

Cartesian closed categories

CCC

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CCC

*typed functional programming
vs.
category theory*

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Recall the definitions of these categories and the constructions of products in each of them

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$$\mathbf{K} \xrightarrow{- \times B} \mathbf{K} \quad C$$

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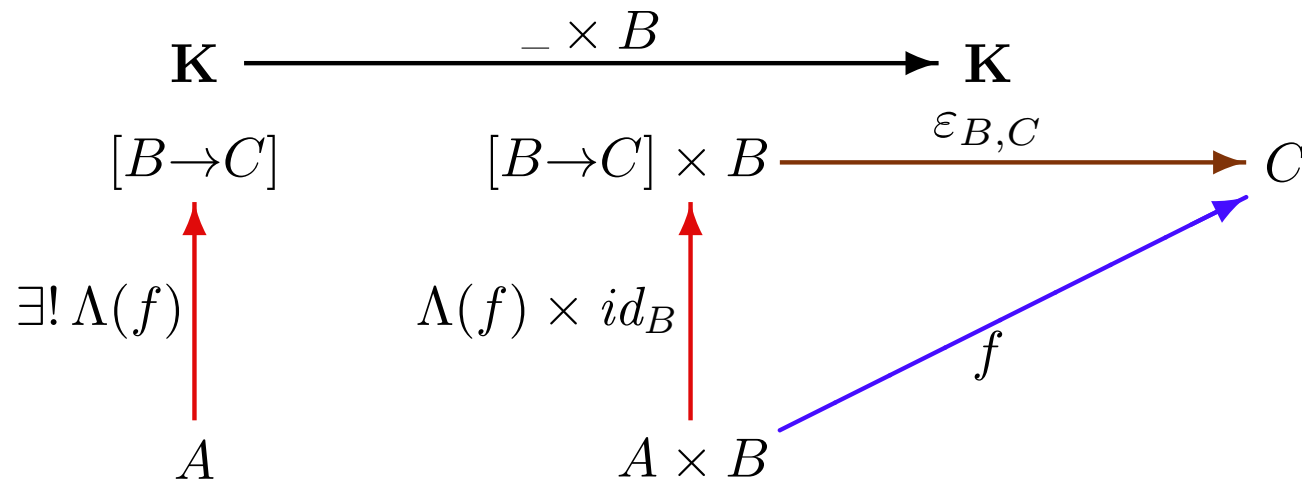
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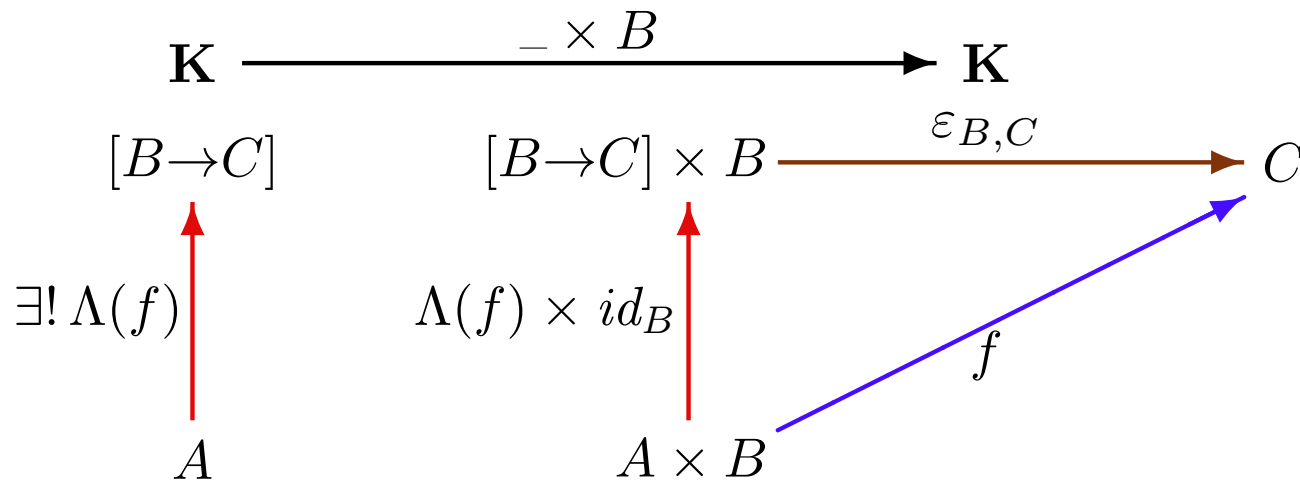
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BTW: for $g: A \rightarrow A'$ and $h: B \rightarrow B'$

$$g \times h = \langle \pi_{A,B}; g, \pi'_{A,B}; h \rangle: A \times B \rightarrow A' \times B'$$

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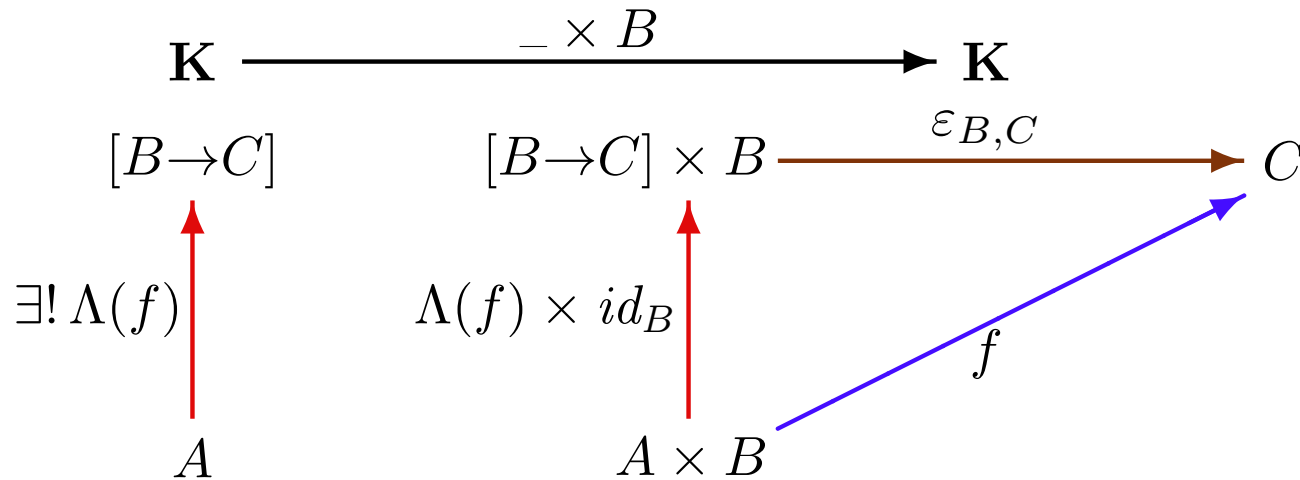
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Non-examples: **Pfn**, $T_{\Sigma, \Phi}^{op}$.

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- for each $B \in |\mathbf{K}|$, $-\times B : \mathbf{K} \rightarrow \mathbf{K}$ has right adjoint $[B \rightarrow -] : \mathbf{K} \rightarrow \mathbf{K}$, with counit given by $\varepsilon_{B,C} : [B \rightarrow C] \times B \rightarrow C$, for $C \in |\mathbf{K}|$.

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Contexts

Contexts Γ are of the form:

- $x_1:\tau_1, \dots, x_n:\tau_n$,
where $n \geq 0$, $x_1, \dots, x_n$ are distinct variables, and $\tau_1, \dots, \tau_n \in \mathcal{T}$

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Weakening – context extension:

If $\Gamma \vdash M : \tau$ then $\Gamma, \Gamma' \vdash M : \tau$

- $\llbracket M \rrbracket^\Gamma : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$,
- $\llbracket M \rrbracket^{\Gamma, \Gamma'} : \llbracket \Gamma, \Gamma' \rrbracket \rightarrow \llbracket \tau \rrbracket$, and

$$\llbracket M \rrbracket^{\Gamma, \Gamma'} = \rho; \pi_{\llbracket \Gamma \rrbracket, \llbracket \Gamma' \rrbracket}; \llbracket M \rrbracket^\Gamma$$

where $\rho : \llbracket \Gamma, \Gamma' \rrbracket \rightarrow \llbracket \Gamma \rrbracket \times \llbracket \Gamma' \rrbracket$
is the obvious isomorphism.

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Semantics of λ -terms

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Easy: $\rho; \llbracket M \rrbracket: \llbracket \Gamma' \rrbracket \times \llbracket \tau_i \rrbracket \rightarrow \llbracket \tau \rrbracket$, hence $\Lambda(\rho; \llbracket M \rrbracket): \llbracket \Gamma' \rrbracket \rightarrow \llbracket \tau_i \rrbracket \rightarrow \llbracket \tau \rrbracket$.

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Easy: $\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rightarrow \tau' \rrbracket \times \llbracket \tau \rrbracket$,

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Easy: $\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rightarrow \tau' \rrbracket \times \llbracket \tau \rrbracket$, which can be composed with

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Easy: $\langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rightarrow \tau' \rrbracket \times \llbracket \tau \rrbracket$, which can be composed with

$\varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket}: (\llbracket \tau \rrbracket \rightarrow \llbracket \tau' \rrbracket) \times \llbracket \tau \rrbracket \rightarrow \llbracket \tau' \rrbracket$.

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$$\frac{\Gamma \vdash M: \tau \quad \Gamma \vdash N: \tau'}{\Gamma \vdash \langle M, N \rangle: \tau \times \tau'}$$

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-

$$\frac{\Gamma \vdash M: \tau \quad \Gamma \vdash N: \tau'}{\Gamma \vdash \langle M, N \rangle: \tau \times \tau'}$$

Given $\llbracket M \rrbracket: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$ and $\llbracket N \rrbracket: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau' \rrbracket$,

Semantics of λ -terms

- $\llbracket x_i \rrbracket = \pi_i: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau_i \rrbracket$ for $\Gamma = x_1:\tau_1, \dots, x_n:\tau_n$, π_i is the obvious projection.
- $\llbracket \lambda x_i:\tau_i. M \rrbracket = \Lambda(\rho; \llbracket M \rrbracket): \llbracket \Gamma' \rrbracket \rightarrow \llbracket \tau_i \rightarrow \tau \rrbracket$ for $\Gamma = x_1:\tau_1, \dots, x_n:\tau_n$, $\Gamma' = x_1:\tau_1, \dots, x_{i-1}:\tau_{i-1}, x_{i+1}:\tau_{i+1}, \dots, x_n:\tau_n$, and $\Gamma \vdash M: \tau$, where $\rho: \llbracket \Gamma' \rrbracket \times \llbracket \tau_i \rrbracket \rightarrow \llbracket \Gamma \rrbracket$ is the obvious isomorphism.
- $\llbracket MN \rrbracket = \langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle; \varepsilon_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket}: \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau' \rrbracket$ for $\Gamma \vdash M: \tau \rightarrow \tau'$ and $\Gamma \vdash N: \tau$.
- $\llbracket \langle \rangle \rrbracket = \langle \rangle_{\llbracket \Gamma \rrbracket}: \llbracket \Gamma \rrbracket \rightarrow 1$
-

$$\frac{\Gamma \vdash M: \tau \quad \Gamma \vdash N: \tau'}{\Gamma \vdash \langle M, N \rangle: \tau \times \tau'}$$

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-

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Hence $\Lambda(\pi'_{\llbracket \Gamma \rrbracket, \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket}; \pi_{\llbracket \tau \rrbracket, \llbracket \tau' \rrbracket}): \llbracket \Gamma \rrbracket \rightarrow \llbracket \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket \rightarrow \llbracket \tau \rrbracket$.

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Equational β, η -calculus

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Judgements:

$$\Gamma \vdash M = N : \tau$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

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Rules: reflexivity, symmetry, transitivity, congruence.

Soundness

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Given $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$

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Proof: :-)

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Warning: The “real work” is in the proof of soundness for (β), where induction on the structure of terms is needed.

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Associativity and identity properties: *easy!*

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SUMMING UP:

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