Cartesian closed categories

CCC

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typed functional programming
vs.
category theory

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Recall the definitions of these categories and the constructions of products in each of them

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$$C$$

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Definition: *A Cartesian category* $\mathbf{K}$ *is closed if for all* $B, C \in |\mathbf{K}|$ *we indicate* $[B{\to}C] \in |\mathbf{K}|$ *and* $\varepsilon_{B,C} : [B{\to}C] \times B \to C$

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BTW: for $g \colon A \to A'$ and $h \colon B \to B'$

$$\boxed{g \times h = \langle \pi_{A,B};g, \pi'_{A,B};h \rangle \colon A \times B \to A' \times B'}$$

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Non-examples: $\mathbf{Pfn}$, $T^{op}_{\Sigma,\Phi}$.

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- for each $B \in |\mathbf{K}|$, $\_ \times B\colon \mathbf{K} \to \mathbf{K}$ has right adjoint $[B \to \_]\colon \mathbf{K} \to \mathbf{K}$, with counit given by $\varepsilon_{B,C}\colon [B \to C] \times B \to C$, for $C \in |\mathbf{K}|$.

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    - for $h : A \to [B\to C]$, $\Lambda((h \times id_B);\varepsilon_{B,C}) = h$.

Typed λ-calculus with products

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Contexts Γ are of the form:

- $x_1{:}\tau_1, \ldots, x_n{:}\tau_n$,
  where $n \geq 0$, $x_1, \ldots, x_n$ are distinct variables, and $\tau_1, \ldots, \tau_n \in \mathcal{T}$

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Weakening – context extension:

If $\Gamma \vdash M : \tau$ then $\Gamma, \Gamma' \vdash M : \tau$

- $[\![M]\!]^\Gamma \colon [\![\Gamma]\!] \to [\![\tau]\!]$,
- $[\![M]\!]^{\Gamma,\Gamma'} \colon [\![\Gamma,\Gamma']\!] \to [\![\tau]\!]$, and

$$\boxed{[\![M]\!]^{\Gamma,\Gamma'} = \rho;\pi_{[\![\Gamma]\!],[\![\Gamma']\!]};[\![M]\!]^\Gamma}$$

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Easy: $\rho;[\![M]\!] \colon [\![\Gamma']\!] \times [\![\tau_i]\!] \to [\![\tau]\!]$,

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Easy: $\rho;[\![M]\!] \colon [\![\Gamma']\!] \times [\![\tau_i]\!] \to [\![\tau]\!]$, hence $\Lambda(\rho;[\![M]\!]) \colon [\![\Gamma']\!] \to [[\![\tau_i]\!]{\to}[\![\tau]\!]]$.

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–

$$\frac{\Gamma \vdash M : \tau \qquad \Gamma \vdash N : \tau'}{\Gamma \vdash \langle M, N \rangle : \tau{\times}\tau'}$$

Semantics of λ-terms

– $\boxed{[\![x_i]\!] = \pi_i\colon [\![\Gamma]\!] \to [\![\tau_i]\!]}$ for $\Gamma = x_1{:}\tau_1, \ldots, x_n{:}\tau_n$, π_i is the obvious projection.

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$\Gamma' = x_1{:}\tau_1, \ldots, x_{i-1}{:}\tau_{i-1}, x_{i+1}{:}\tau_{i+1}, \ldots, x_n{:}\tau_n$, and $\Gamma \vdash M \colon \tau$, where
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Semantics of λ-terms

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$$\frac{}{\Gamma \vdash \pi_{\tau,\tau'} : \tau \times \tau' \to \tau}$$

Semantics of λ-terms

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Define $[\![\pi_{\tau,\tau'}]\!]\colon [\![\Gamma]\!] \to [\![\tau{\times}\tau'{\to}\tau]\!]$.

Semantics of λ-terms

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  Define $[\![\pi_{\tau,\tau'}]\!]\colon [\![\Gamma]\!] \to [\![\tau{\times}\tau'{\to}\tau]\!]$. We have $\pi'_{[\![\Gamma]\!],[\![\tau]\!]\times[\![\tau']\!]}\colon [\![\Gamma]\!] \times ([\![\tau]\!] \times [\![\tau']\!]) \to [\![\tau]\!] \times [\![\tau']\!]$, and $\pi_{[\![\tau]\!],[\![\tau']\!]}\colon [\![\tau]\!] \times [\![\tau']\!] \to [\![\tau]\!]$.

Semantics of λ-terms

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Define $[\![\pi_{\tau,\tau'}]\!]\colon [\![\Gamma]\!] \to [\![\tau \times \tau' \to \tau]\!]$. We have $\pi'_{[\![\Gamma]\!],[\![\tau]\!]\times[\![\tau']\!]}\colon [\![\Gamma]\!] \times ([\![\tau]\!] \times [\![\tau']\!]) \to [\![\tau]\!] \times [\![\tau']\!]$, and $\pi_{[\![\tau]\!],[\![\tau']\!]}\colon [\![\tau]\!] \times [\![\tau']\!] \to [\![\tau]\!]$.
Hence $\Lambda(\pi'_{[\![\Gamma]\!],[\![\tau]\!]\times[\![\tau']\!]};\pi_{[\![\tau]\!],[\![\tau']\!]})\colon [\![\Gamma]\!] \to [[\![\tau]\!] \times [\![\tau']\!] \to [\![\tau]\!]]$.

Semantics of λ-terms

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Equational β, η-calculus

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Judgements:

$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

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Axioms:

Equational β, η-calculus

Judgements:

$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

Axioms:

(β) $\Gamma \vdash (\lambda x{:}\tau.M)N = M[N/x] : \tau'$, for $\Gamma \vdash \lambda x{:}\tau.M : \tau{\to}\tau'$, $\Gamma \vdash N : \tau$

Equational β, η-calculus

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(η) $\Gamma \vdash \lambda x{:}\tau.Mx = M : \tau \to \tau'$, for $\Gamma \vdash M : \tau \to \tau'$, $x \notin dom(\Gamma)$

Equational β, η-calculus

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- $\Gamma \vdash M = \langle\rangle : 1$, for $\Gamma \vdash M : 1$

Equational β, η-calculus

Judgements:

$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

Axioms:

(β) $\Gamma \vdash (\lambda x{:}\tau.M)N = M[N/x] : \tau'$, for $\Gamma \vdash \lambda x{:}\tau.M : \tau{\rightarrow}\tau'$, $\Gamma \vdash N : \tau$

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- $\Gamma \vdash M = \langle\rangle : 1$, for $\Gamma \vdash M : 1$
- $\Gamma \vdash \pi_{\tau,\tau'}\langle M, N\rangle = M : \tau$ and $\Gamma \vdash \pi'_{\tau,\tau'}\langle M, N\rangle = N : \tau'$, for $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau'$

Equational β, η-calculus

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$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

Axioms:

(β) $\Gamma \vdash (\lambda x{:}\tau.M)N = M[N/x] : \tau'$, for $\Gamma \vdash \lambda x{:}\tau.M : \tau \to \tau'$, $\Gamma \vdash N : \tau$

(η) $\Gamma \vdash \lambda x{:}\tau.Mx = M : \tau \to \tau'$, for $\Gamma \vdash M : \tau \to \tau'$, $x \notin dom(\Gamma)$

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Equational β, η-calculus

Judgements:

$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

Axioms:

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Rules: reflexivity, symmetry, transitivity, congruence.

Soundness

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Given $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$

$$\boxed{\text{if } \Gamma \vdash M = N : \tau \text{ then } [\![M]\!] = [\![N]\!]}$$

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Just check that the axioms and rules of the equational β, η-calculus are sound w.r.t. the semantics in any CCC.

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Warning: The “real work” is in the proof of soundness for (β), where induction on the structure of terms is needed.

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Given $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$,

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Associativity and identity properties: *easy!*

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$\boxed{\text{Product and exponent properties: } \textit{easy!}}$

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To wrap this up: add constants of arbitrary types

- Products in $\boldsymbol{\lambda}$:
  - terminal object $1 \in |\boldsymbol{\lambda}|$, with $\langle\rangle_\tau = [\lambda x{:}\tau.\langle\rangle]_\approx$
  - binary product $\tau\times\tau'$, with $\pi_{\tau,\tau'} = [\pi_{\tau,\tau'}]_\approx$, $\pi'_{\tau,\tau'} = [\pi'_{\tau,\tau'}]_\approx$ and pairing $\langle [M]_\approx, [N]_\approx\rangle = [\langle M, N\rangle]_\approx$ for $\vdash M : \tau$, $\vdash N : \tau'$.
- Exponent in $\boldsymbol{\lambda}$: $\tau\to\tau'$, with $\varepsilon_{\tau,\tau'} = [\lambda x{:}(\tau\to\tau')\times\tau.(\pi_{\tau\to\tau',\tau}x)(\pi'_{\tau\to\tau',\tau}x)]_\approx$ and $\Lambda([M]_\approx) = [\lambda x{:}\tau''.\lambda y{:}\tau.M\langle x, y\rangle]_\approx$, for $\vdash M : \tau''\times\tau\to\tau'$.

$\boxed{\boldsymbol{\lambda} \text{ is Cartesian closed}}$

Now: if in every CCC, $[\![M]\!] = [\![N]\!]$, then this holds in particular in $\boldsymbol{\lambda}$, and so $\vdash M = N : \tau$.

To wrap this up: add constants of arbitrary types

SUMMING UP:

CCCs coincide with λ-calculi