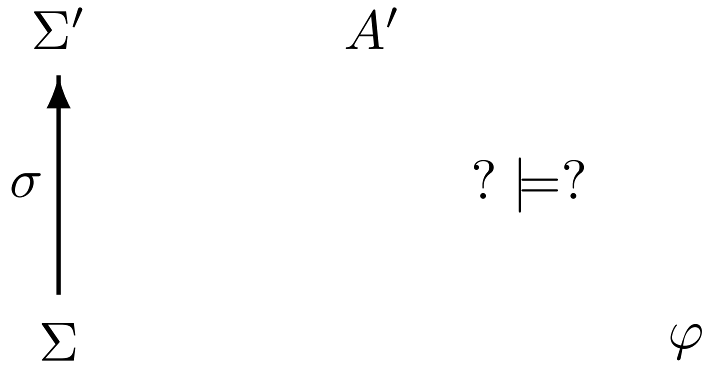

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$$\begin{array}{ccccc} \Sigma' & & A' & & \\ \sigma\uparrow & & \downarrow & & \\ \Sigma & & A'|_\sigma & \models_\Sigma & \varphi \end{array}$$

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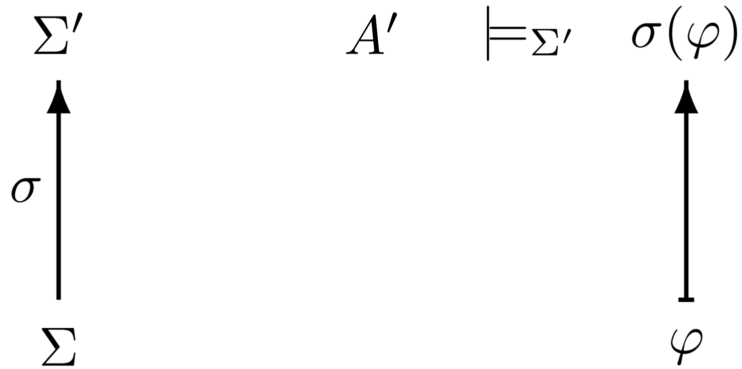

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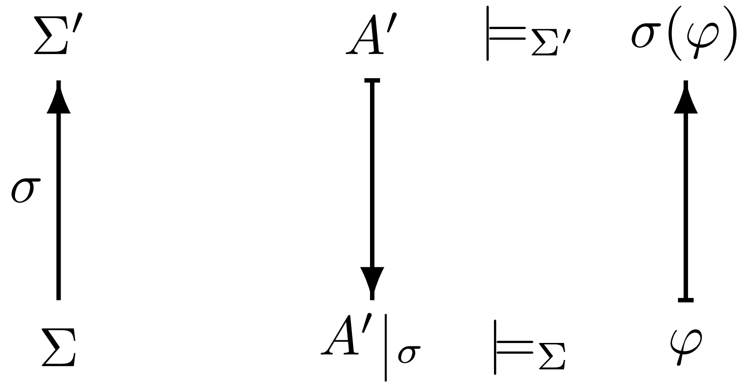

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\Sigma & & A'|_\sigma & \models_{\Sigma} & \varphi
\end{array}
$$

$$
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$$

Proof (idea): for $t \in |T_\Sigma(X)|$ and $v\colon X \to |A'|_\sigma|$, $t_{A'|_\sigma}[v] = \sigma(t)_{A'}[v']$, where $v'\colon X' \to |A'|$ is given by $v'_{\sigma(s)}(x) = v_s(x)$ for $s \in S$, $x \in X_s$.

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$$

$$A'|_{\sigma} \models_{\Sigma} \varphi \iff A' \models_{\Sigma'} \sigma(\varphi)$$

TRUTH is preserved (at least) under:

- *change of notation*
- *restriction/extension of irrelevant context*

Preservation of consequence

Given any signature morphism $\sigma\colon \Sigma \to \Sigma'$, set of Σ-equations Φ and Σ-equation φ:

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In general, the equivalence does not hold!

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Moreover, if $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$ is surjective then:

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Specification morphisms

Specification morphism:

$$\sigma\colon \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$$

is a signature morphism $\sigma\colon \Sigma \to \Sigma'$ such that for all $M' \in \mathbf{Alg}(\Sigma')$:

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Conservativity

A specification morphism:

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- Translation of syntax, reducts of algebras, satisfaction condition, and many other notions and results: similarly as before.

not quite all though...

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- *Partial* Σ*-algebra*:

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as before, but operations $f_A \colon |A|_{s_1} \times \ldots \times |A|_{s_n} \rightharpoonup |A|_s$, for $f \colon s_1 \times \ldots \times s_n \to s$, may now be *partial functions*.

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- $\mathbf{PAlg}(\Sigma)$ stands for the class of all partial Σ-algebras.

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  For $f : s_1 \times \ldots s_n \to s$ and $a_1 \in |A_{sub}|_{s_1}$, ..., $a_n \in |A_{sub}|_{s_n}$
  - *(strong) subalgebra*: if $f_A(a_1, \ldots, a_n)$ is defined then $f_{A_{sub}}(a_1, \ldots, a_n)$ is defined
  - *(full) subalgebra*: if $f_A(a_1, \ldots, a_n)$ is defined and $f_A(a_1, \ldots, a_n) \in |A_{sub}|_s$ then $f_{A_{sub}}(a_1, \ldots, a_n)$ is defined
  - *(weak) subalgebra*: if $f_{A_{sub}}(a_1, \ldots, a_n)$ is defined then $f_A(a_1, \ldots, a_n)$ is defined

  and $f_{A_{sub}}(a_1, \ldots, a_n) = f_A(a_1, \ldots, a_n)$.

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  the natural homomorphism from A to $A/{\equiv}$ is strong if the congruence is strong.

Formulae

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(Strong) equation:

$$\forall X.t \stackrel{s}{=} t'$$

as before

Satisfaction relation

partial Σ-algebra A *satisfies* $\forall X.t \stackrel{s}{=} t'$

$$A \models \forall X.t \stackrel{s}{=} t'$$

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Definedness formula:

$$\forall X.def\ t$$

where X is a set of variables, and $t \in |T_\Sigma(X)|_s$ is a term

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BTW:

- $\forall X.t \stackrel{e}{=} t'$ iff $\forall X.(t \stackrel{s}{=} t' \wedge \mathit{def}\ t)$
- $\forall X.t \stackrel{s}{=} t'$ iff $\forall X.(\mathit{def}\ t \iff \mathit{def}\ t') \wedge (\mathit{def}\ t \implies t \stackrel{e}{=} t')$

Further notions and results

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- signature morphisms, translation of formulae, reducts of partial algebras, satisfaction condition; specification morphisms, conservativity, etc. (easy)
- even more general signature morphisms: $\delta\colon \Sigma \to \Sigma'$ maps sort names to sort names, and operation names $f\colon s_1 \times \ldots s_n \to s$ to sequences $\langle \varphi_i, t_i \rangle_{i \geq 0}$, where φ_i is a Σ'-formula and t_i is a Σ'-term of sort $\delta(s)$, both with variables among $x_1{:}\delta(s_1), \ldots, x_n{:}\delta(s_n)$; syntax does not quite translate, but reducts are well defined...

Example

$$
\begin{array}{l}
\textbf{spec}\ \textsc{NatPred} = \textbf{free}\ \{\ \textbf{sort}\ Nat \\
\qquad \textbf{ops}\ 0 : Nat; \\
\qquad\qquad succ : Nat \to Nat; \\
\qquad\qquad \_+\_ : Nat \times Nat \to Nat \\
\qquad\qquad pred : Nat \to ?\ Nat \\
\qquad \textbf{axioms}\ \forall n{:}Nat \bullet n + 0 = n; \\
\qquad\qquad \forall n, m{:}Nat \bullet n + succ(m) = succ(n + m) \\
\qquad\qquad \forall n{:}Nat \bullet pred(succ(n)) \stackrel{s}{=} n; \\
\}
\end{array}
$$

Example′

$$\textbf{spec}\ \textsc{NatPred}' = \textbf{free type}\ Nat ::= 0 \mid succ(pred :?\ Nat)$$
$$\textbf{op}\ \_+\_ : Nat \times Nat \to Nat$$
$$\textbf{axioms}\ \forall n{:}Nat \bullet n + 0 = n;$$
$$\forall n, m{:}Nat \bullet n + succ(m) = succ(n + m)$$

$$\boxed{\textsc{NatPred} \equiv \textsc{NatPred}'}$$