Universal constructions: limits and colimits

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Consider and arbitrary but fixed category $\mathbf{K}$ for a while.

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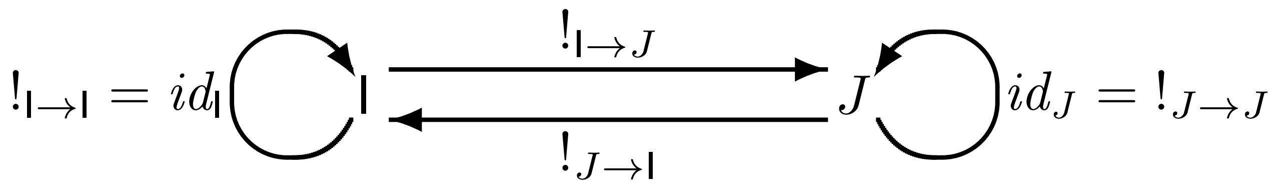

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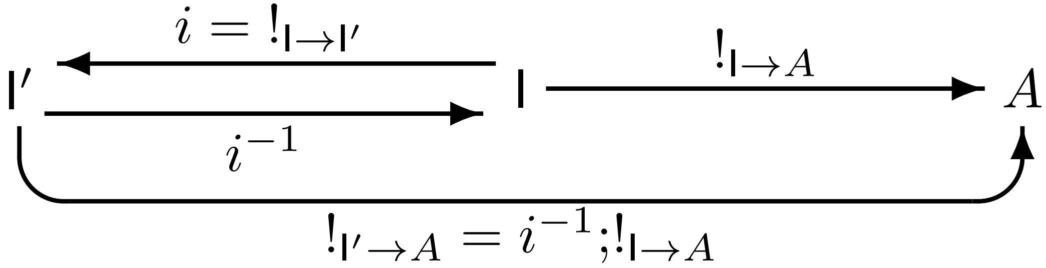

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- Look for categories where there is an object which is both initial and terminal.

Products

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$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

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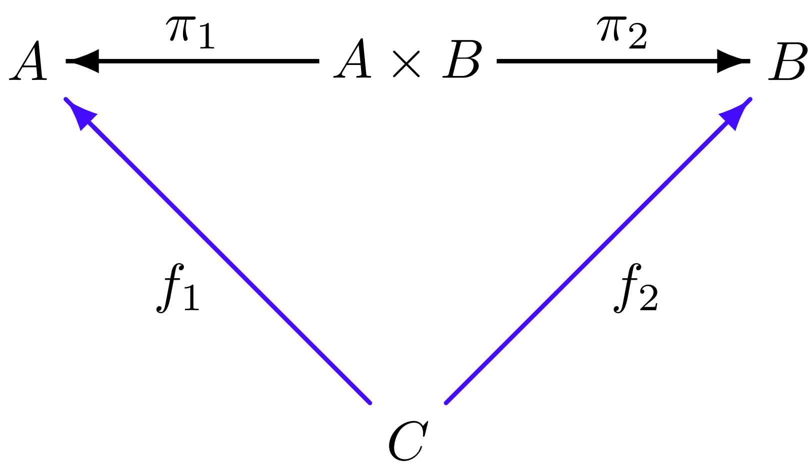

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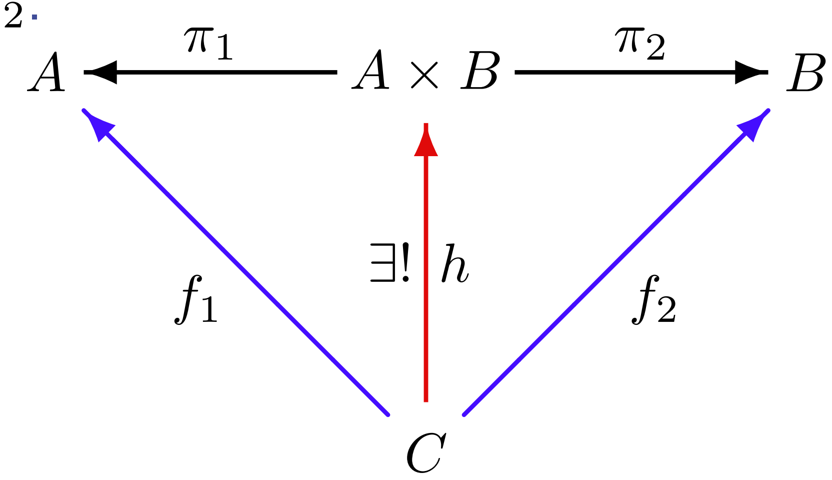

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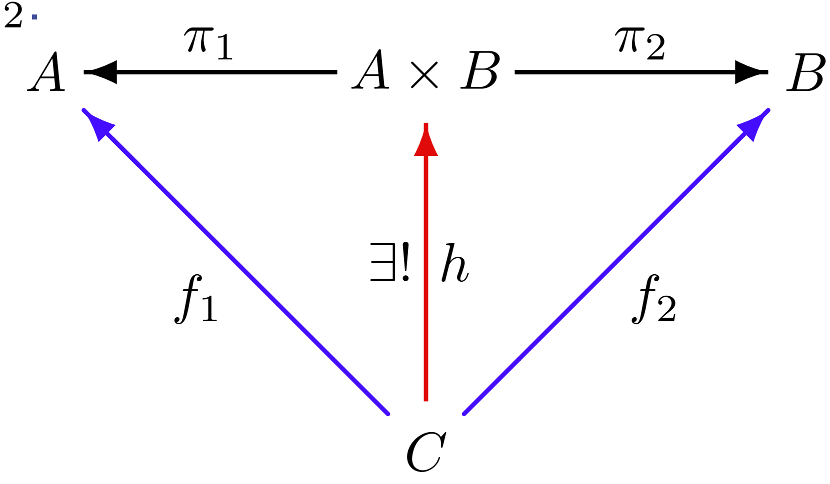

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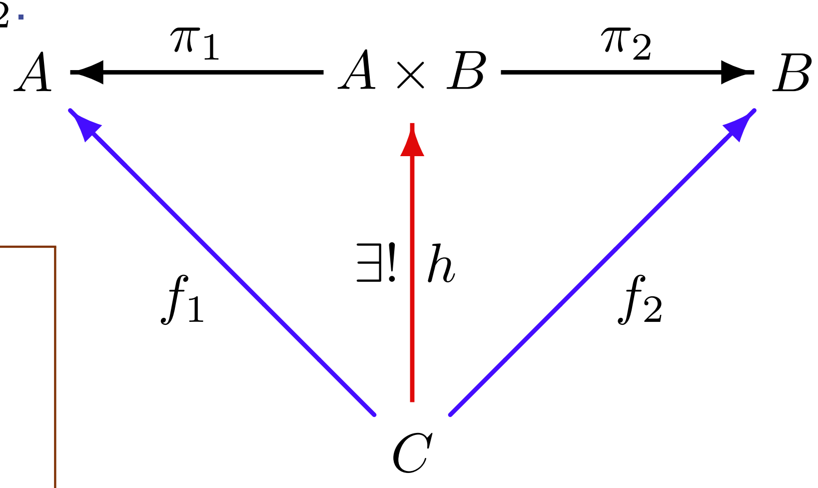

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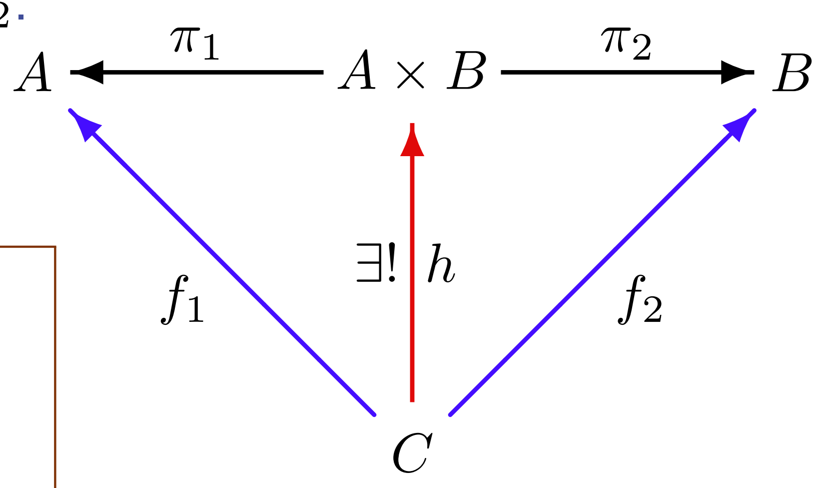

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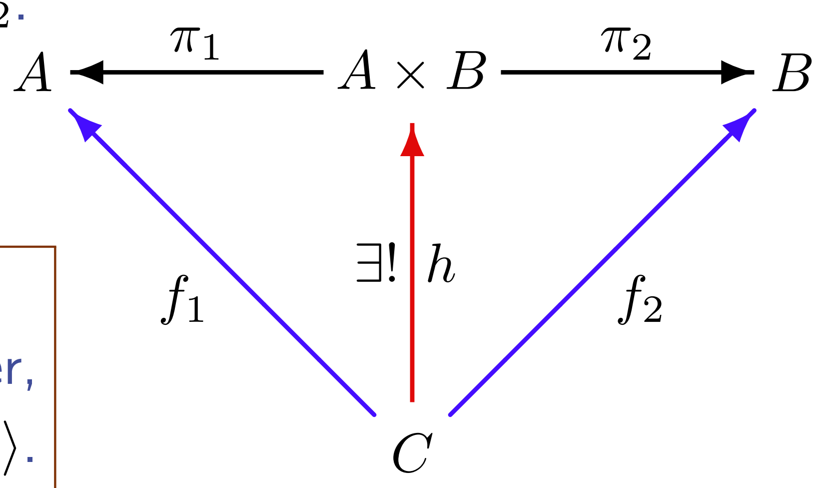

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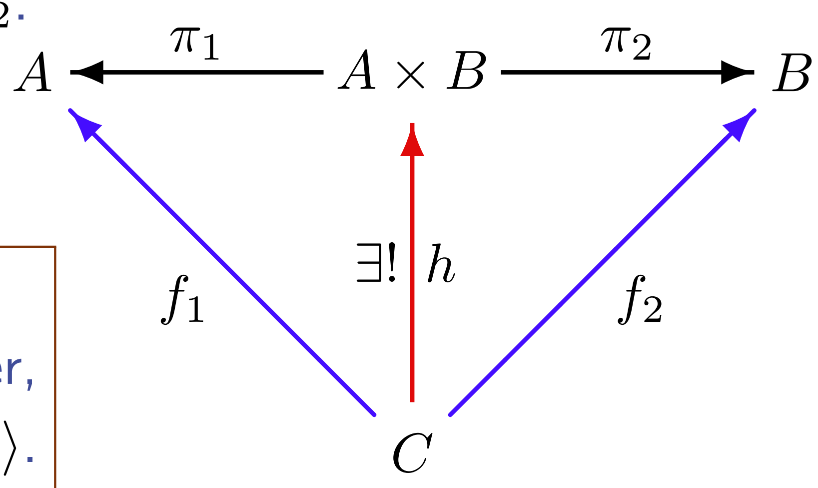

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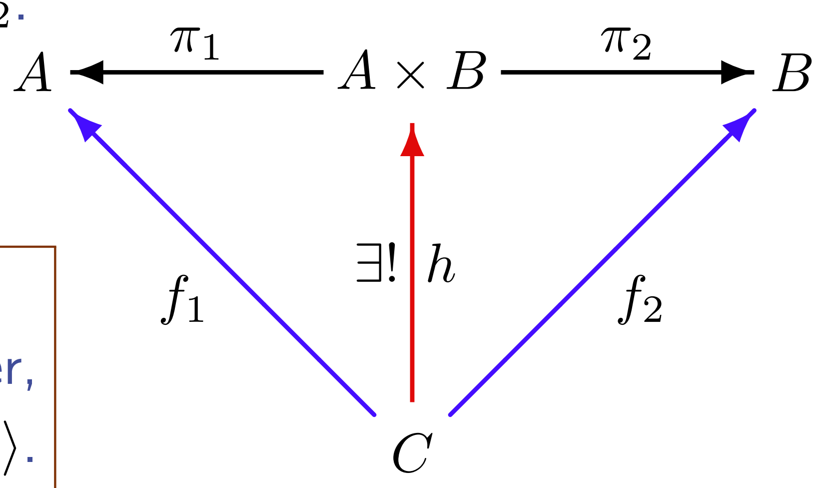

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Theorem: *Products are defined to within an isomorphism (which commutes with projections).*

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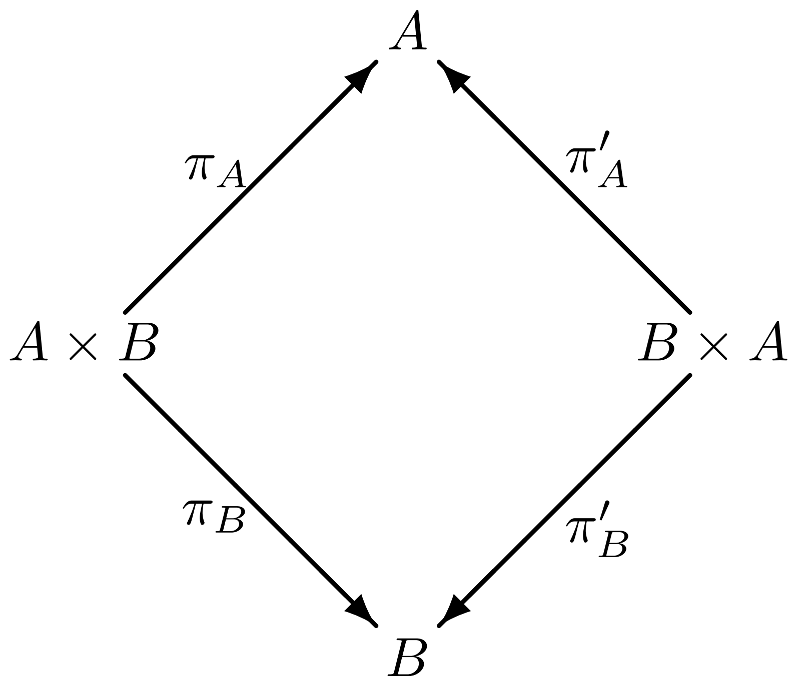

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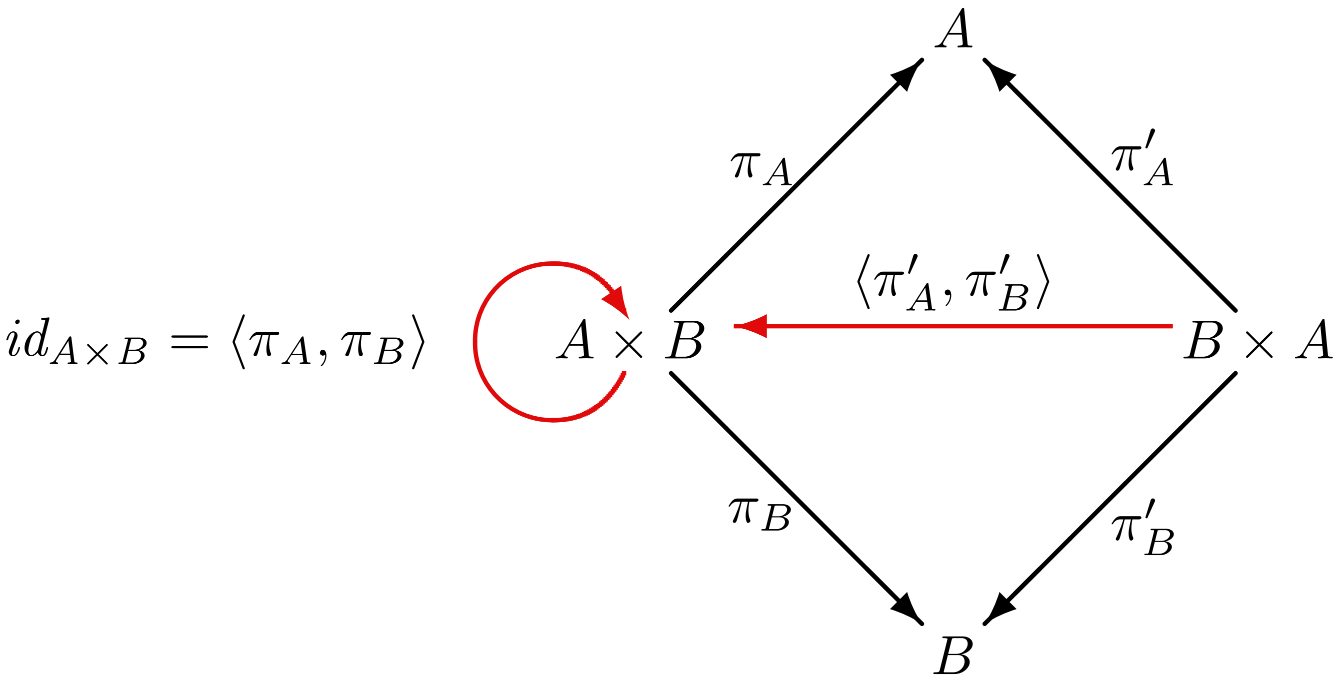

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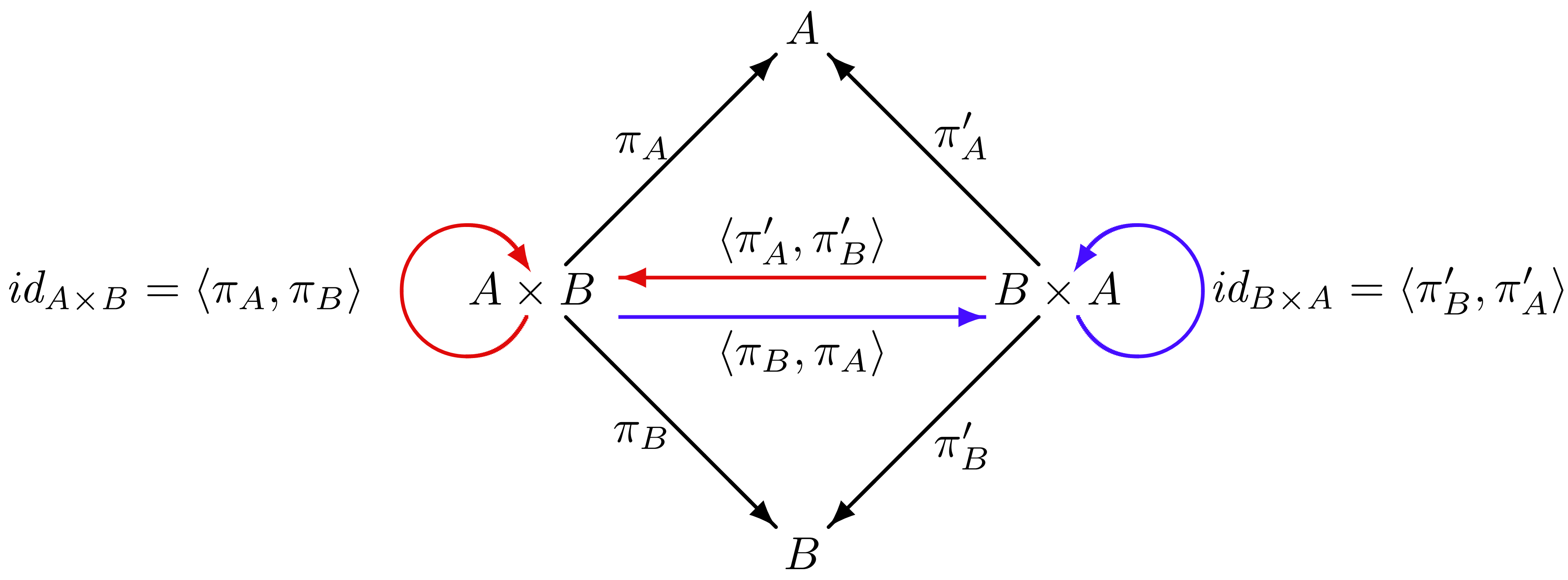

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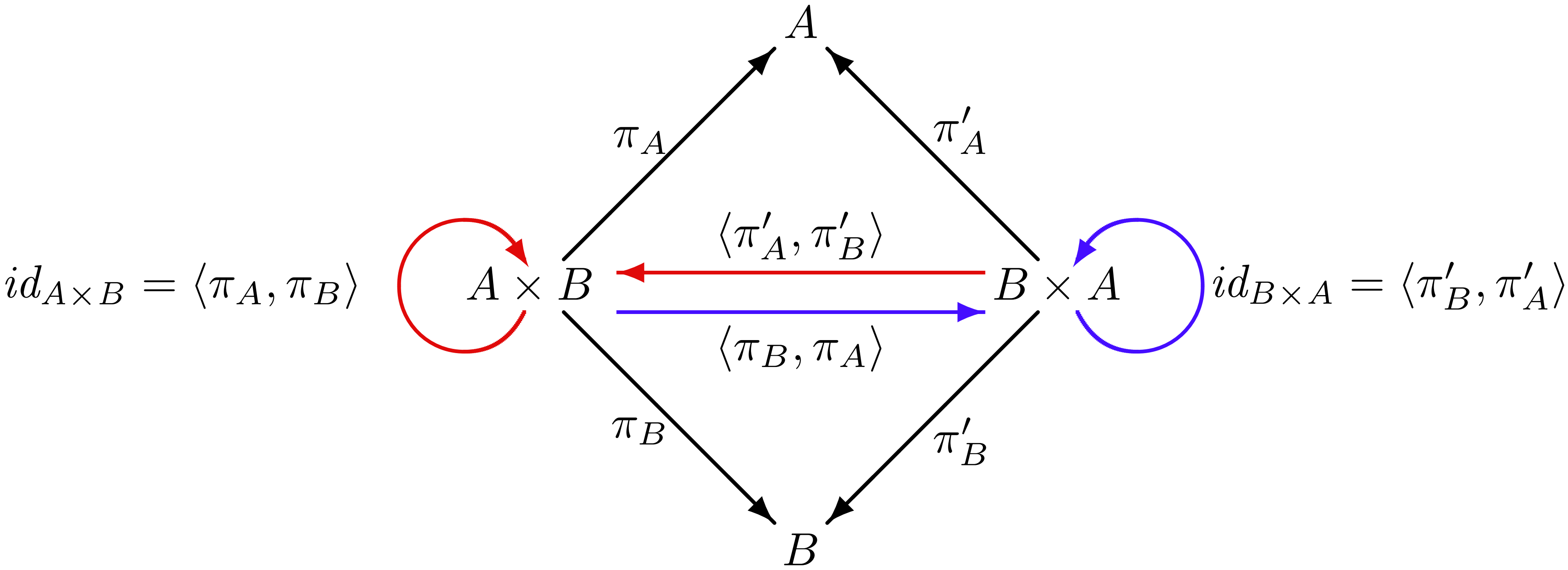

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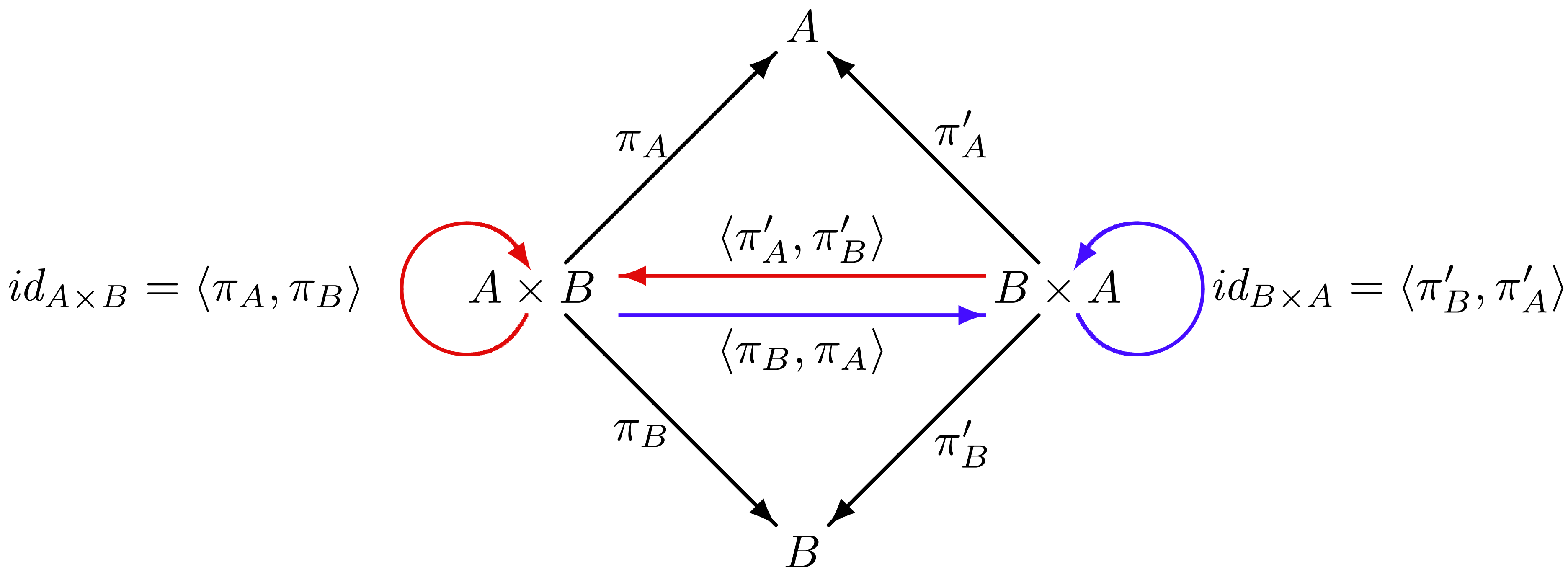

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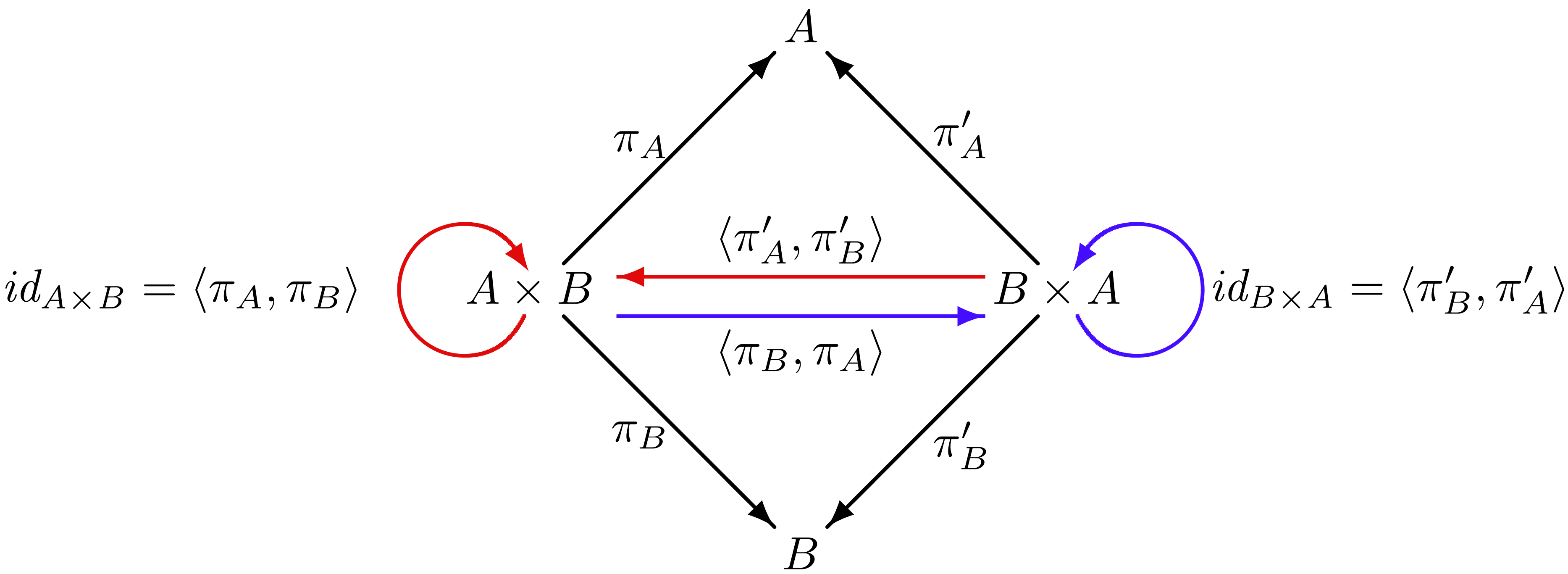

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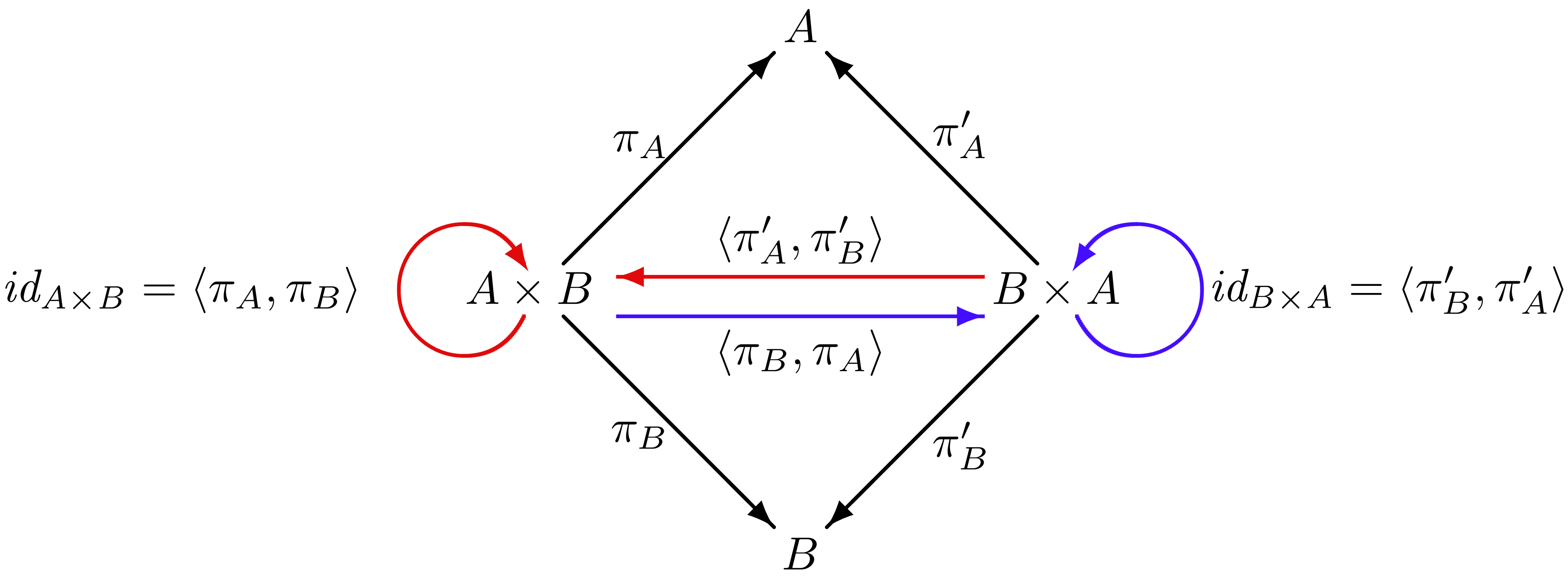

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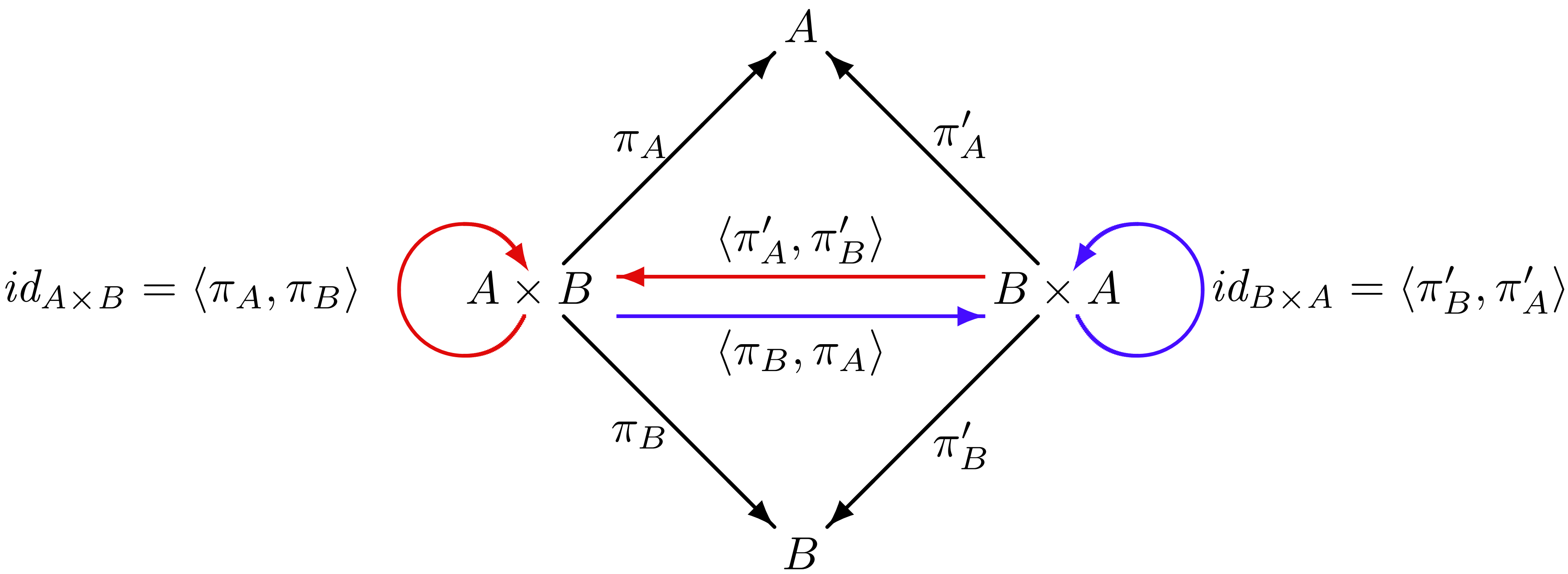

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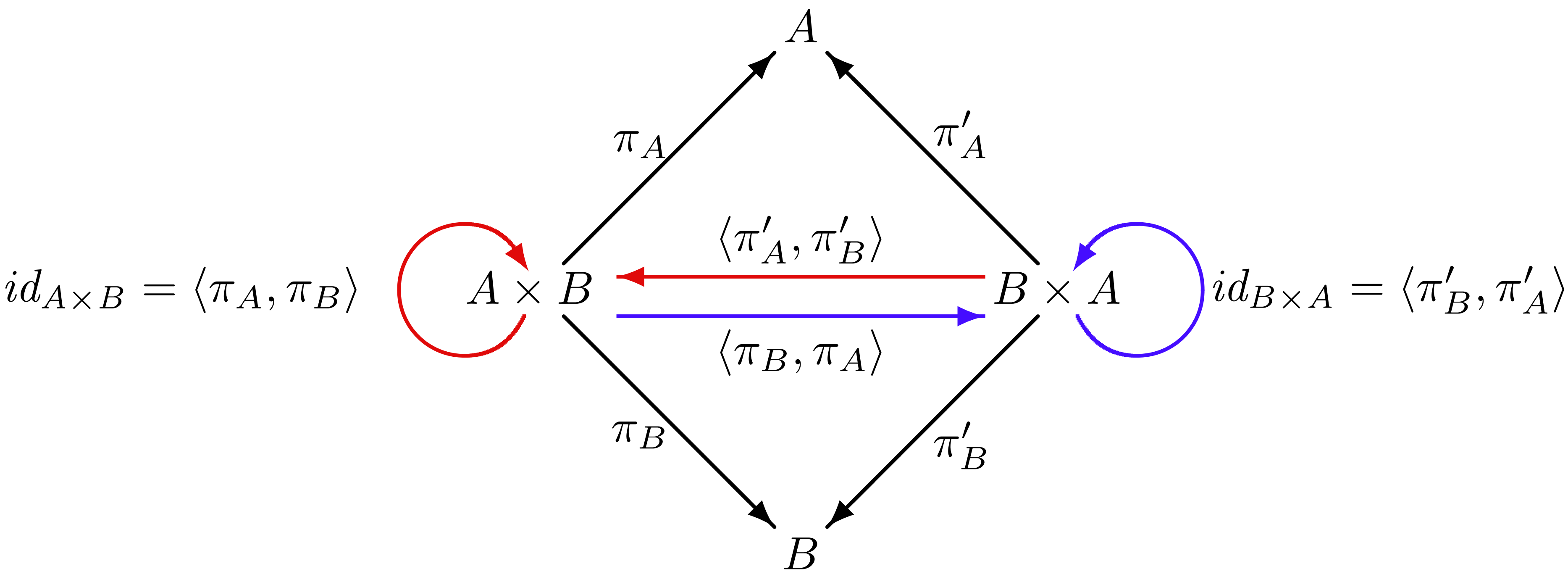

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  – Thus: $\langle \pi_B, \pi_A \rangle;\langle \pi'_A, \pi'_B \rangle = \langle \pi_A, \pi_B \rangle = id_{A\times B}$

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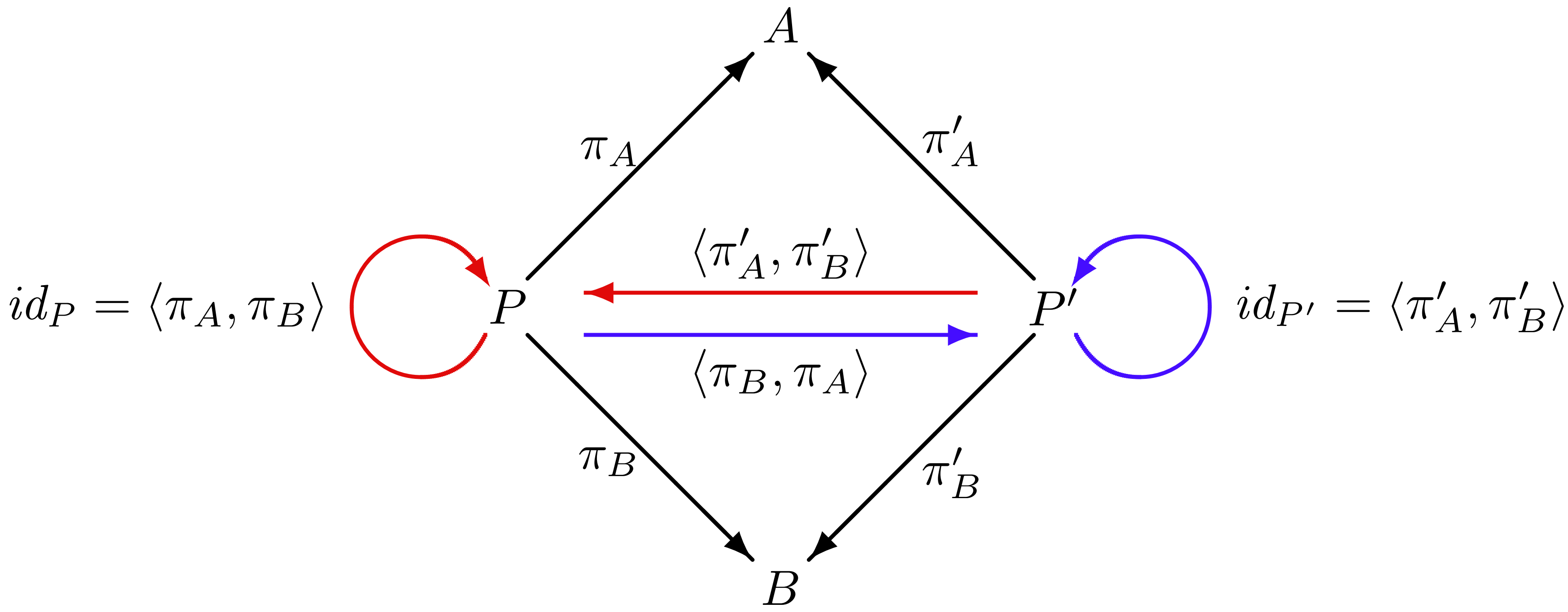

– By much the same argument, any two products of A and B are isomorphic.

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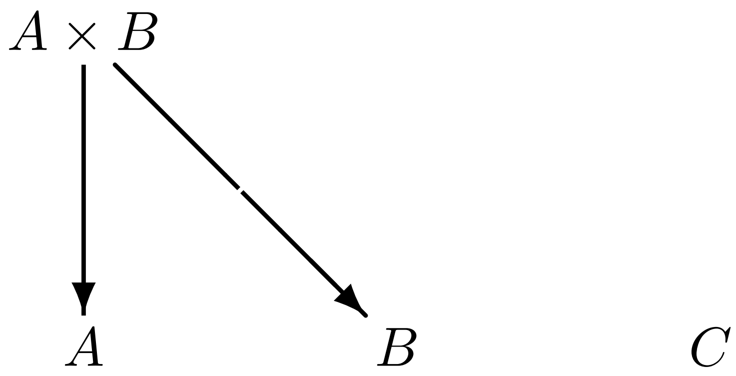

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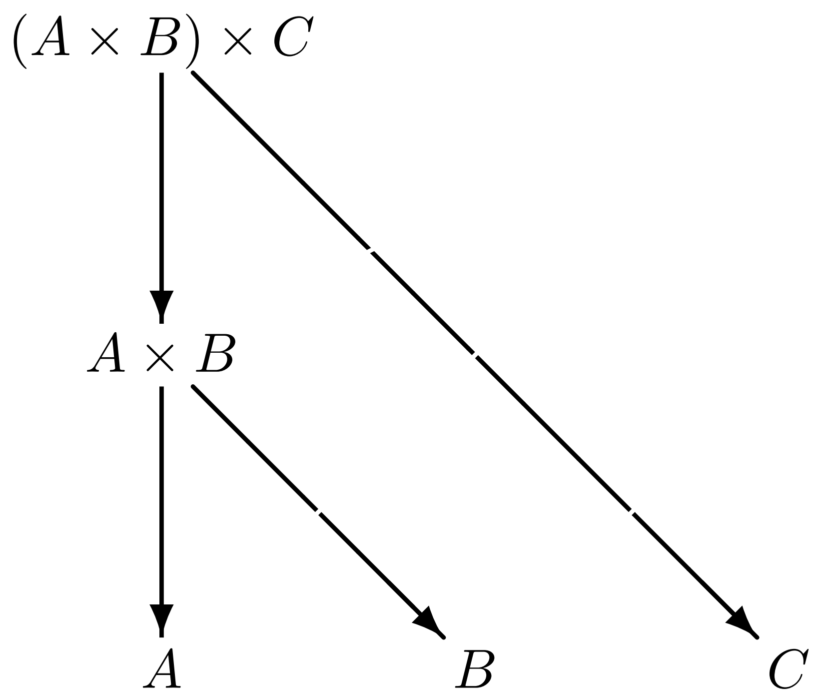

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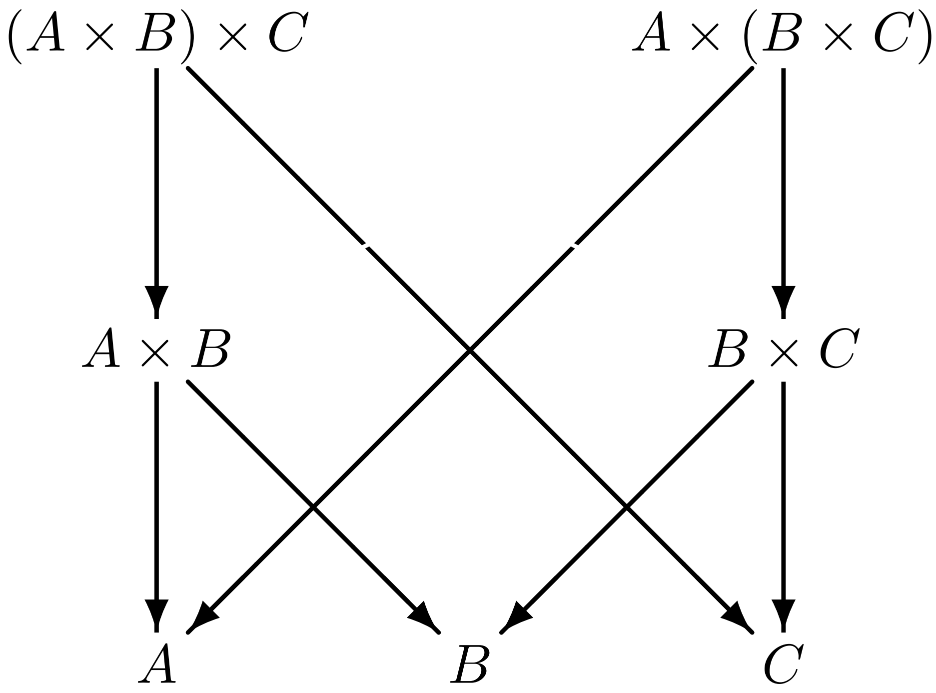

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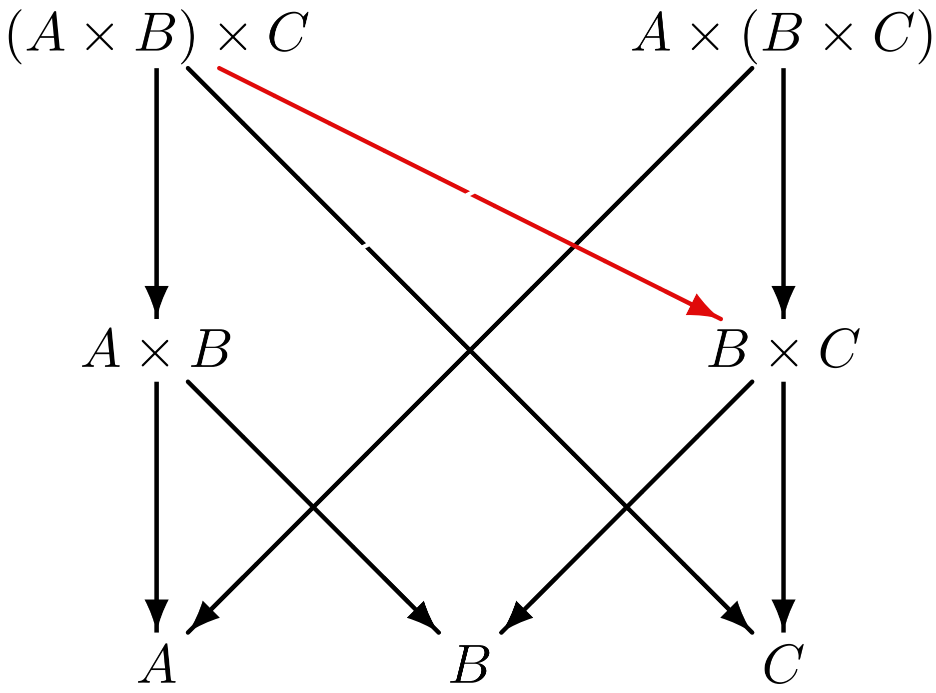

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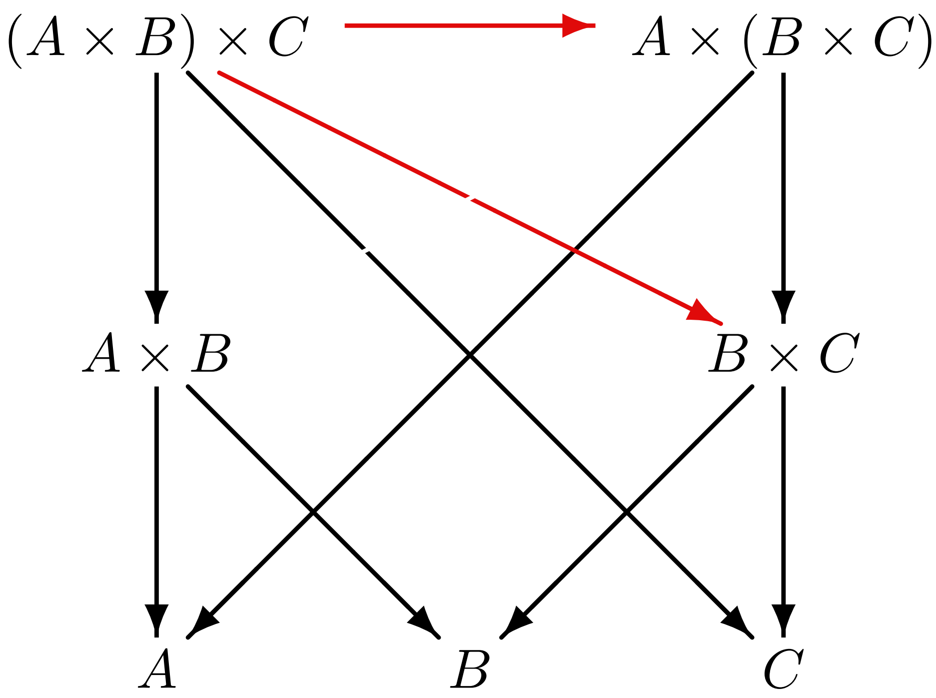

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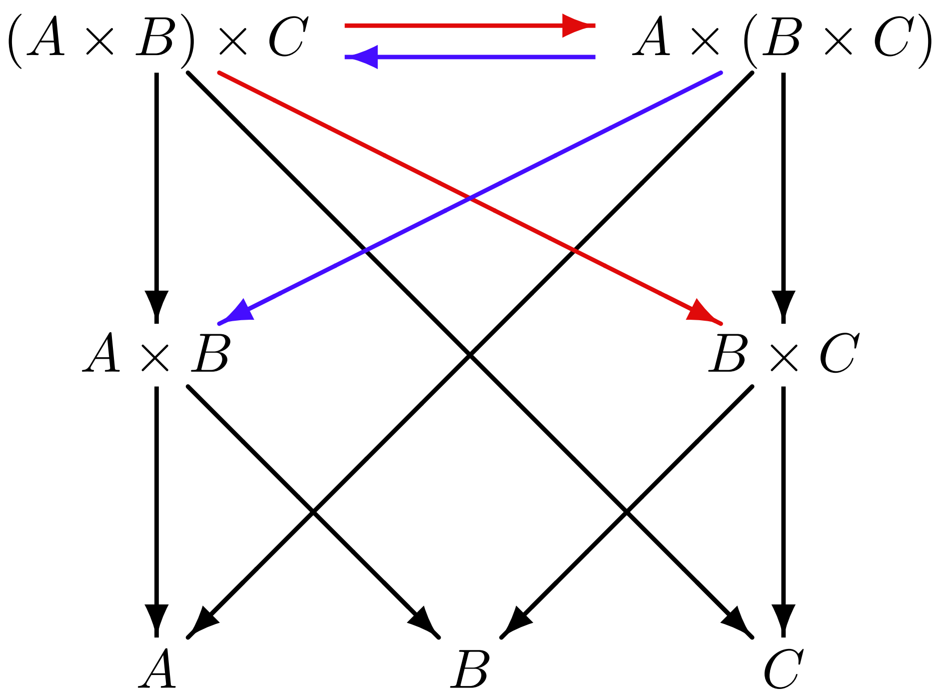

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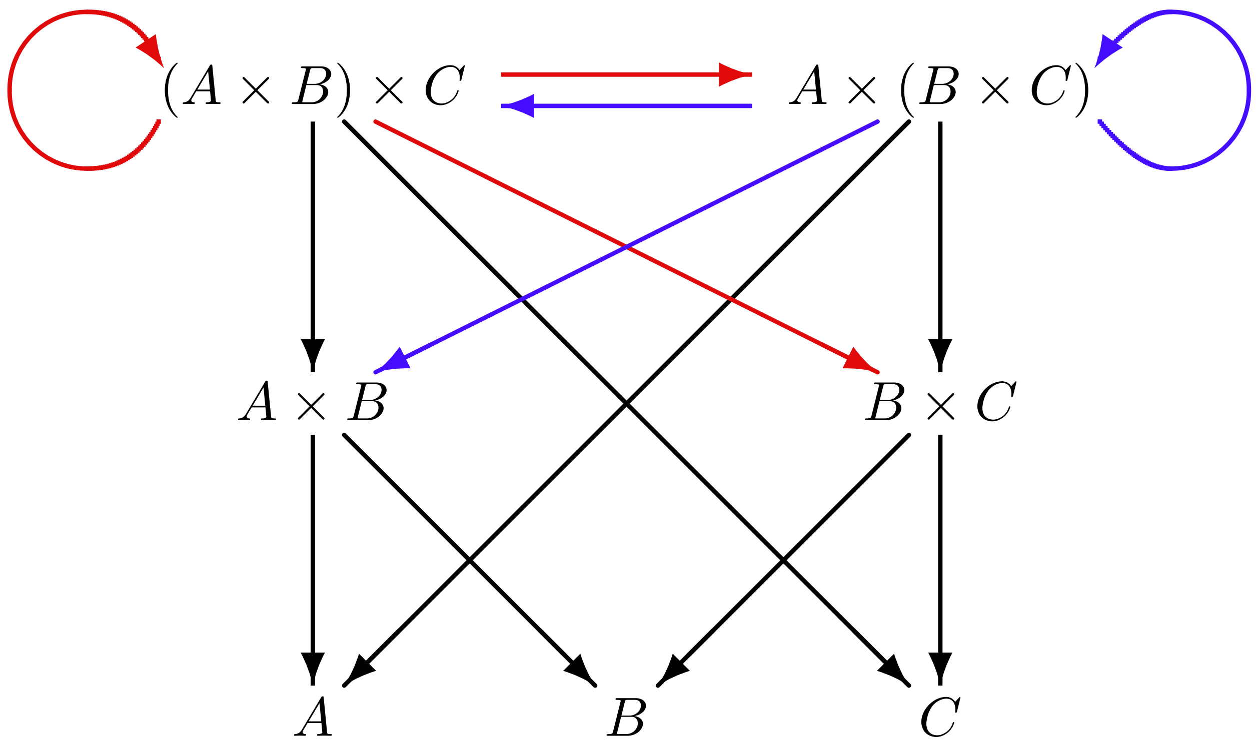

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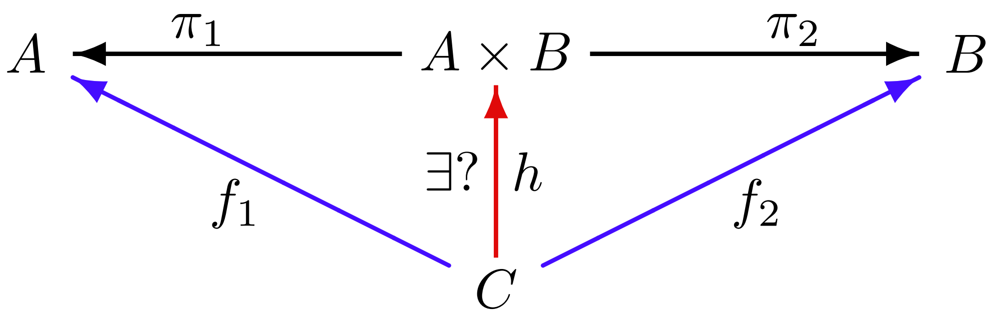

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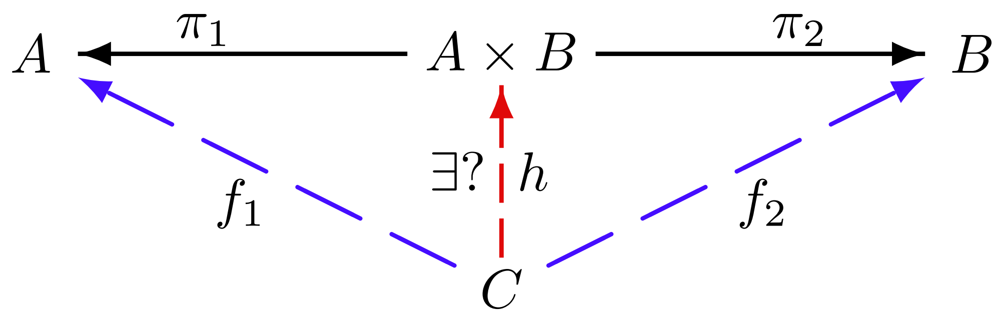

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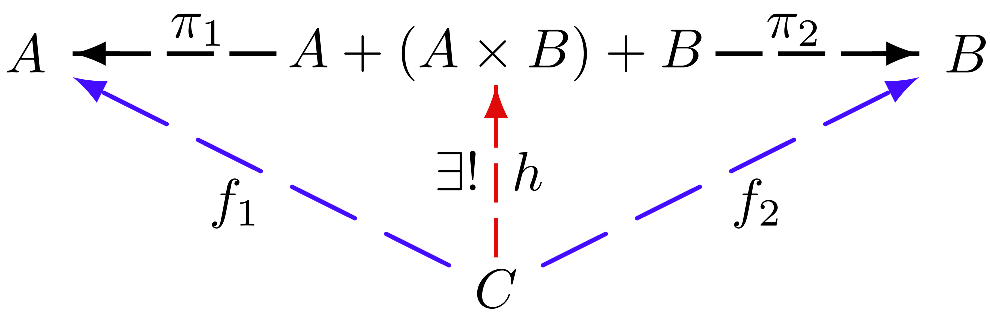

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  - BTW: What about products in $\mathbf{Rel}^{op}$?

Coproducts

coproduct $=$ co-product

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A B

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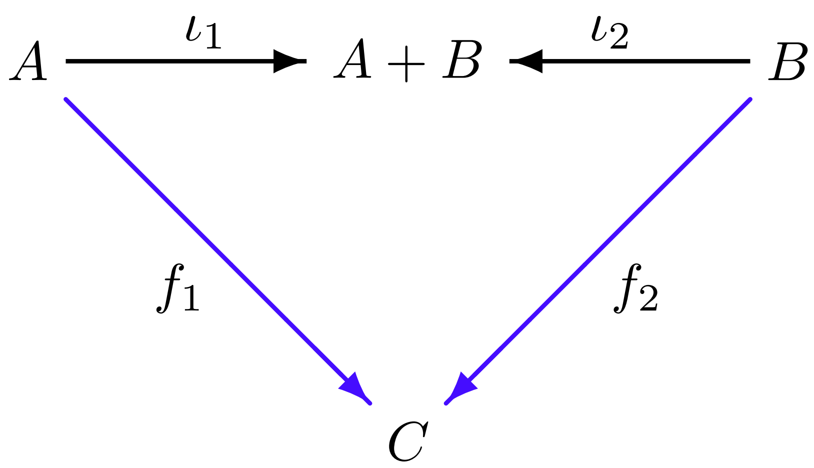

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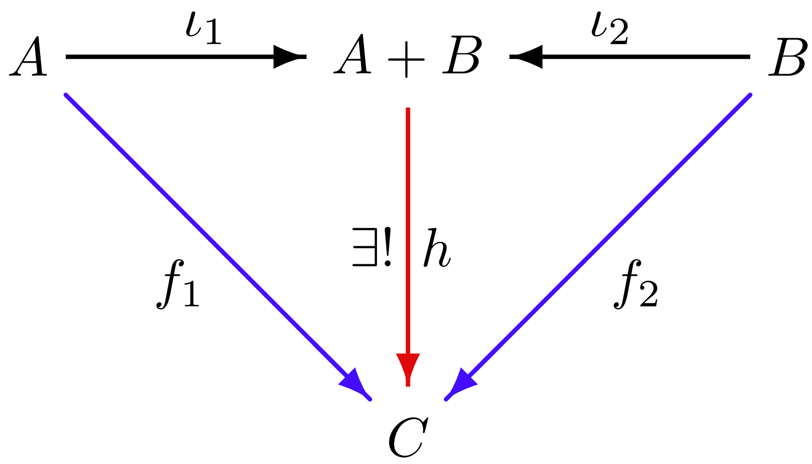

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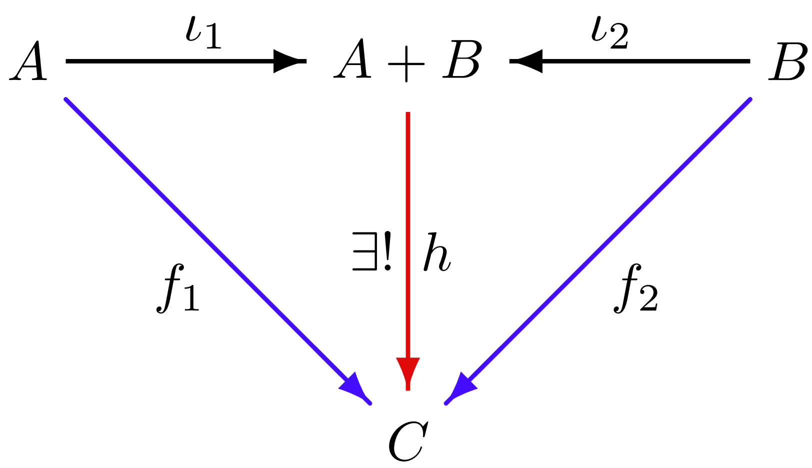

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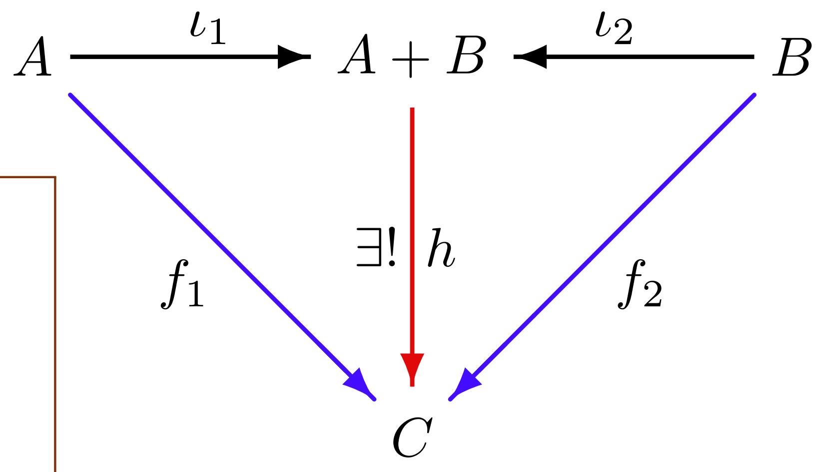

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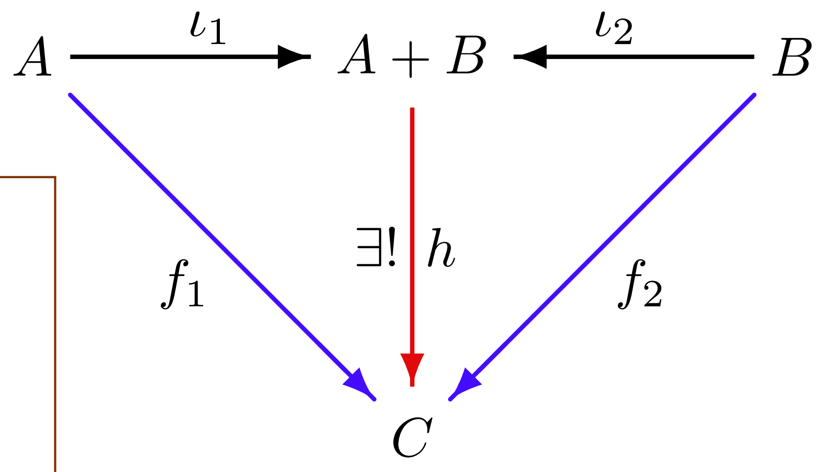

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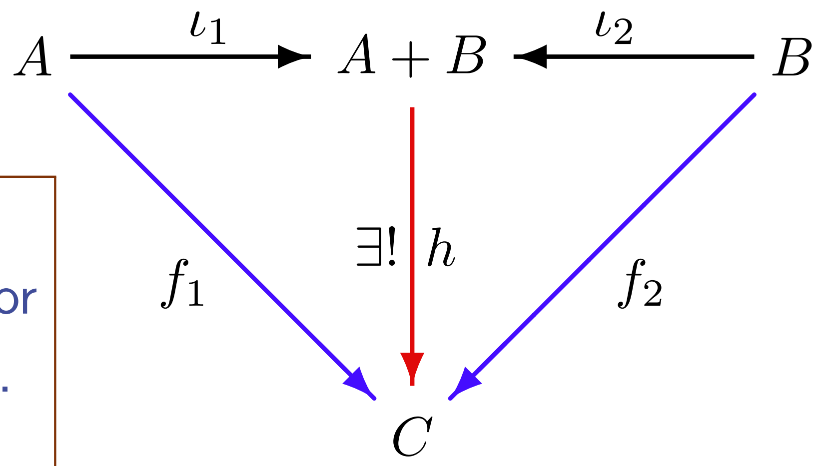

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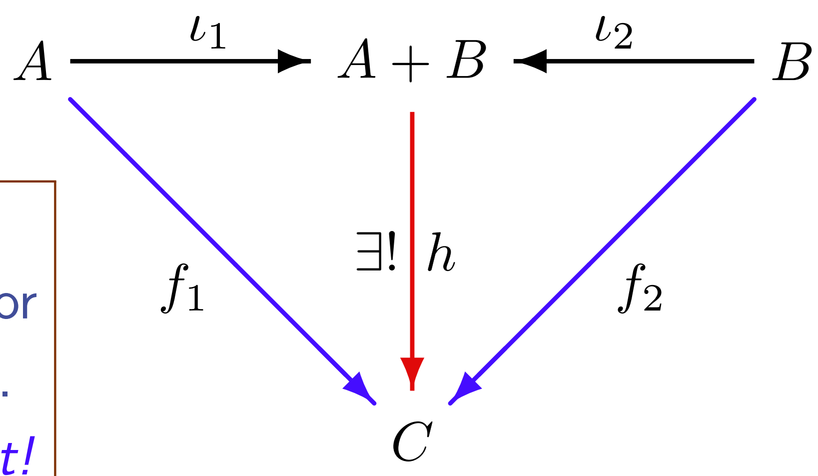

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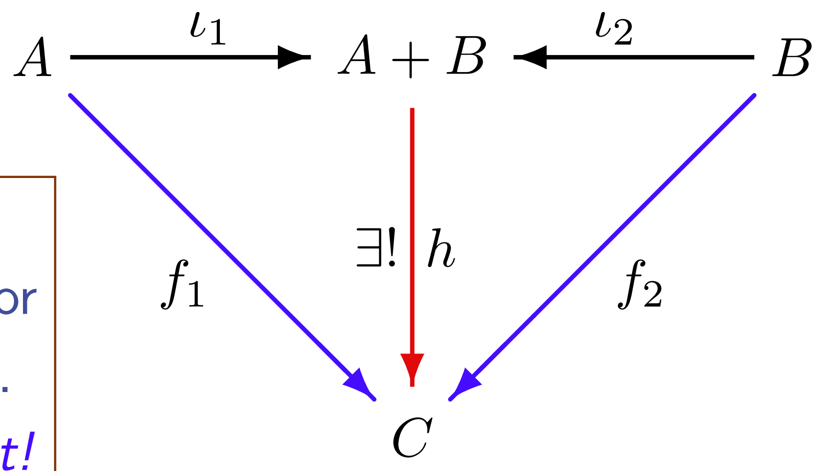

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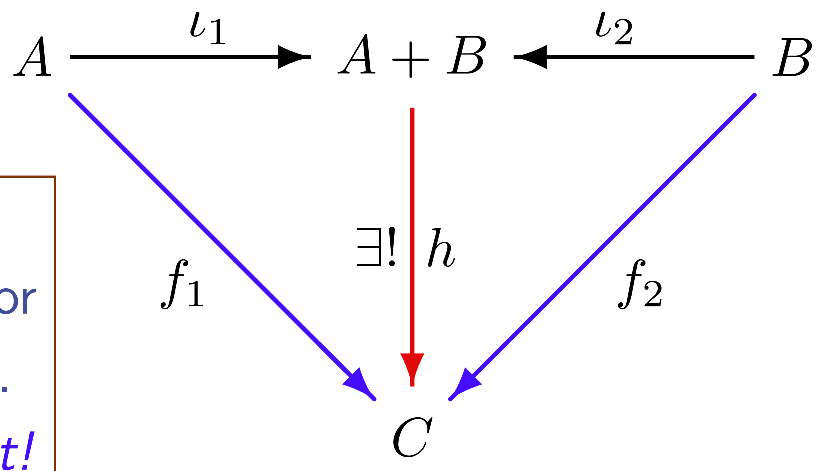

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Exercises: Dualise!

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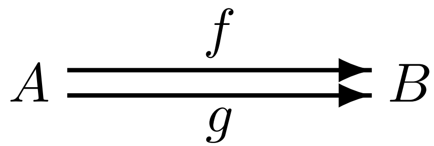
An *equaliser* of two "parallel" morphisms $f, g \colon A \to B$



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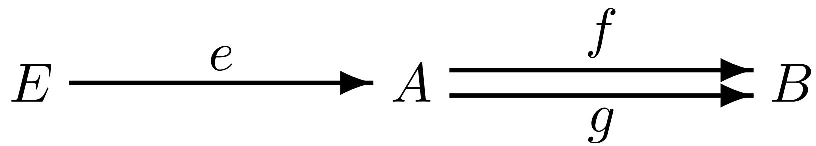

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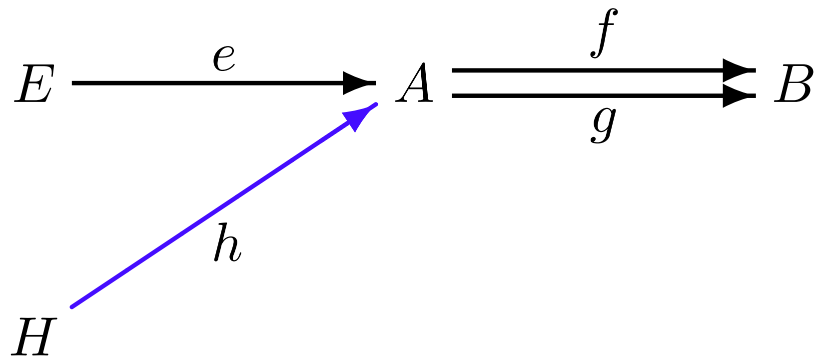

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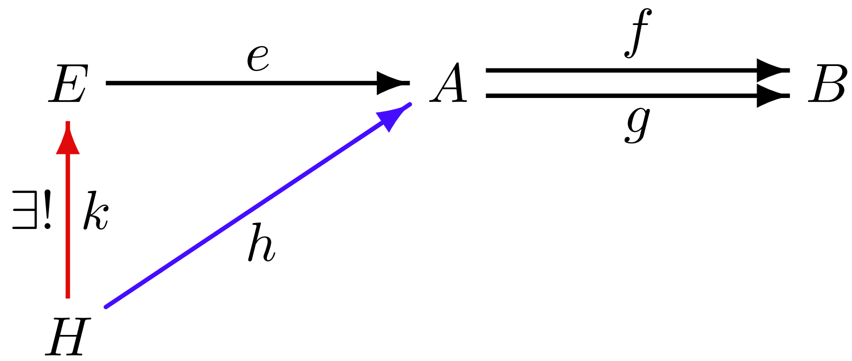

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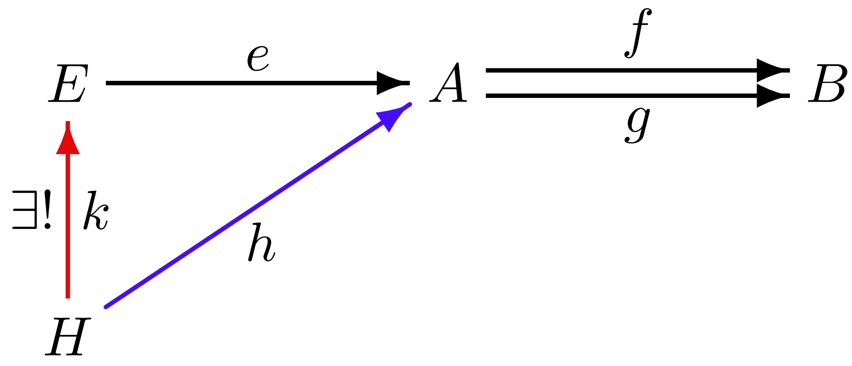

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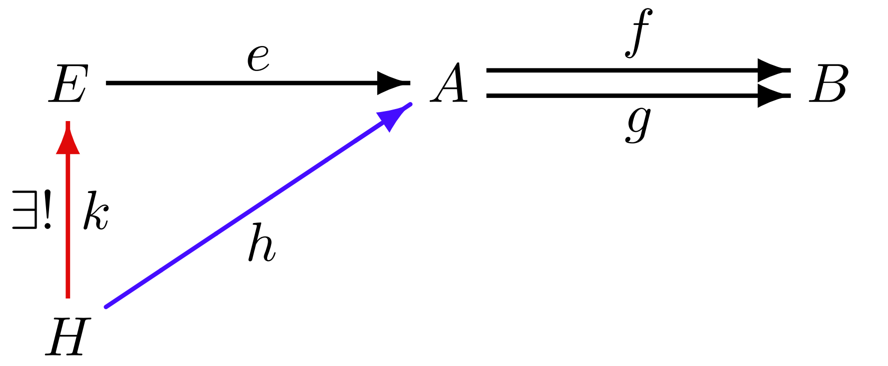

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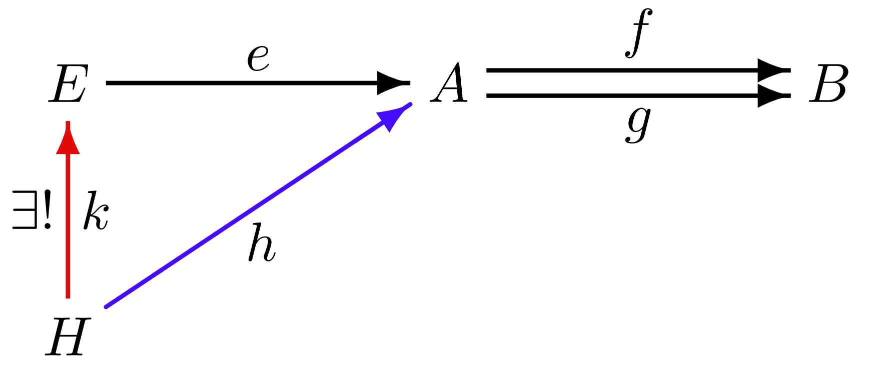

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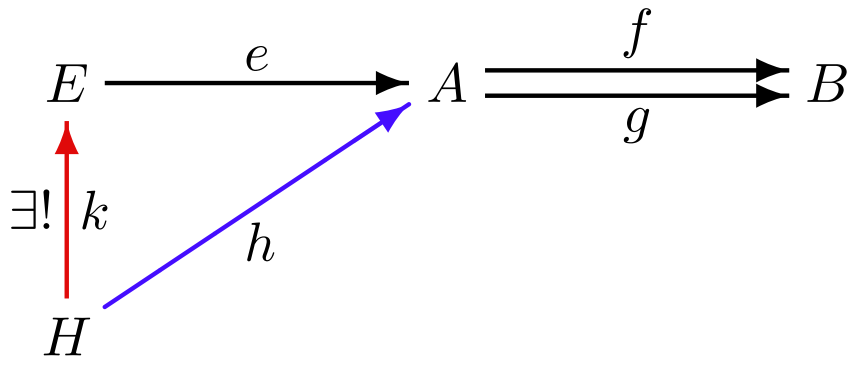

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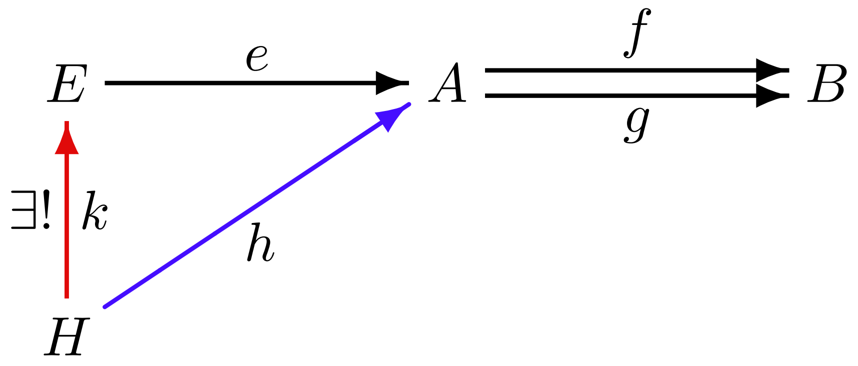

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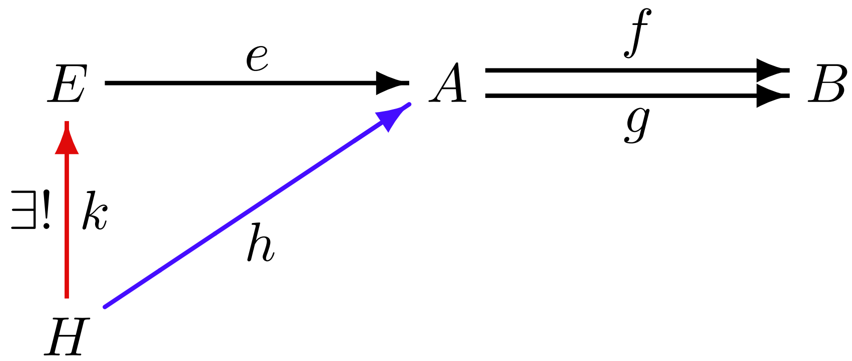

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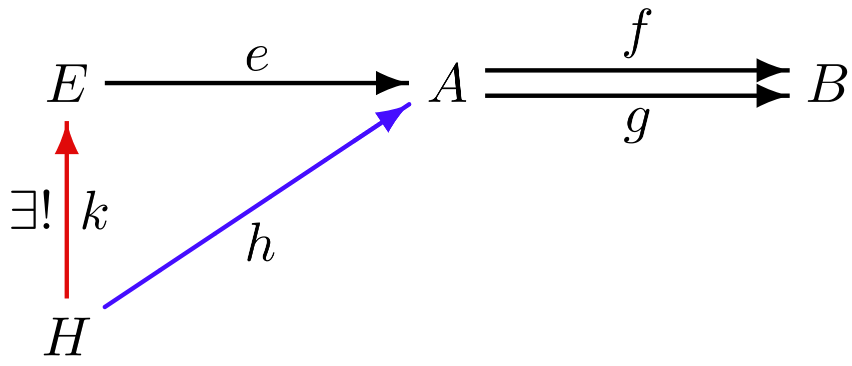

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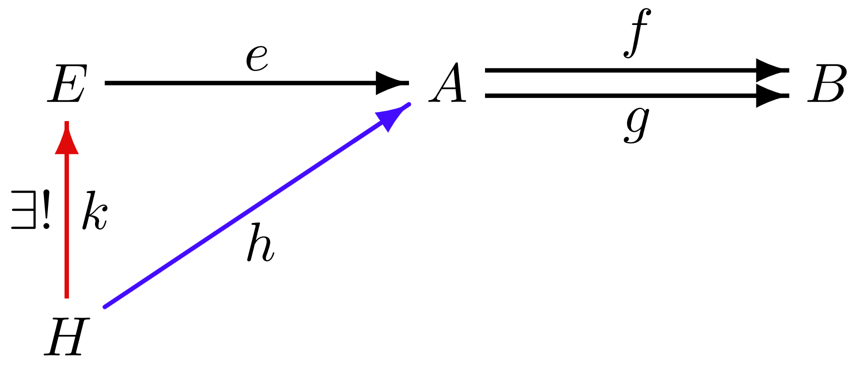

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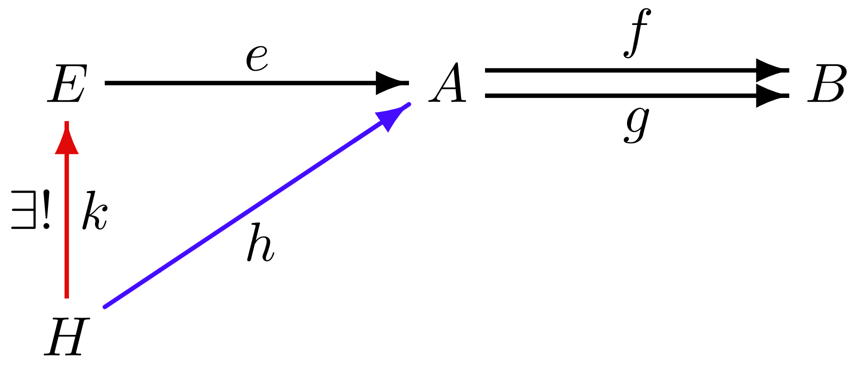

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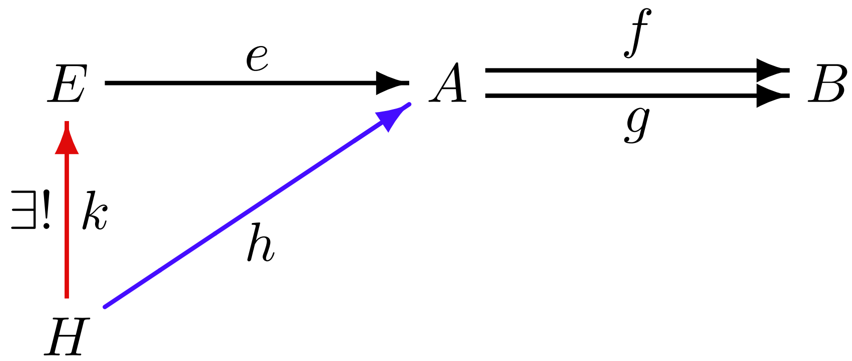

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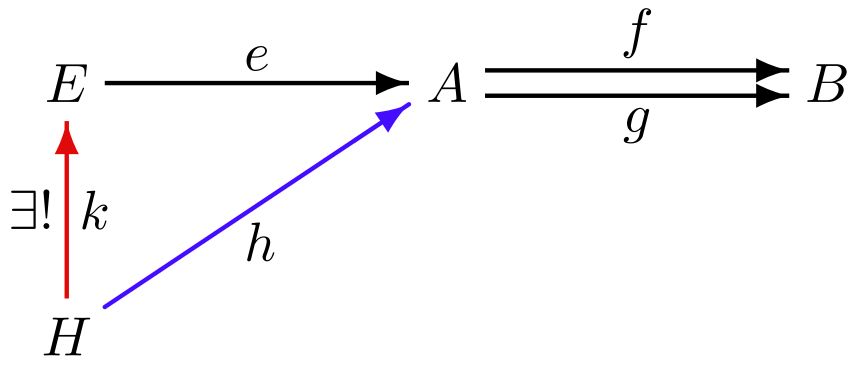

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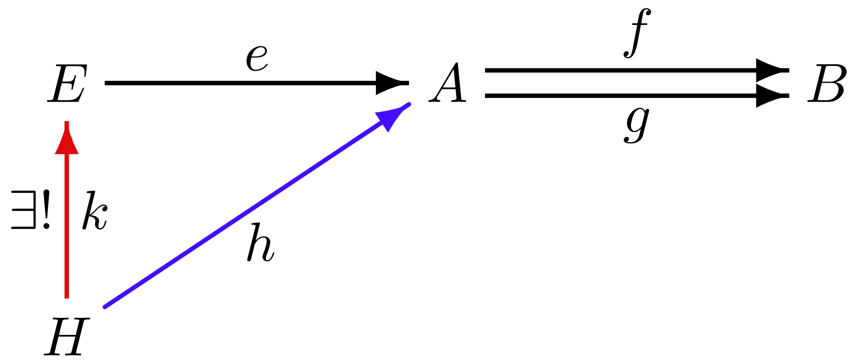

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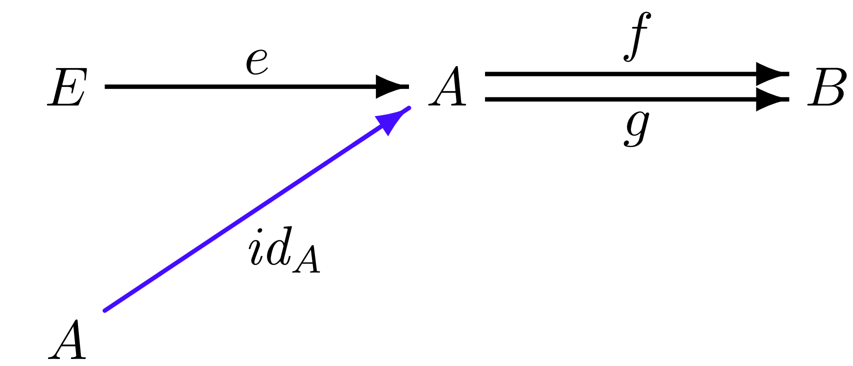

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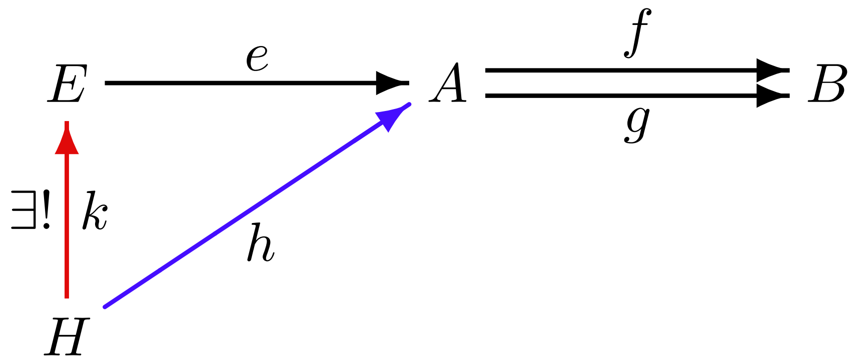

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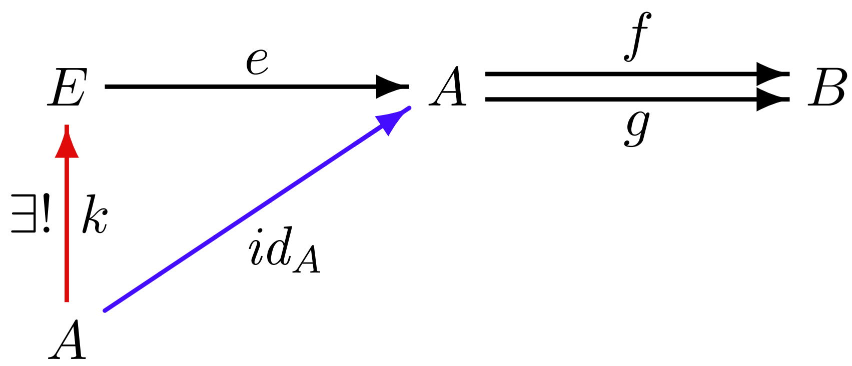

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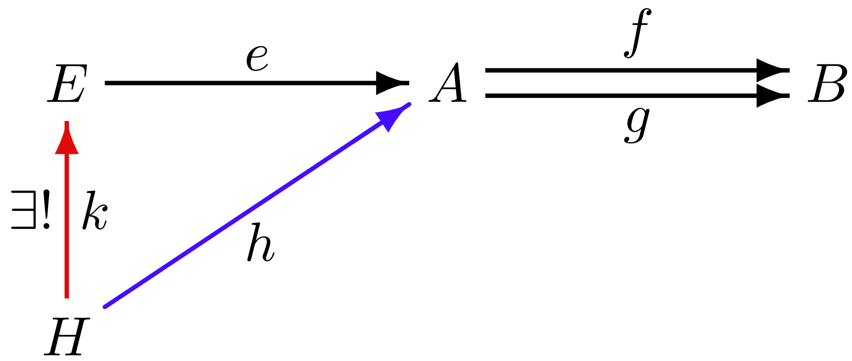

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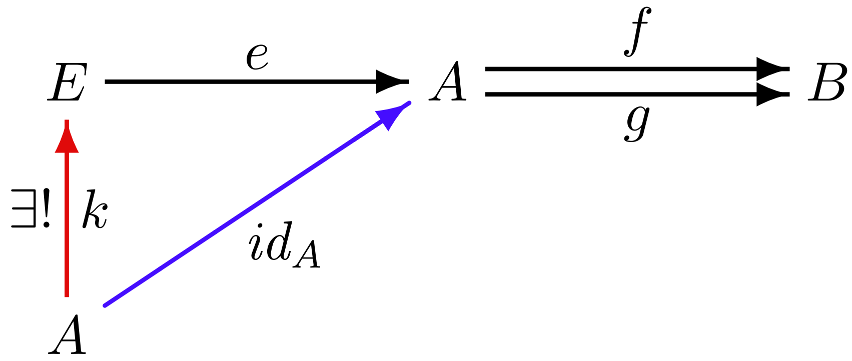
— and so is iso.



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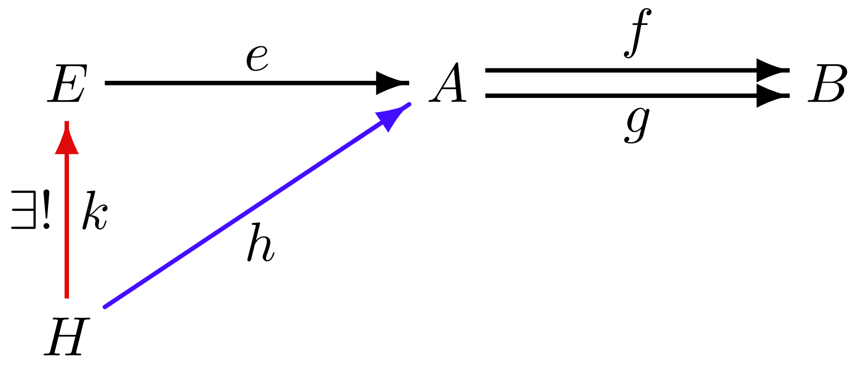

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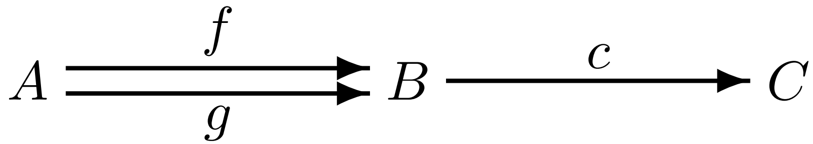

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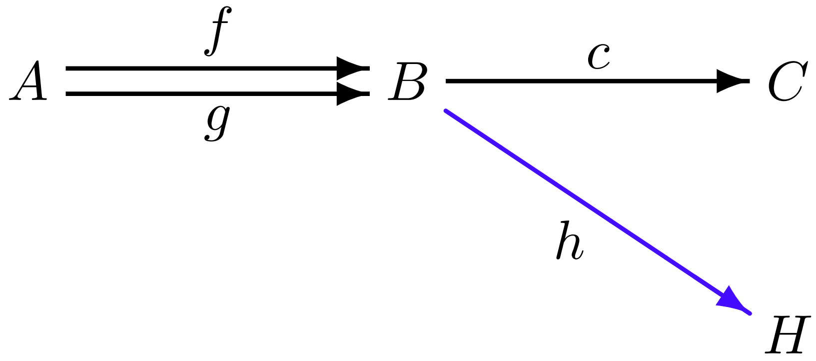

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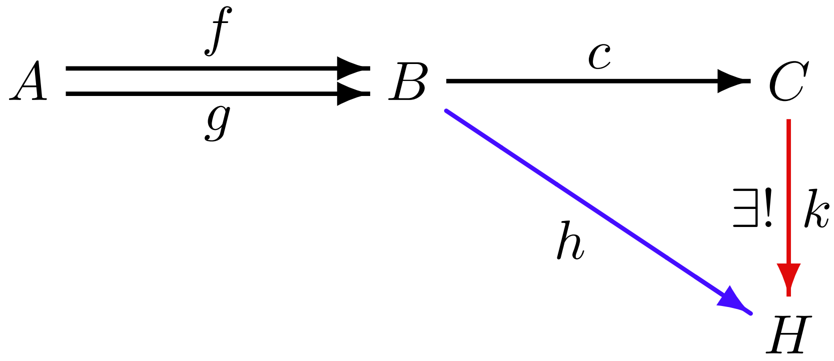

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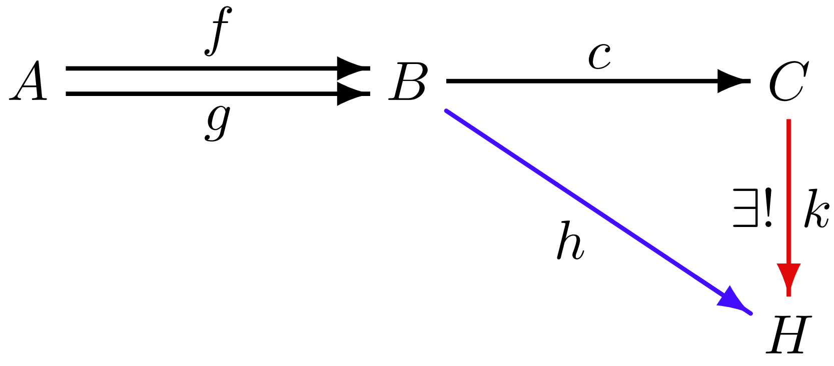

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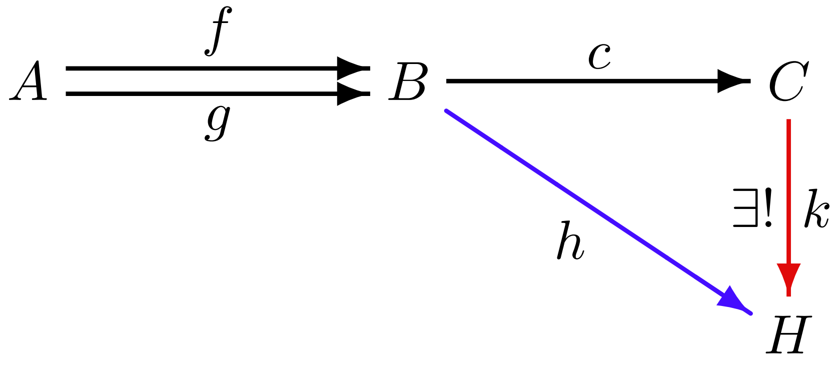

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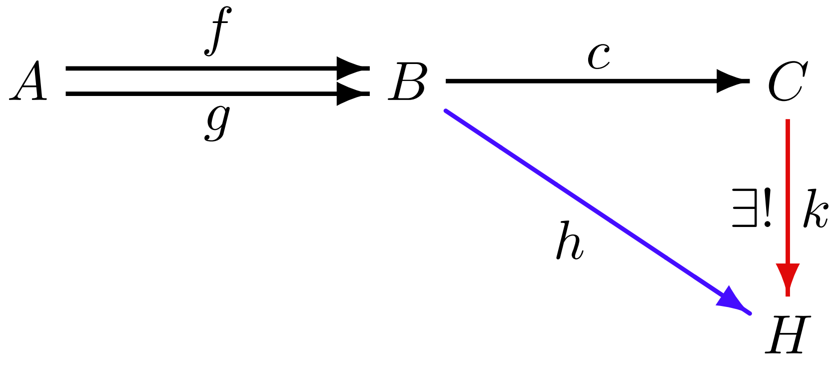

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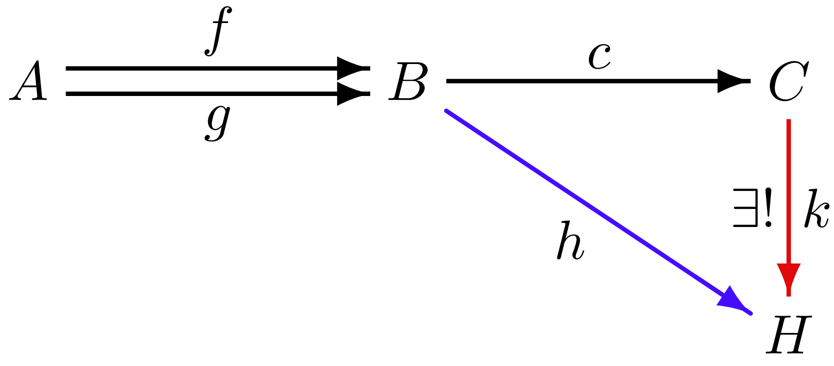

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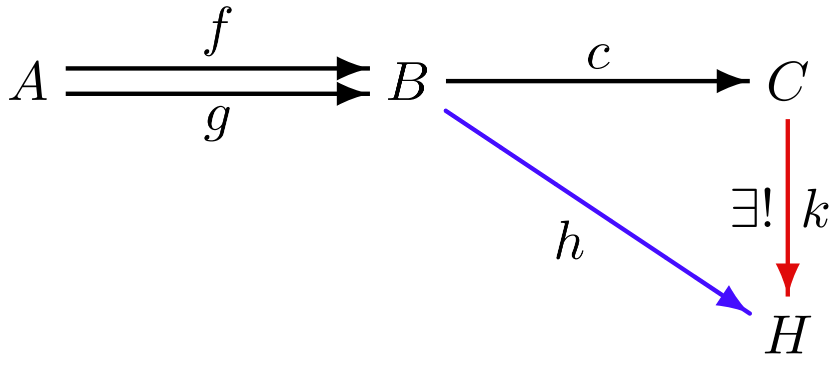

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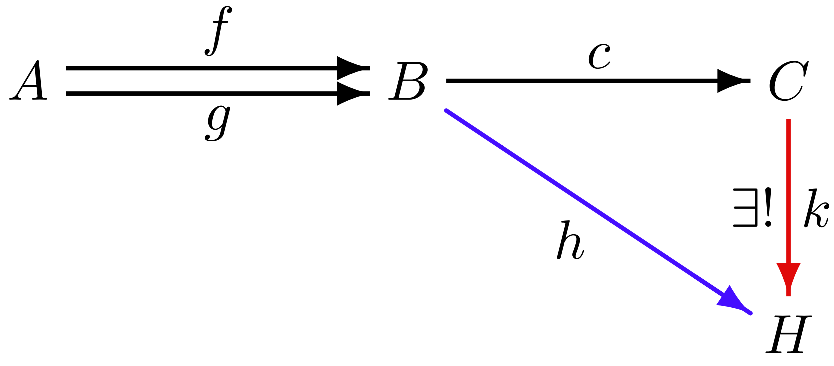

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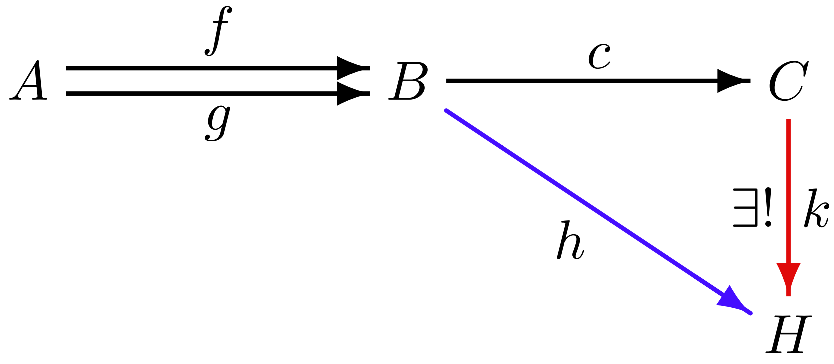

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Pullbacks

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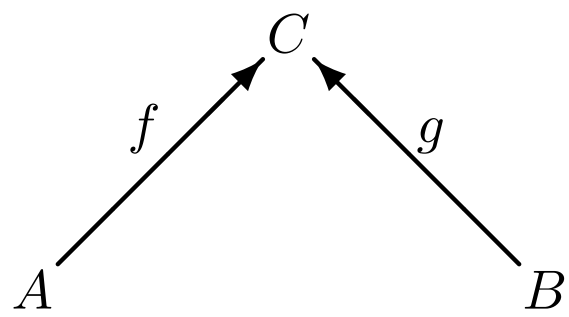
A *pullback* of two morphisms with common target $f: A \to C$ and $g: B \to C$



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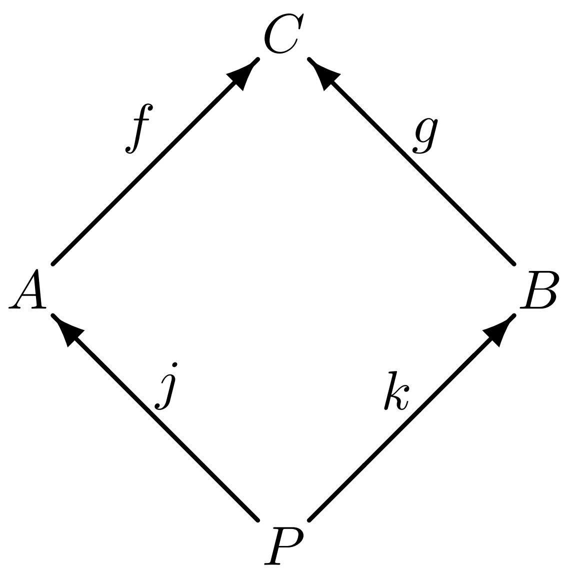

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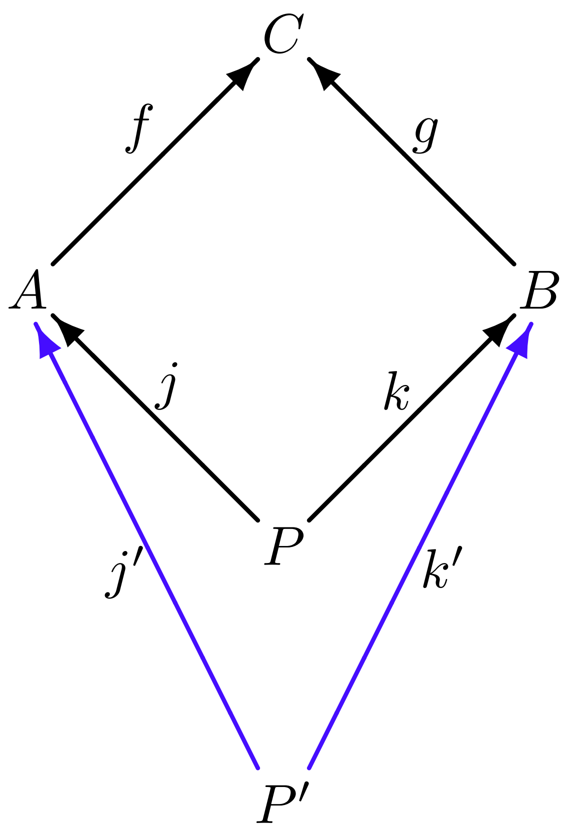

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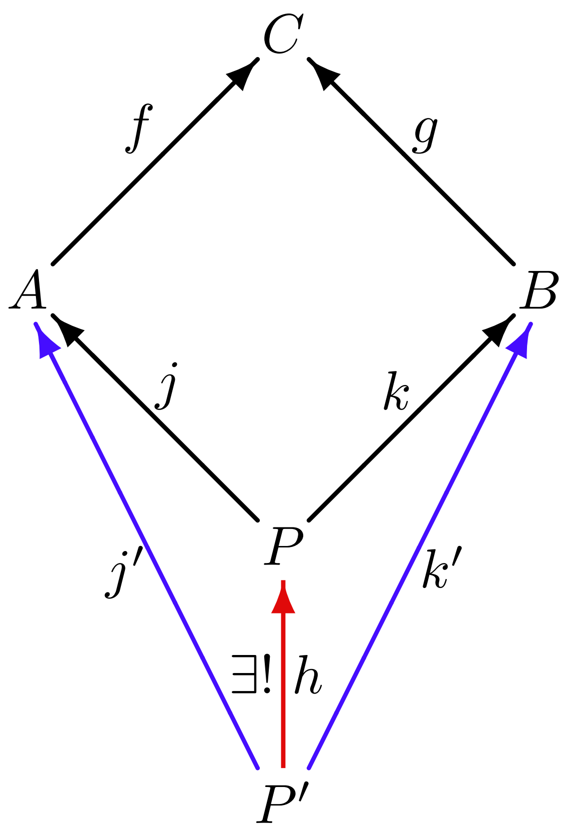

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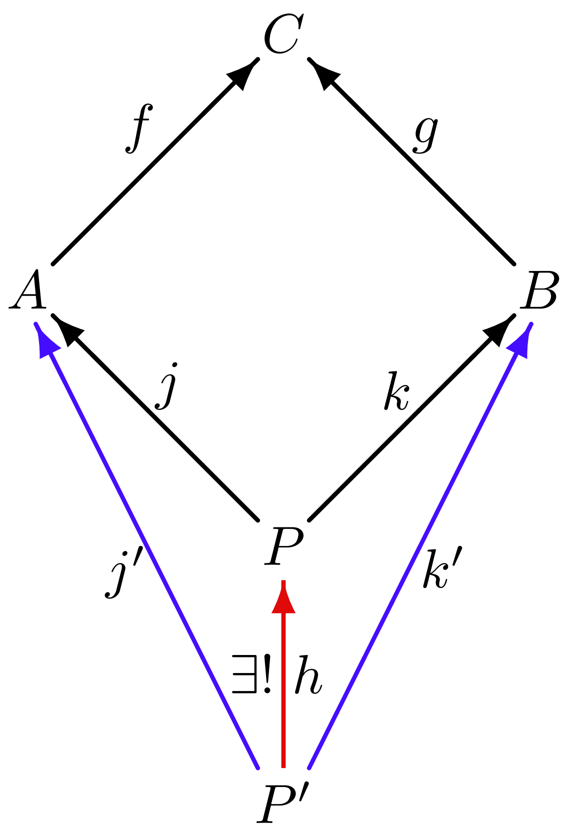

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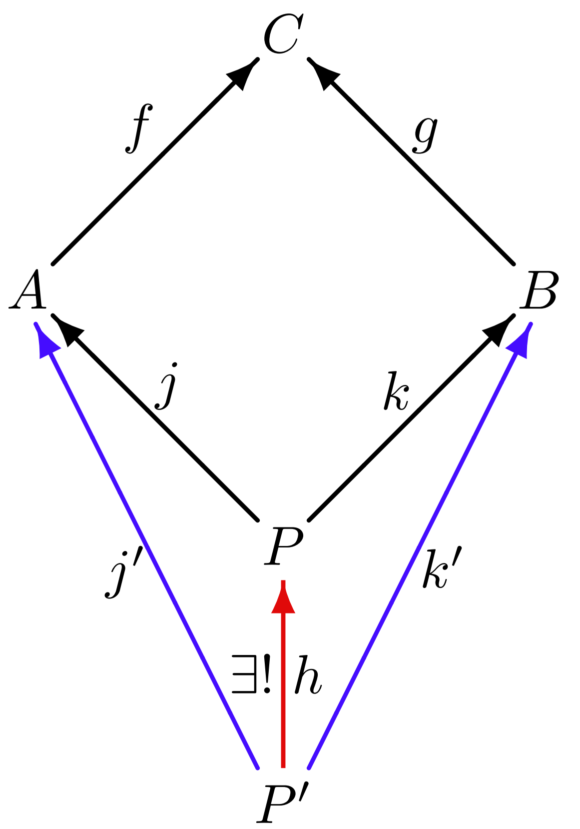

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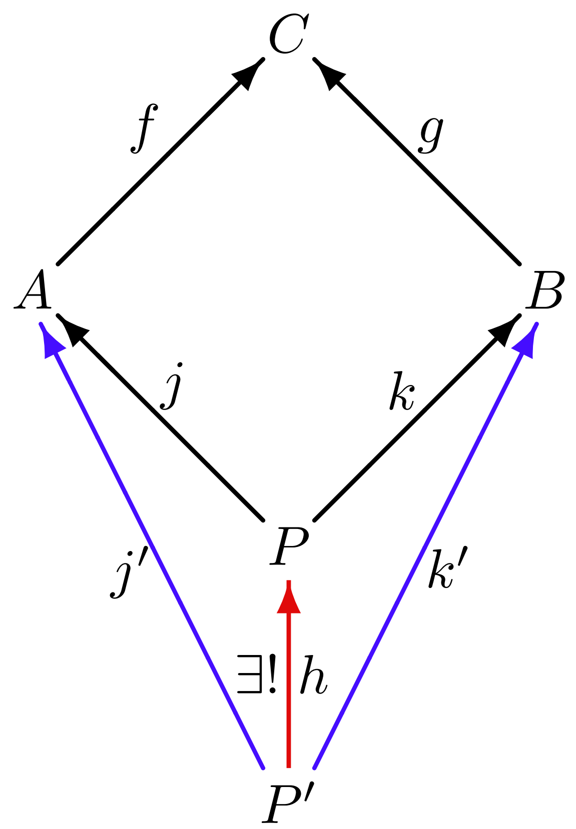

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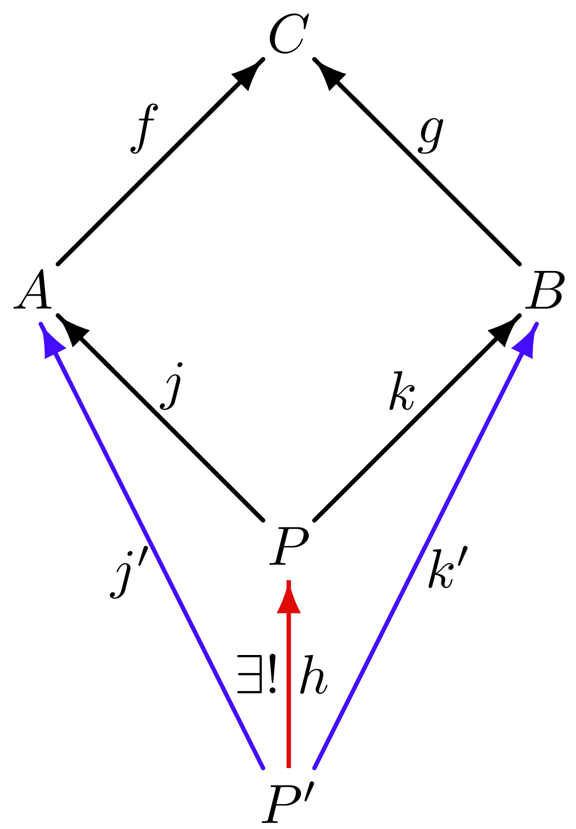

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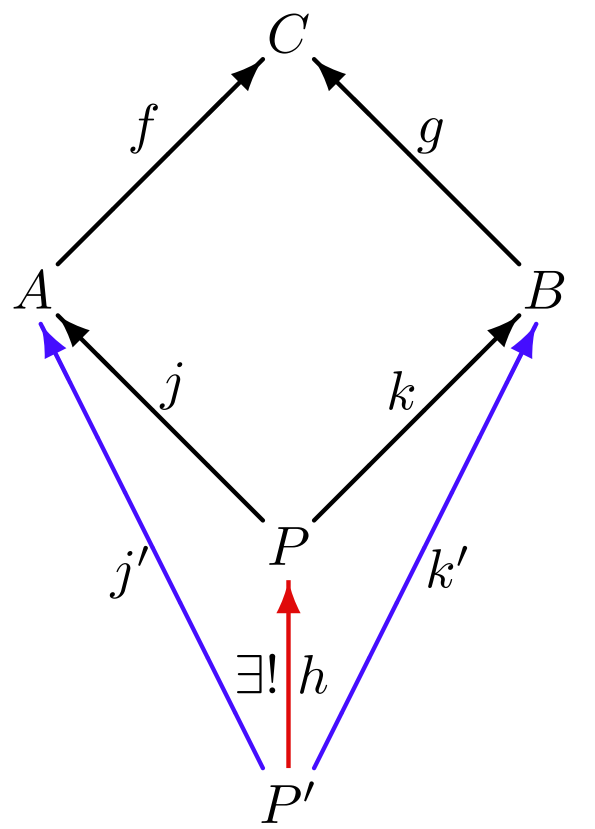

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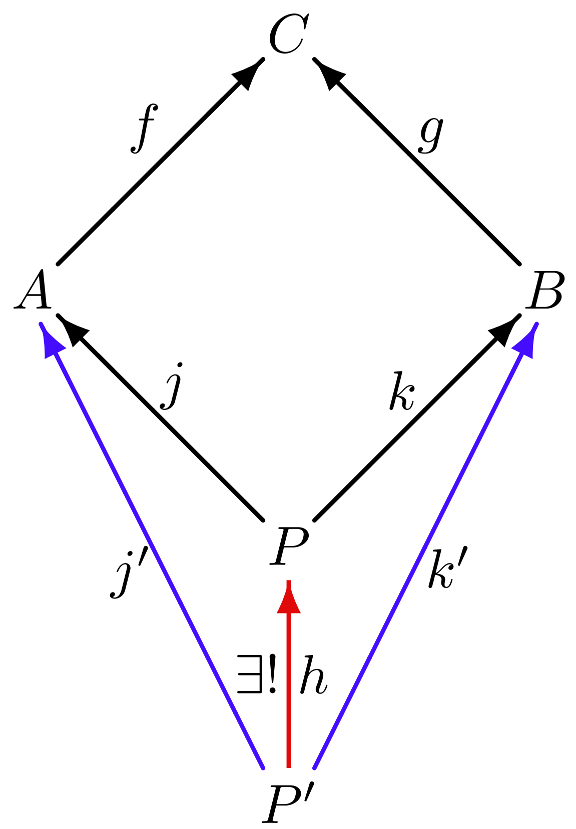
Wait for a hint to come...



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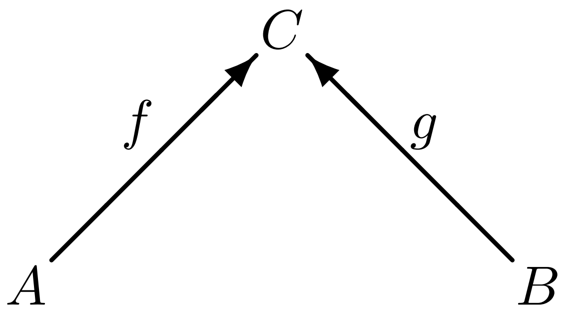

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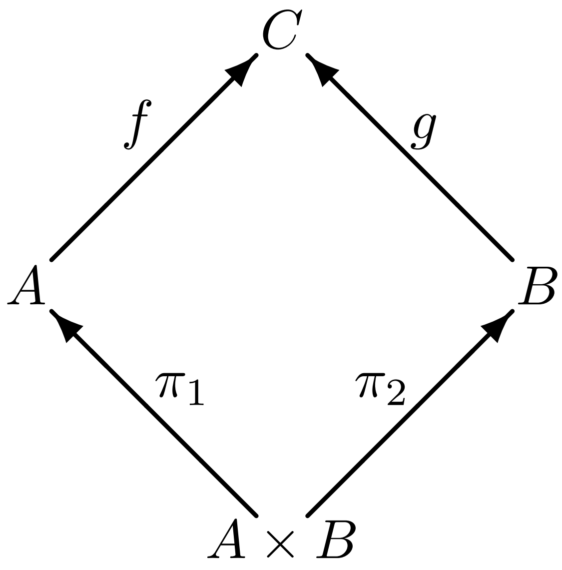

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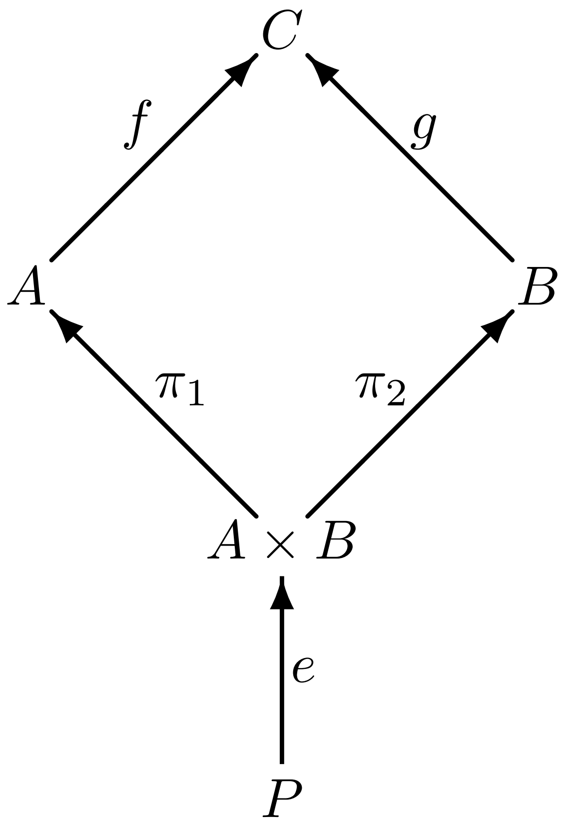

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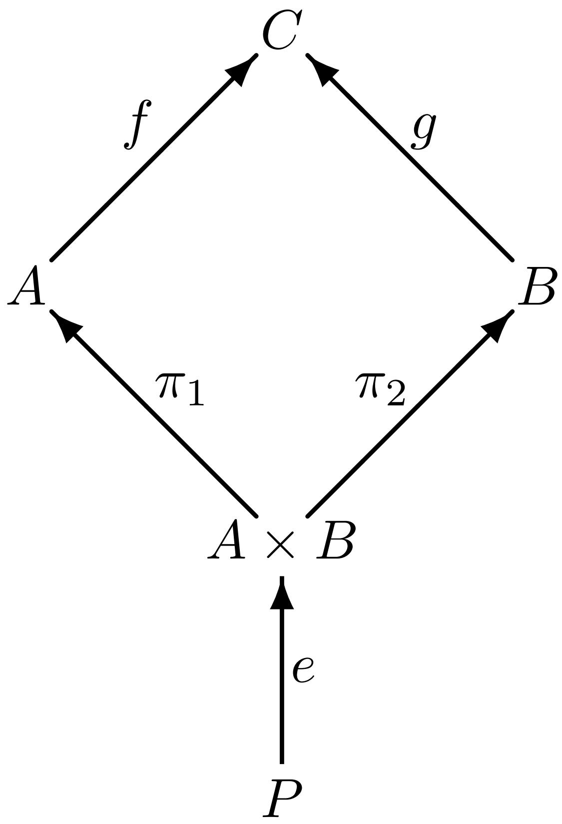

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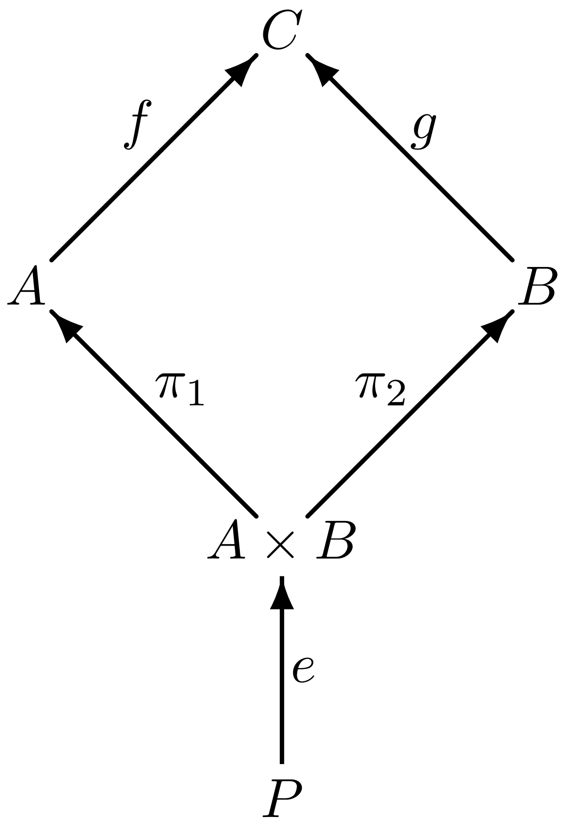

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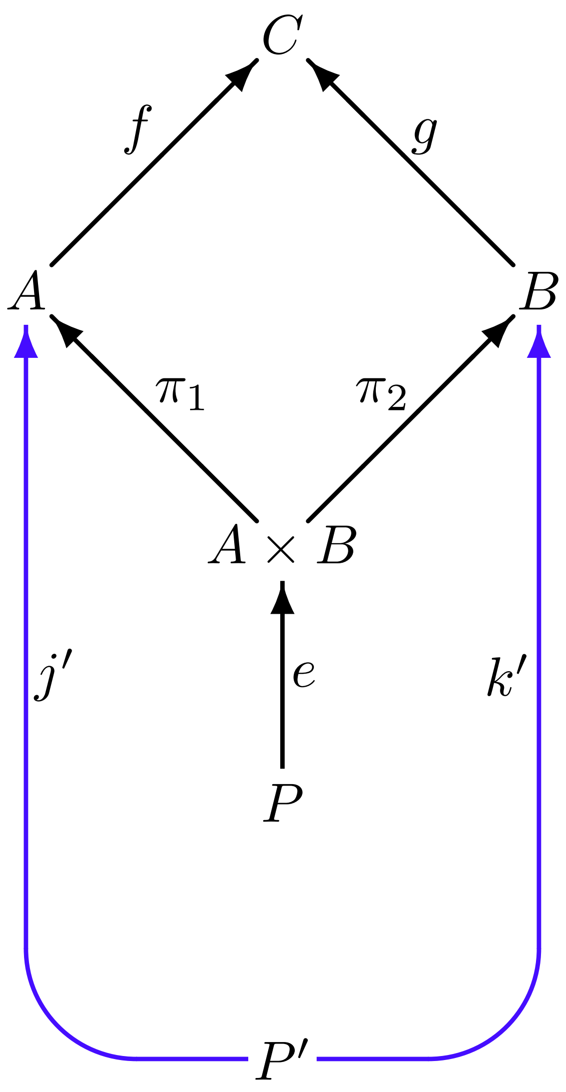

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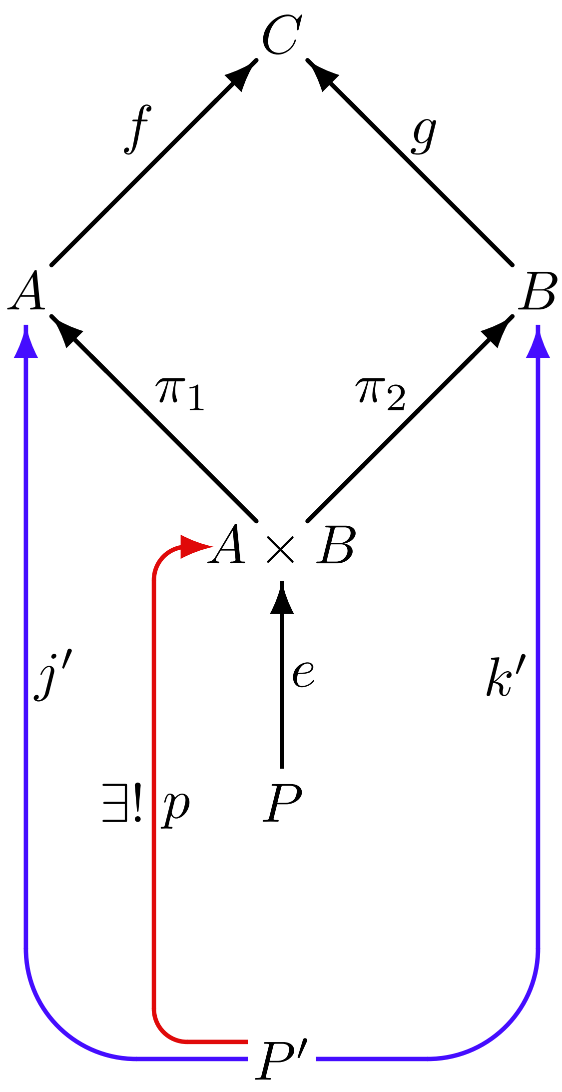

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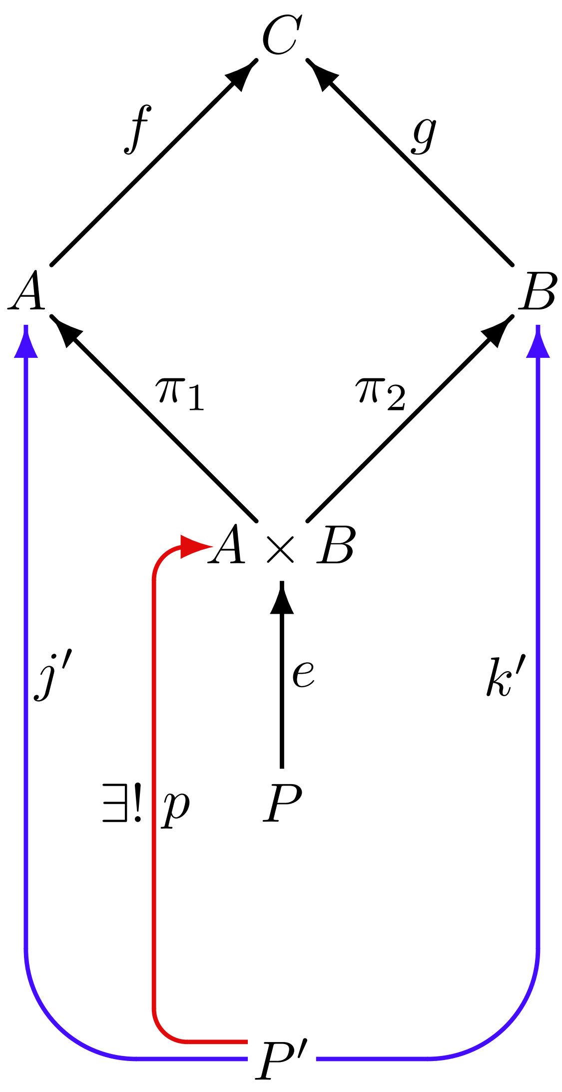

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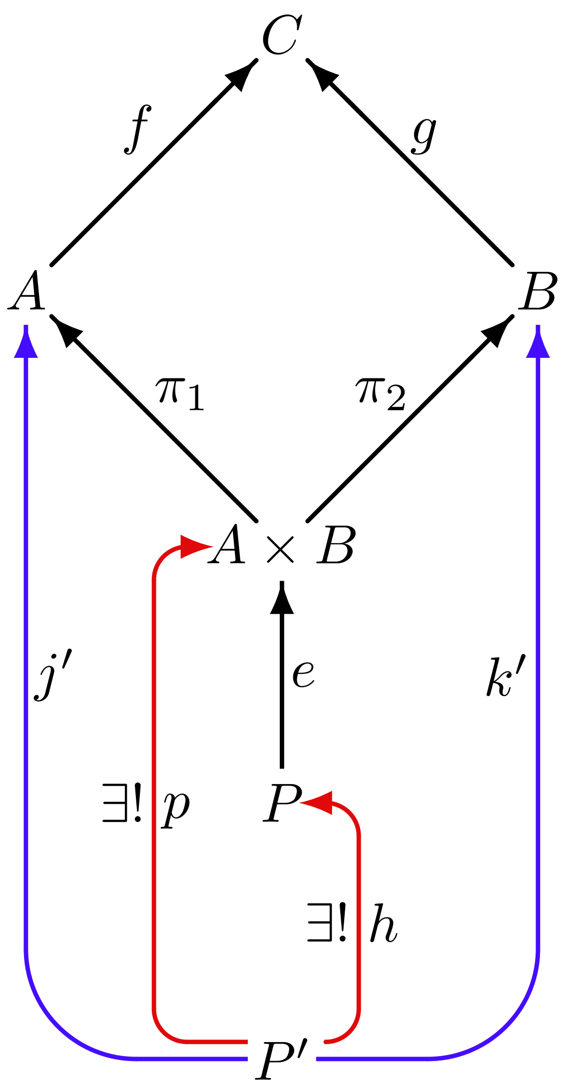

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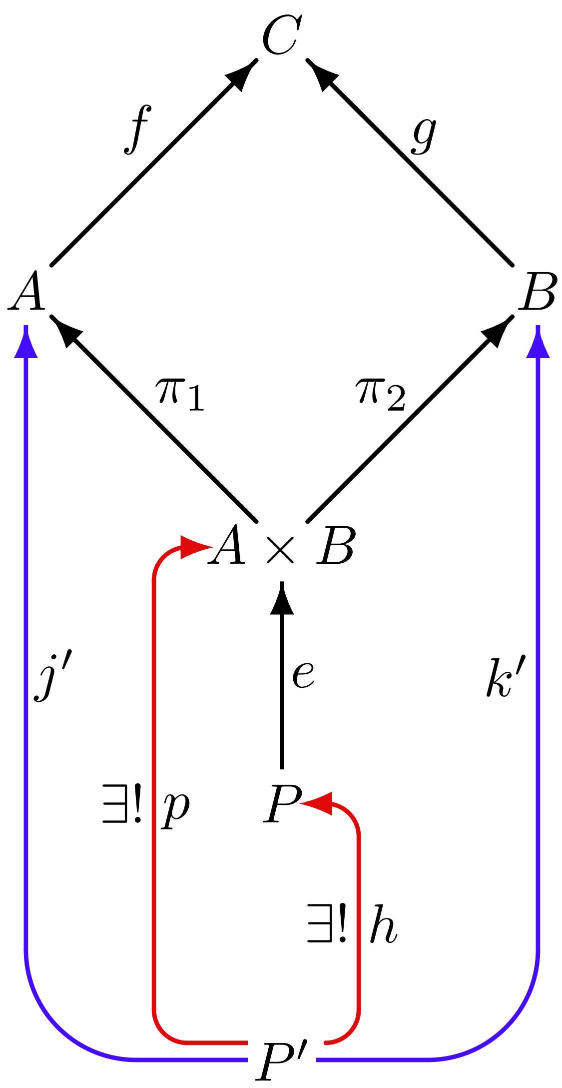

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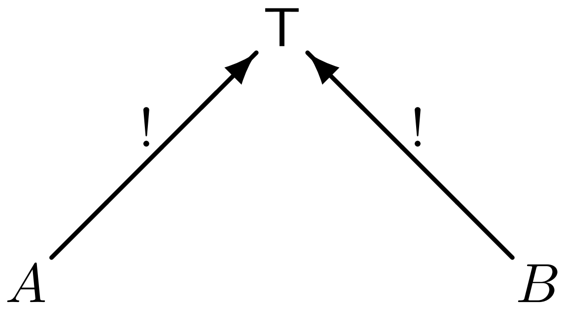

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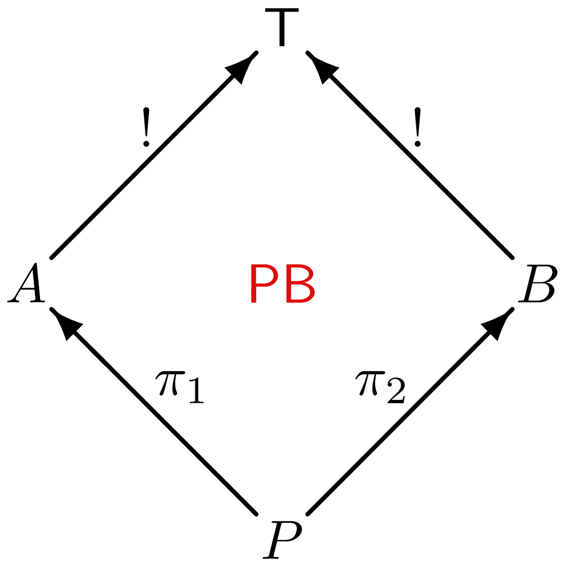

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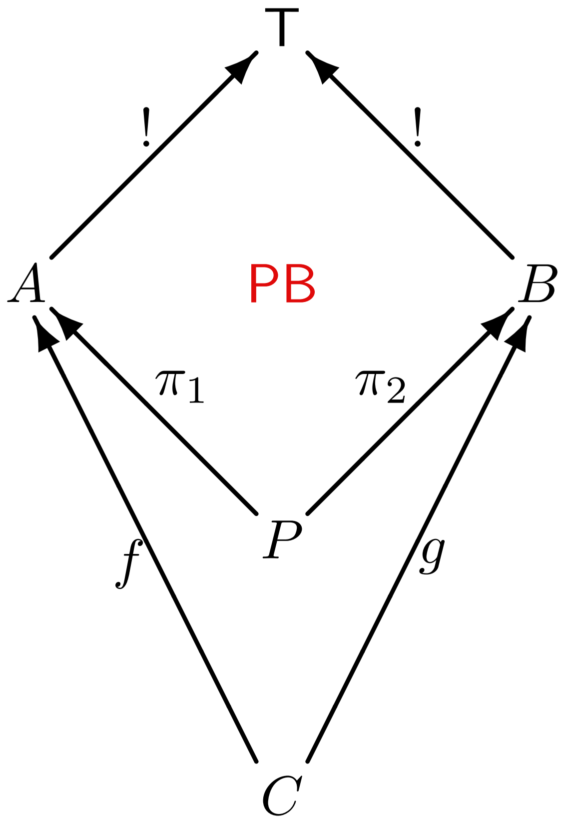

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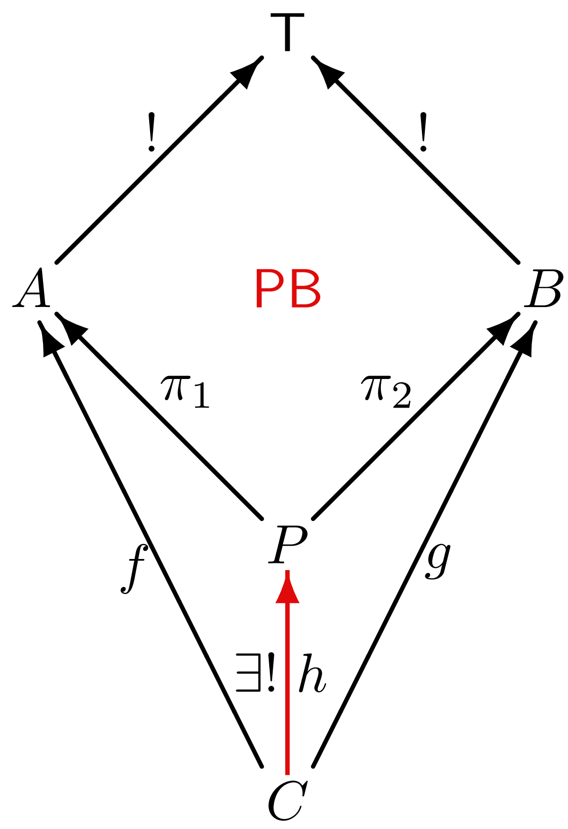

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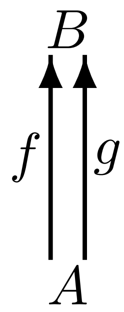

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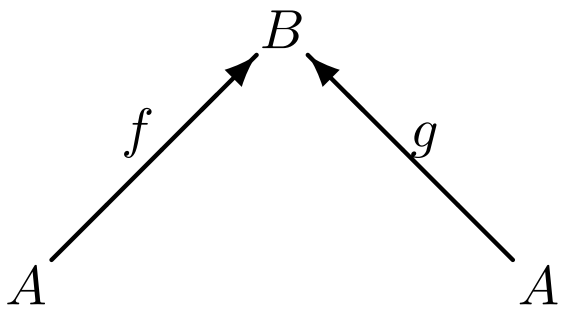

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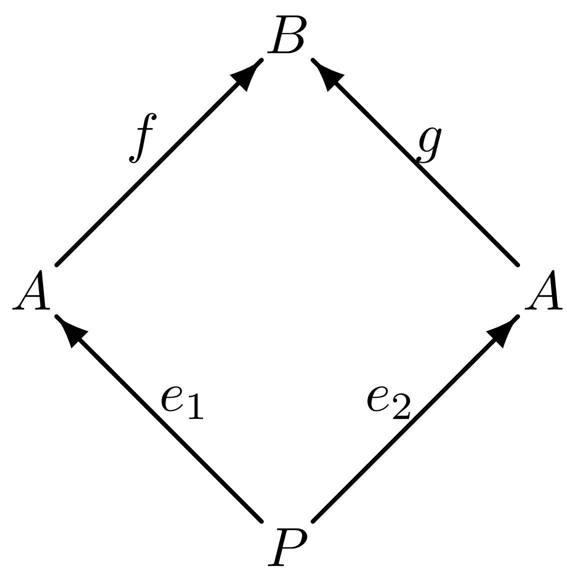

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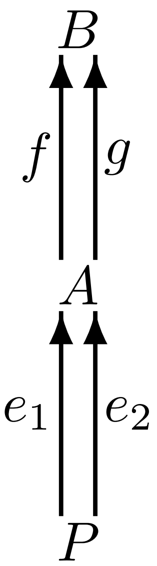

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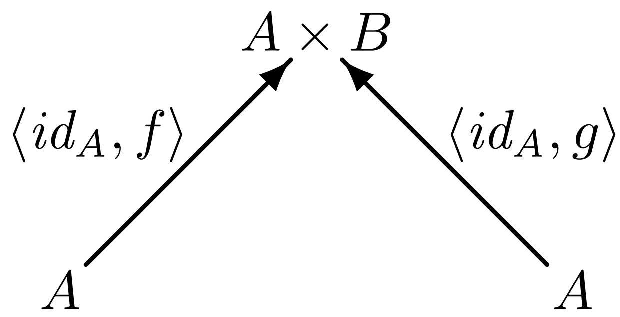

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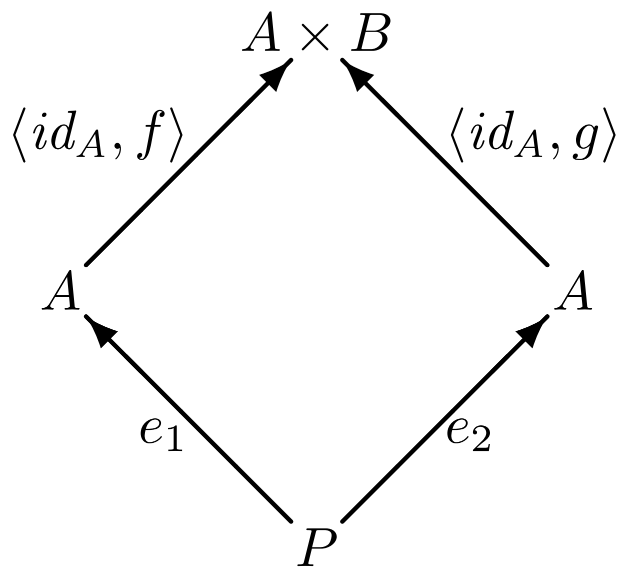

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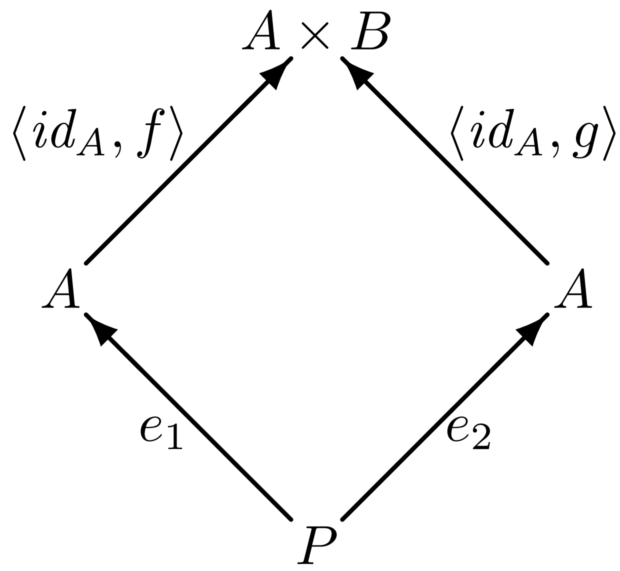

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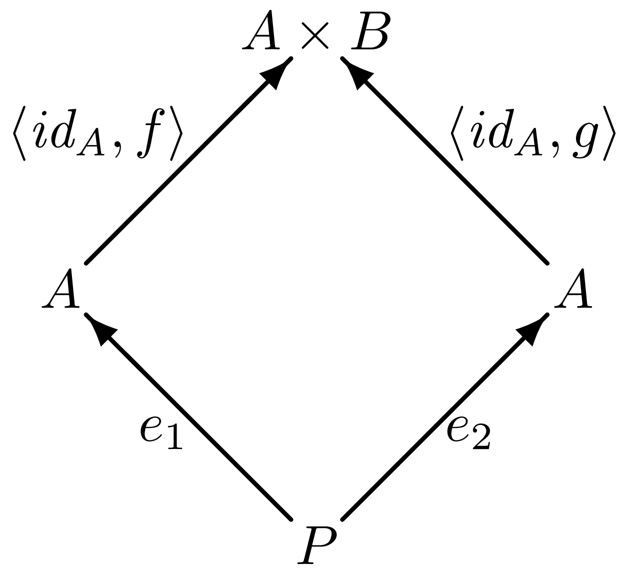

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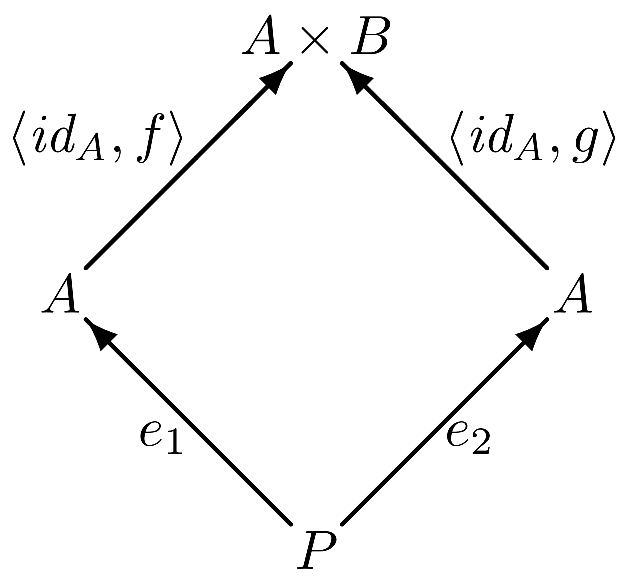

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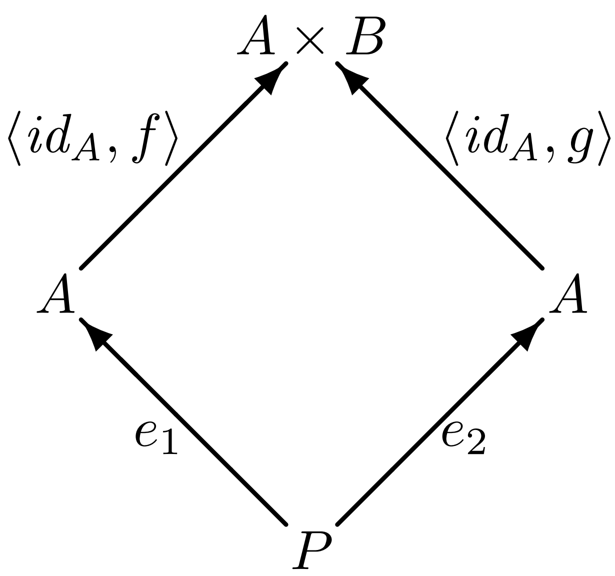

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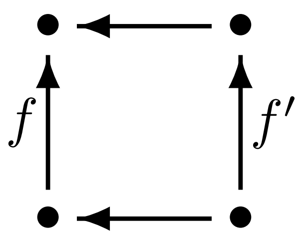

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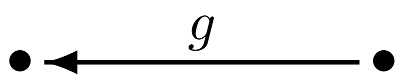

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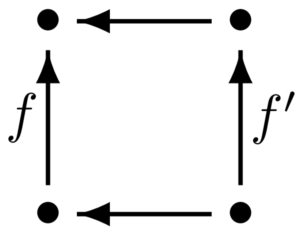

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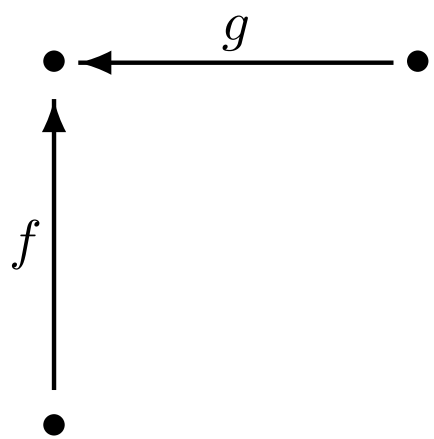

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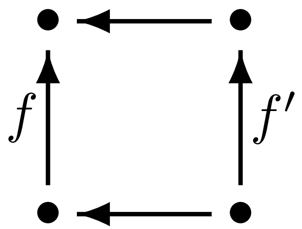

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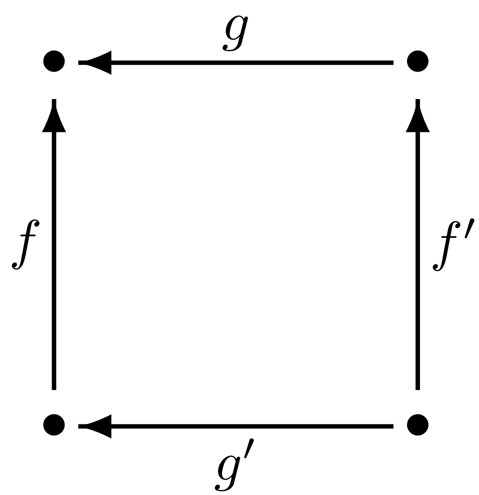

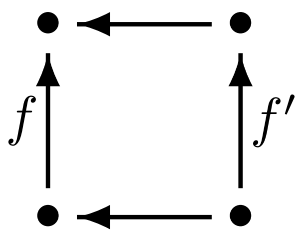

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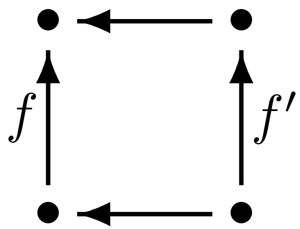

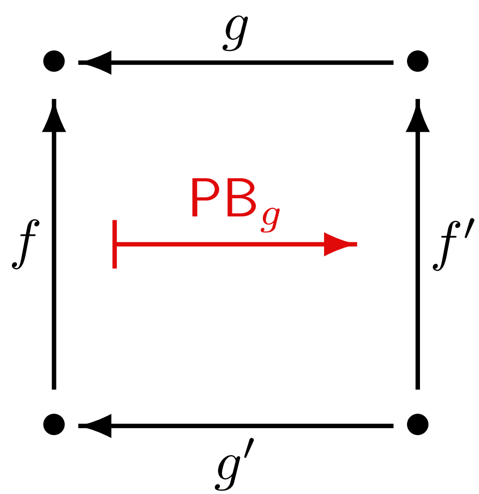

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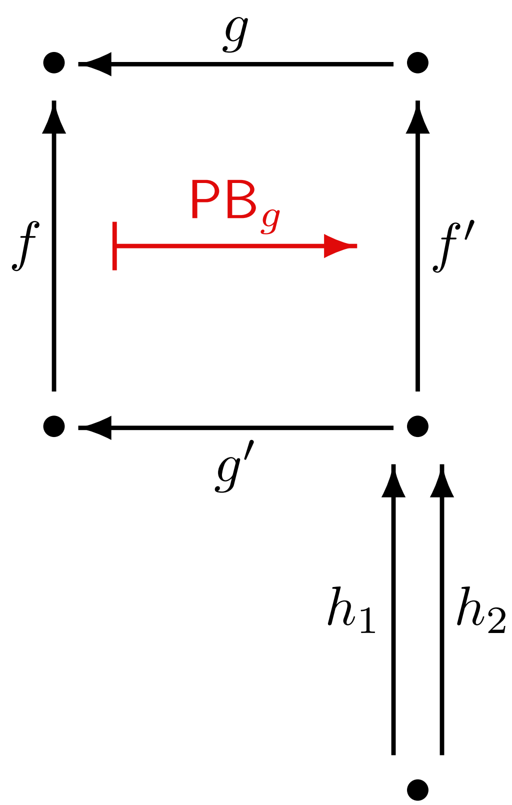

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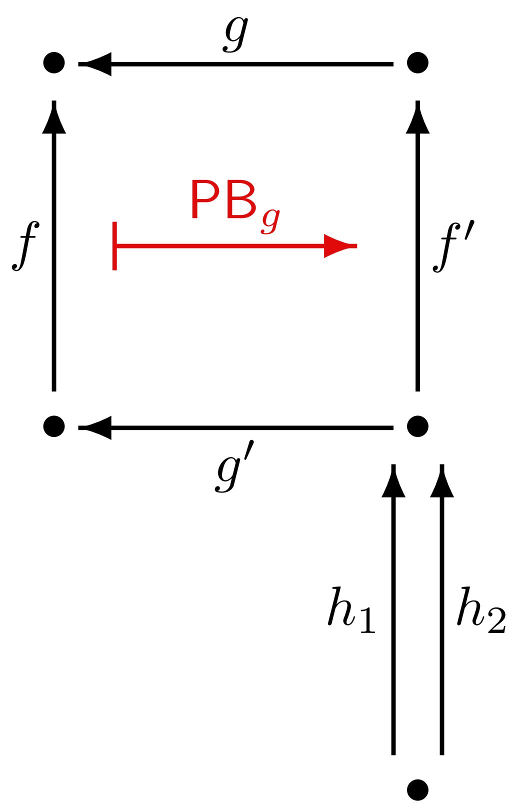

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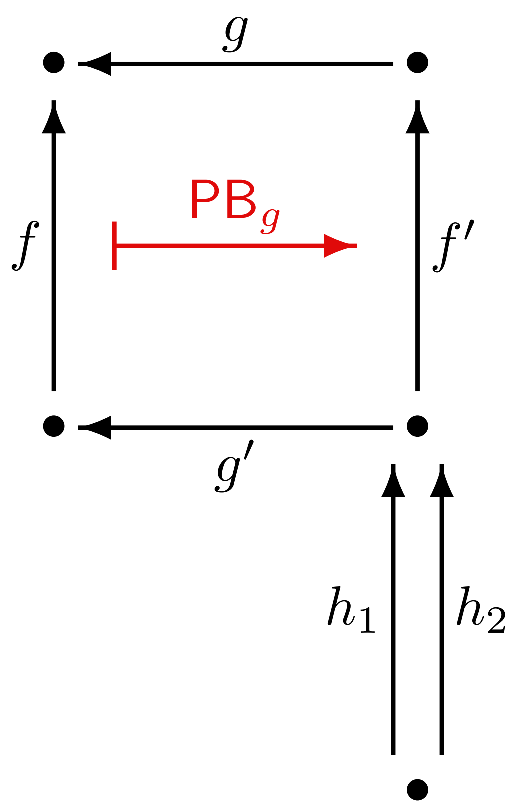

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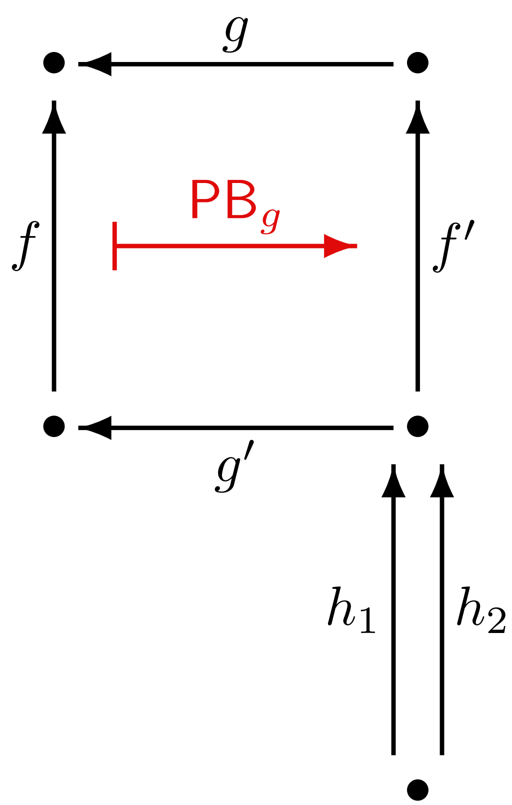

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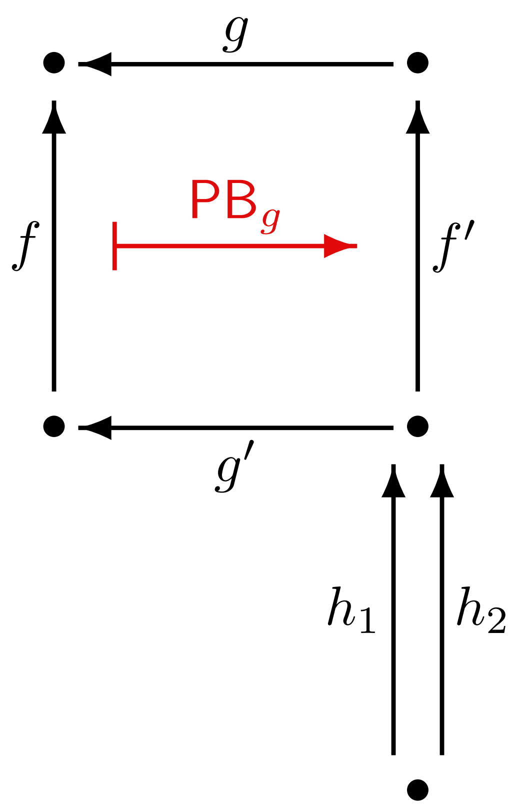

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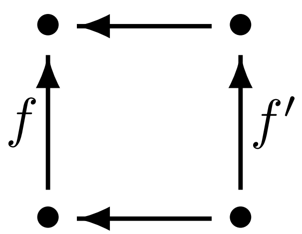

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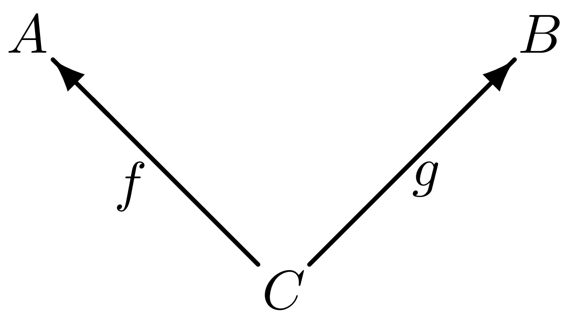

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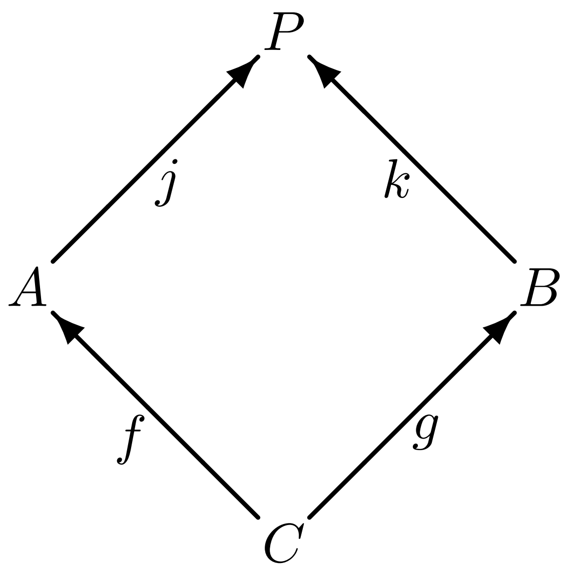

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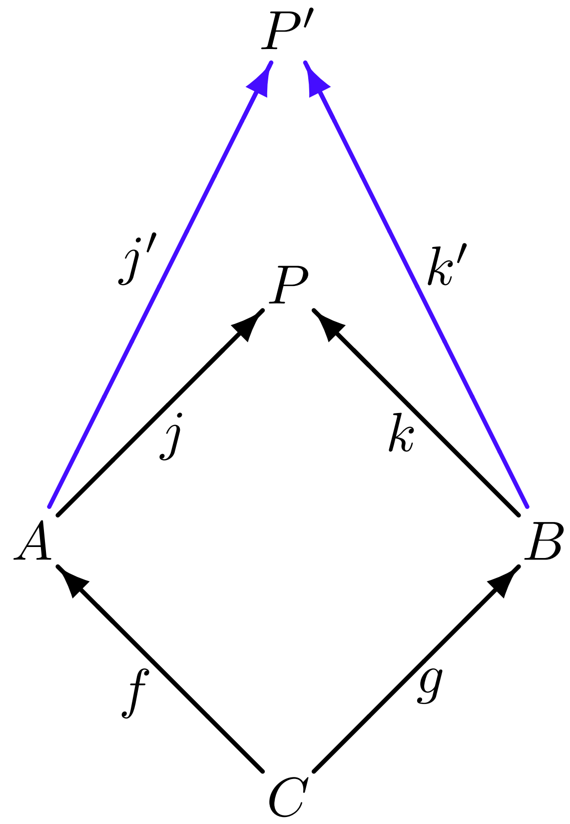

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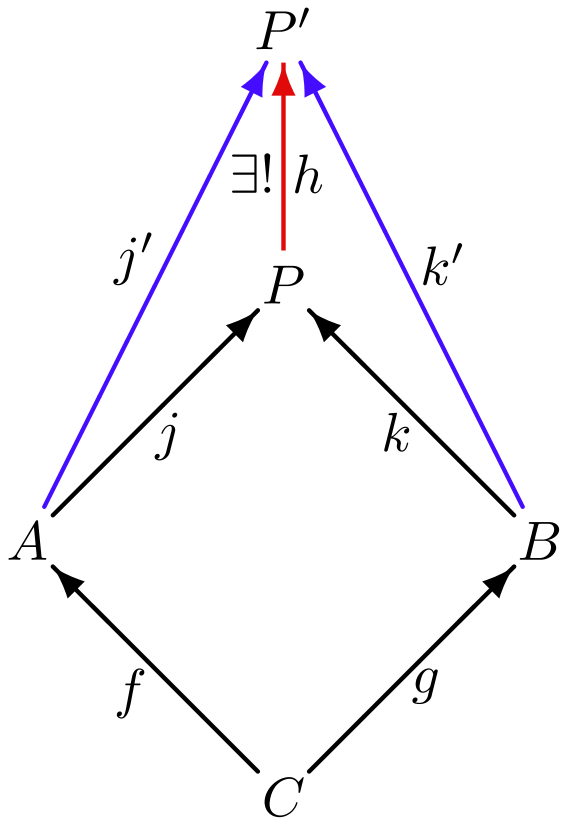

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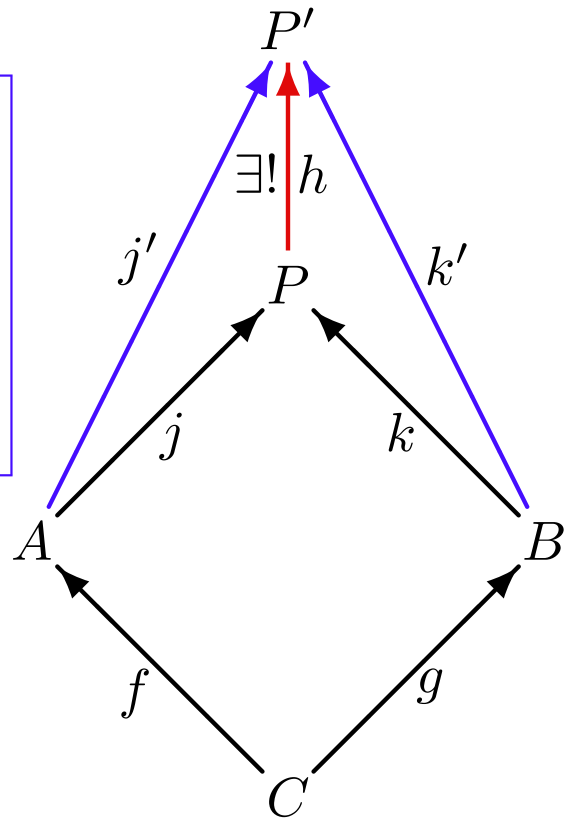

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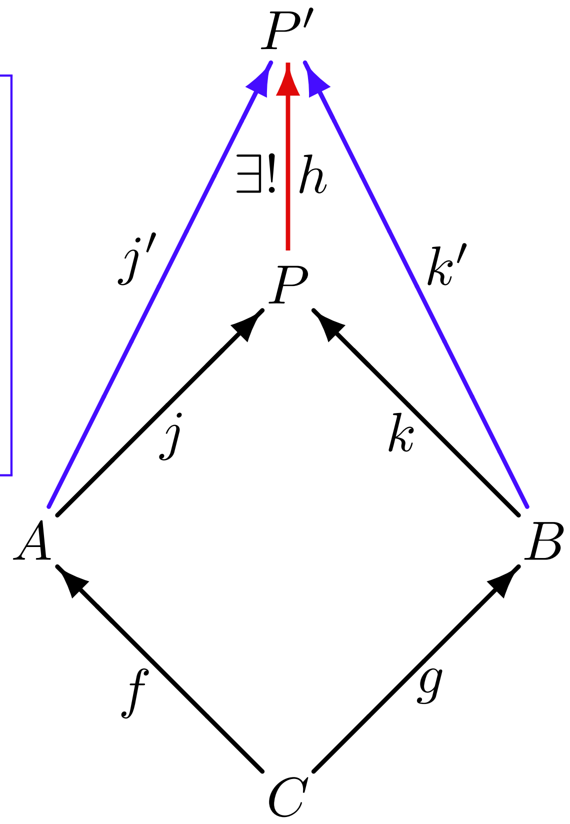

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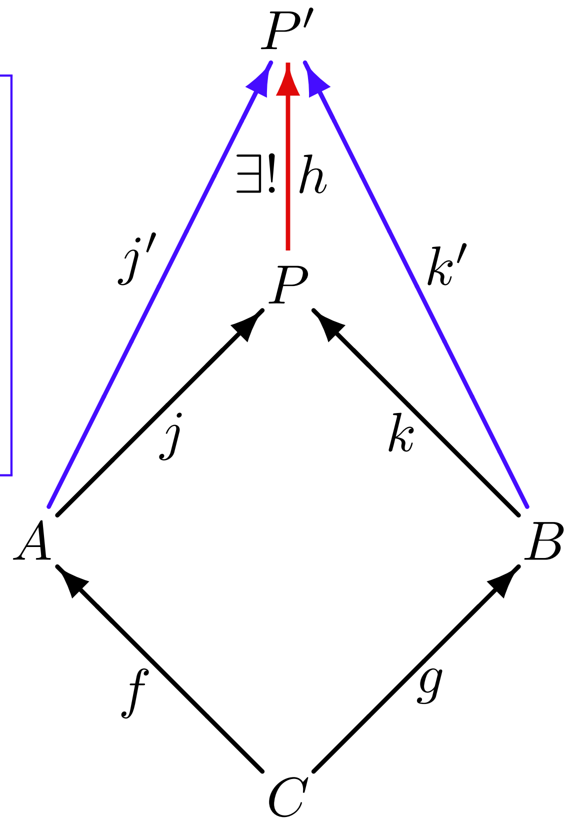

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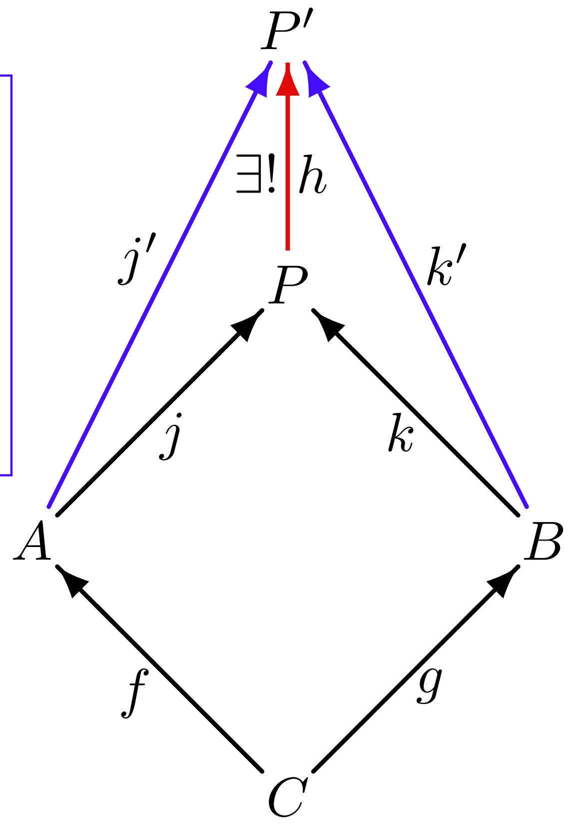

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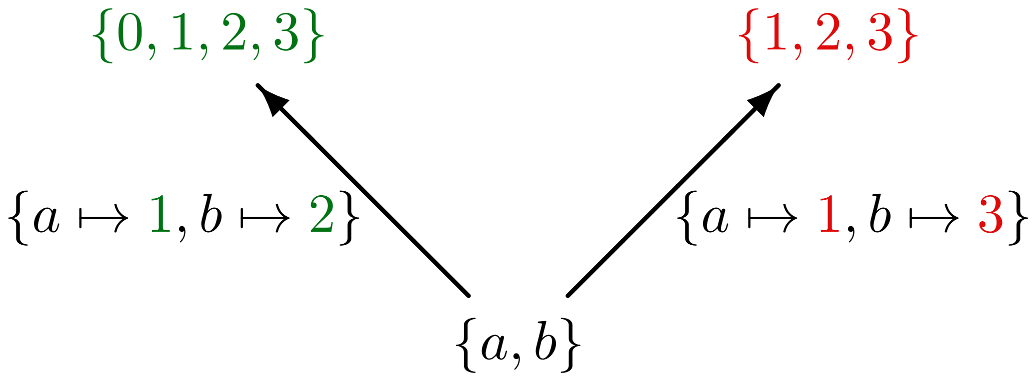

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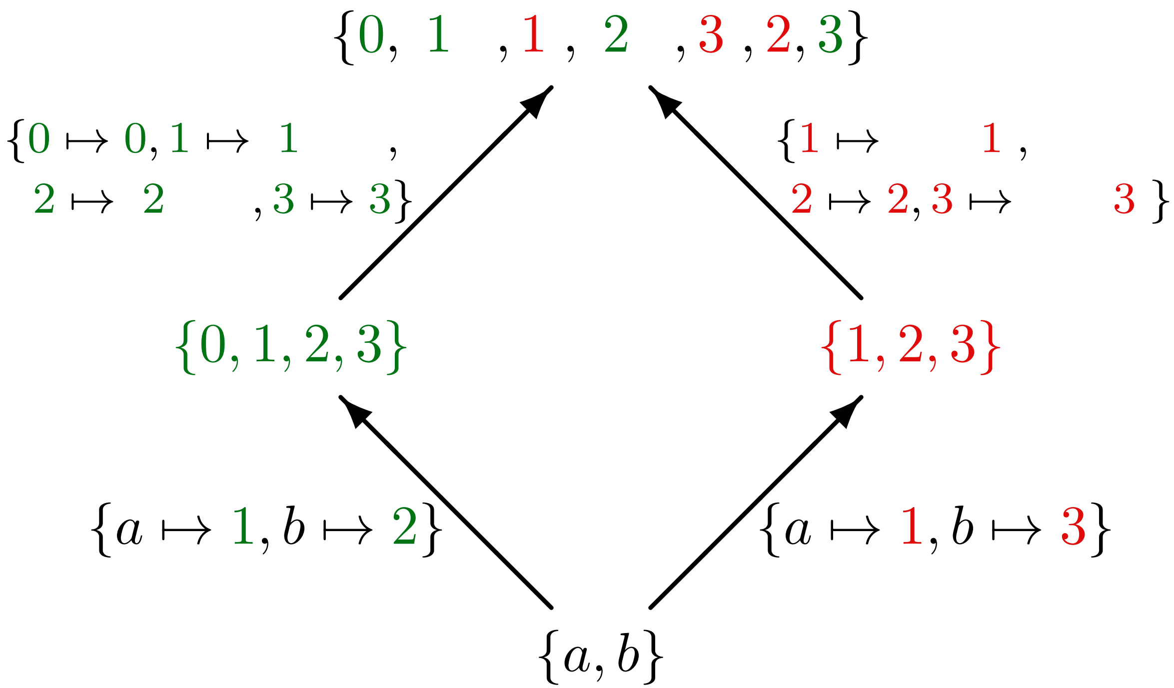

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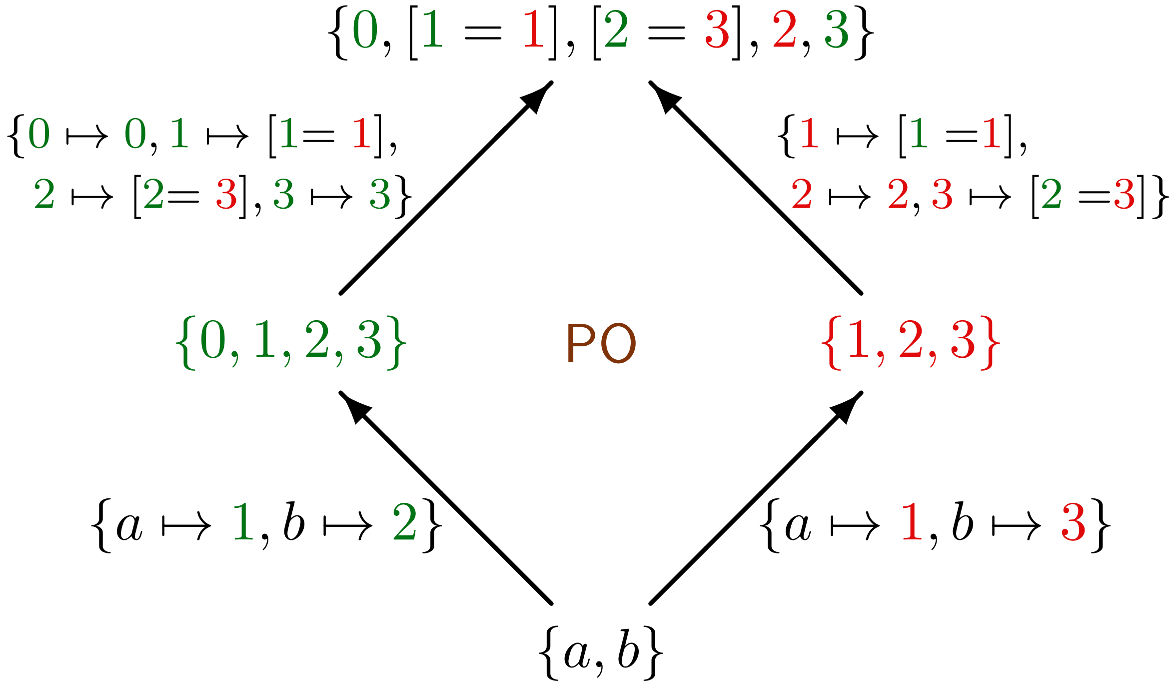
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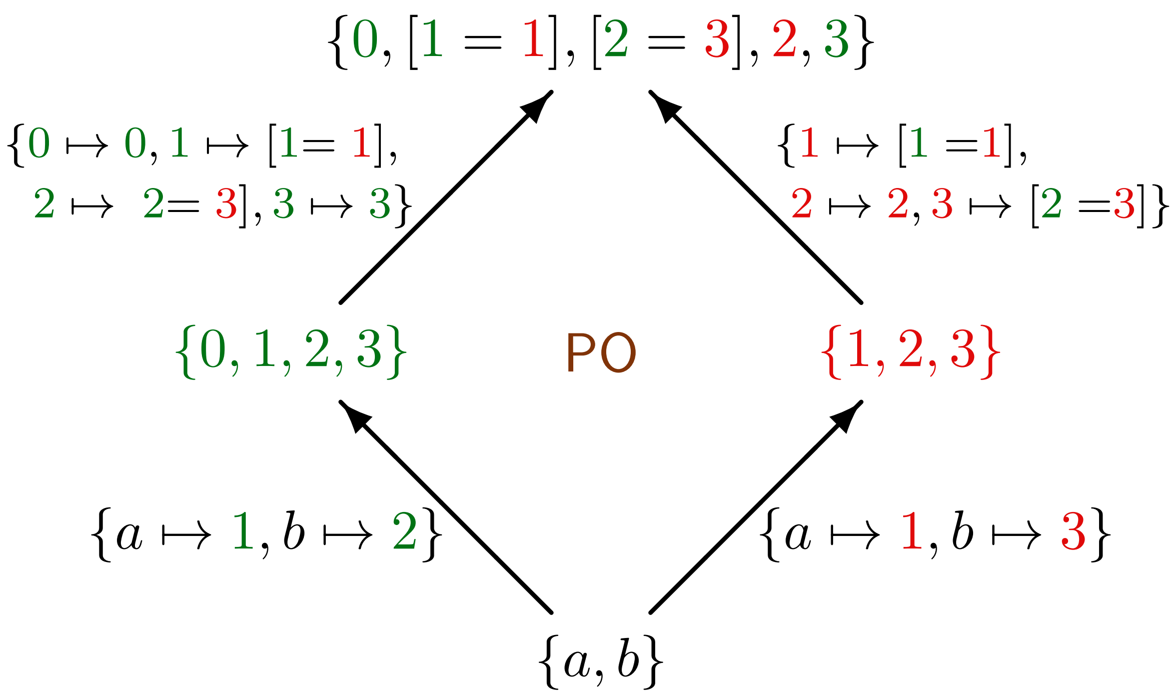

Pushouts put objects together taking account of the indicated sharing.

Example in $\mathbf{AlgSig}$

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ops $a, \ldots, z \colon String;$
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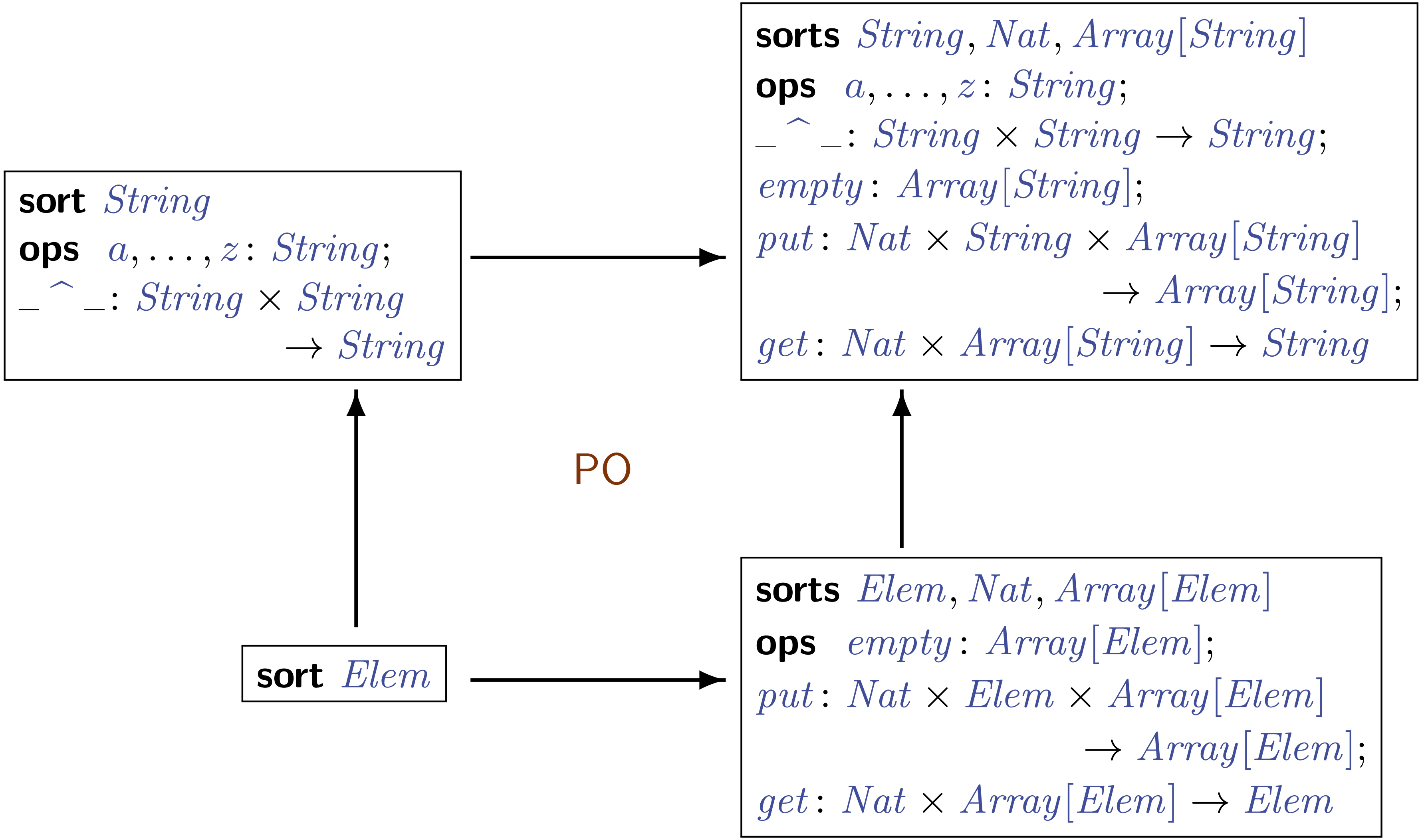
↑

sort $Elem$ ⟶

sorts $Elem, Nat, Array[Elem]$
ops $empty \colon Array[Elem]$;
$put \colon Nat \times Elem \times Array[Elem] \to Array[Elem]$;
$get \colon Nat \times Array[Elem] \to Elem$

Example in AlgSig



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- morphisms: *paths* in G, i.e., sequences $n_0e_1n_1 \ldots n_{k-1}e_kn_k$ of nodes $n_0, \ldots, n_k \in |G|_{nodes}$ and edges $e_1, \ldots, e_k \in |G|_{edges}$ such that $source(e_i) = n_{i-1}$ and $target(e_i) = n_i$ for $i = 1, \ldots, k$.

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BTW: A diagram D *commutes* (or is *commutative*) if for any two paths in $\mathcal{G}(D)$ with common source and target, the compositions of morphisms that label the edges of each of them coincide.

Diagram categories

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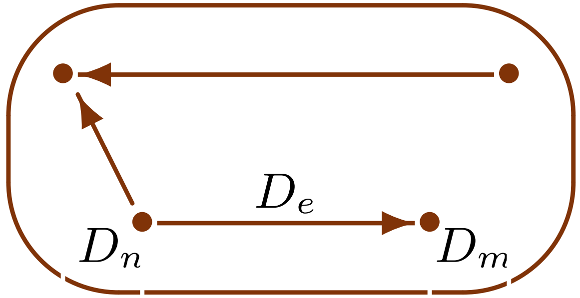

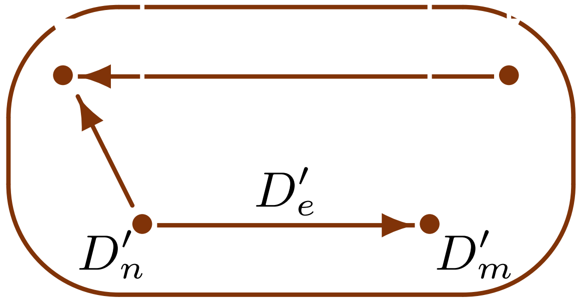

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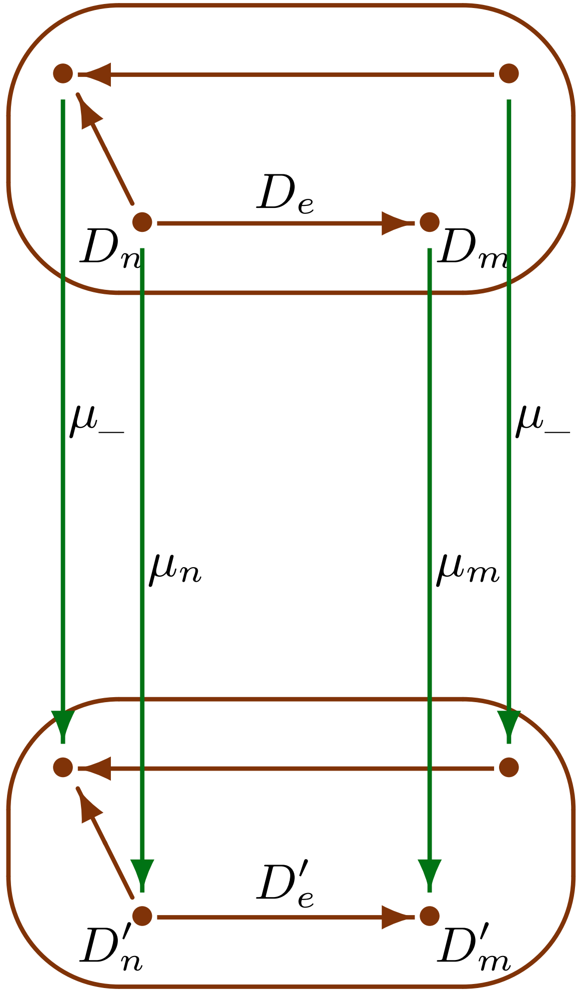

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$$\mu_n;D'_e = D_e;\mu_m$$

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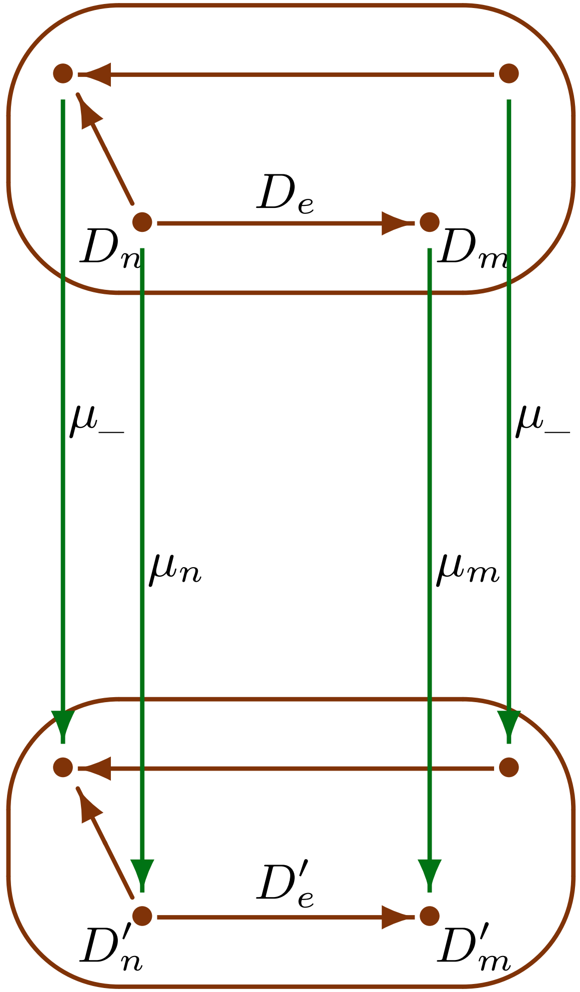

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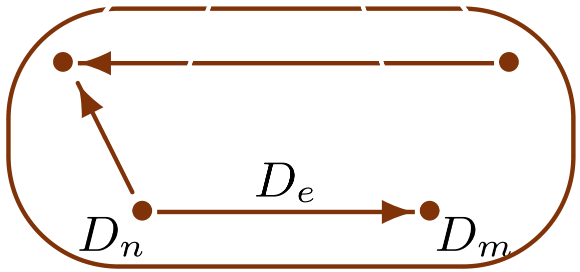

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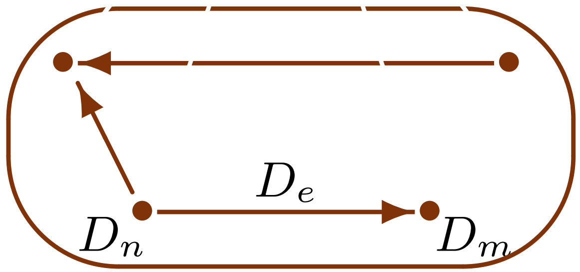
Cones and cocones



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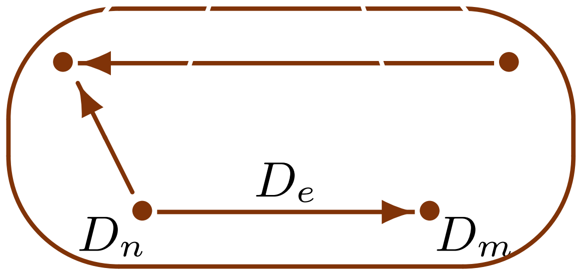

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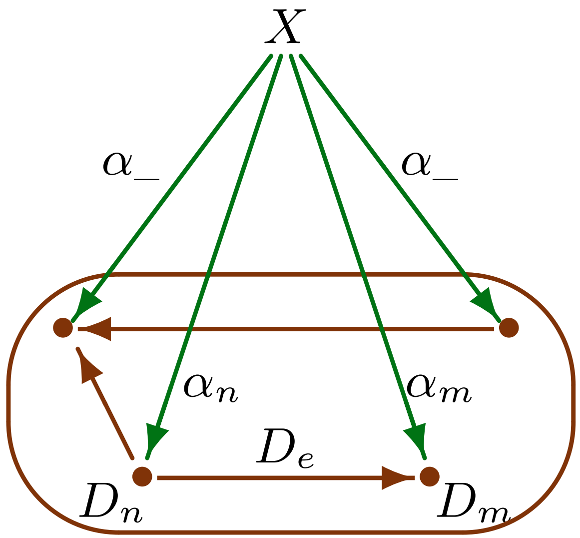

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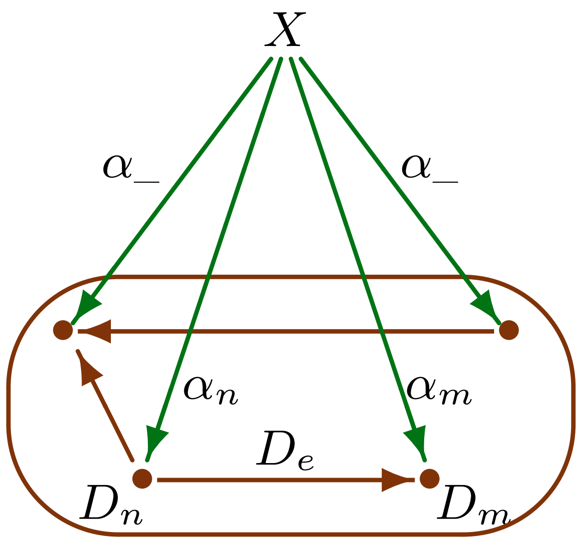

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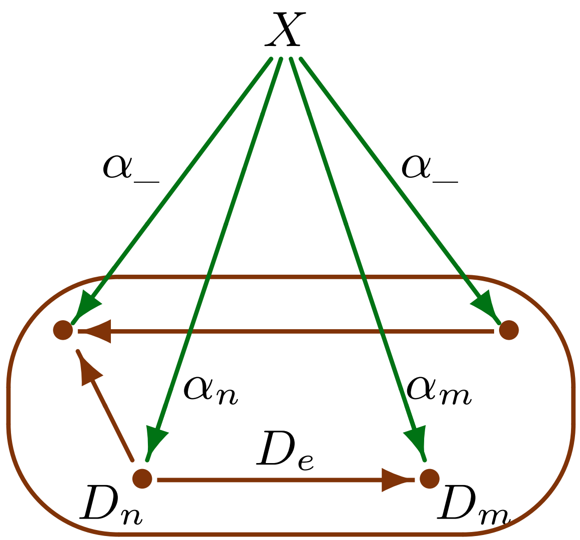

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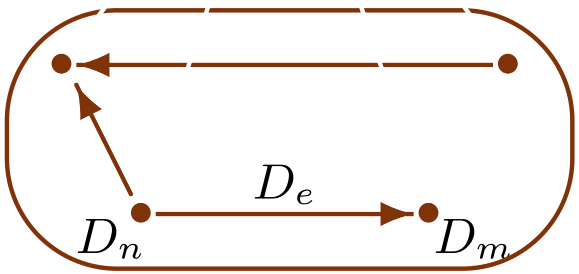

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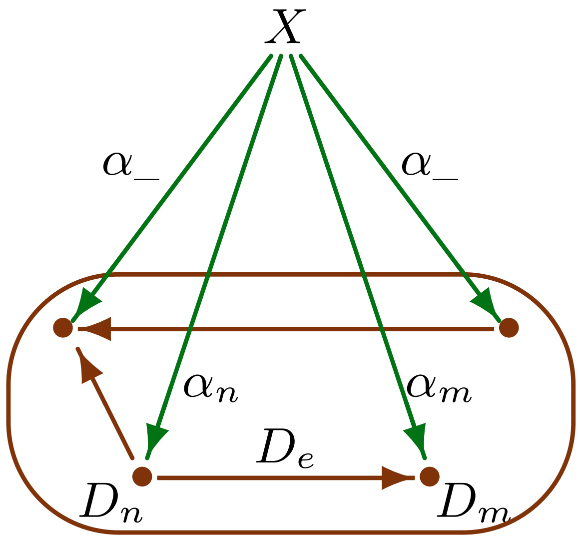

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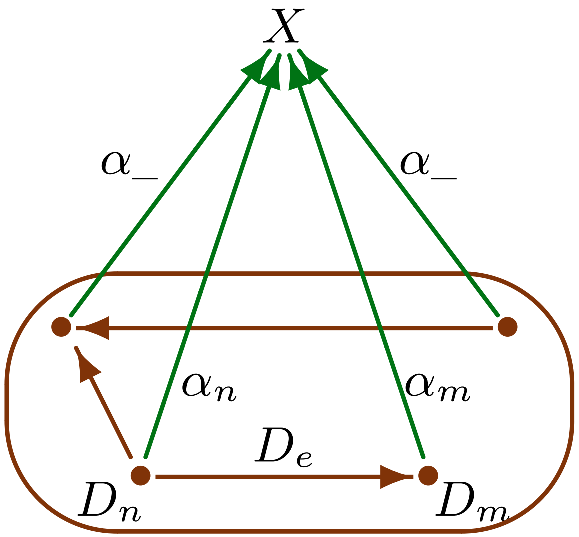

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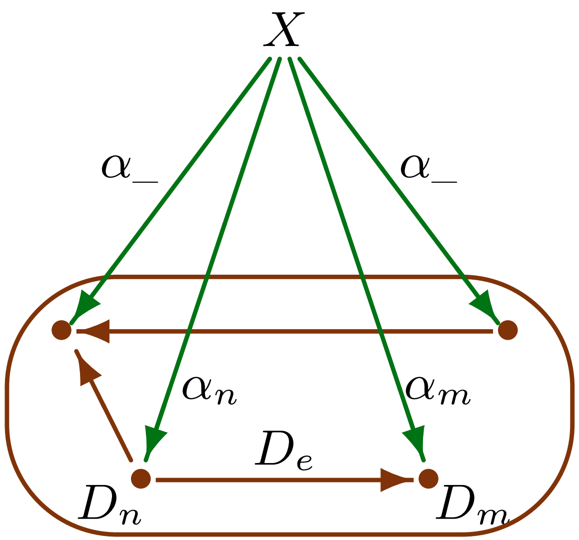

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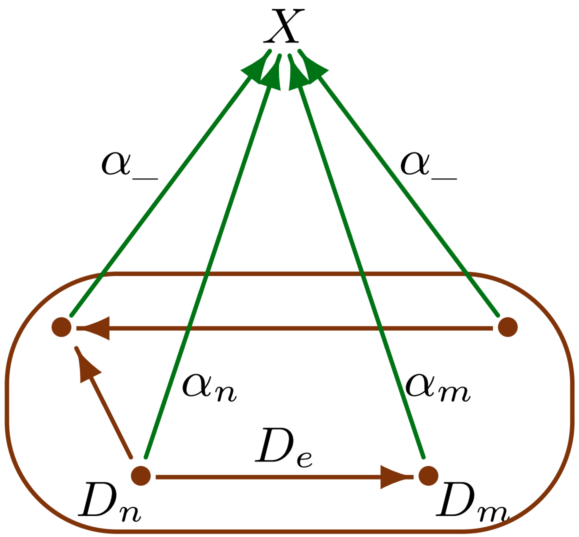

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Limits and colimits

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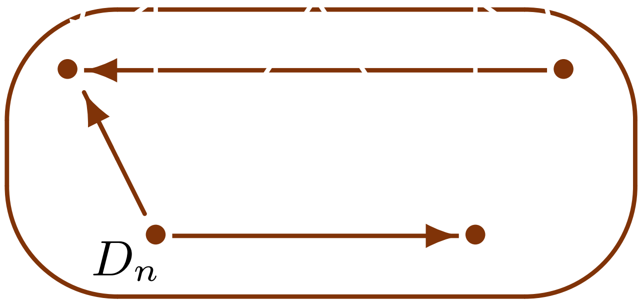

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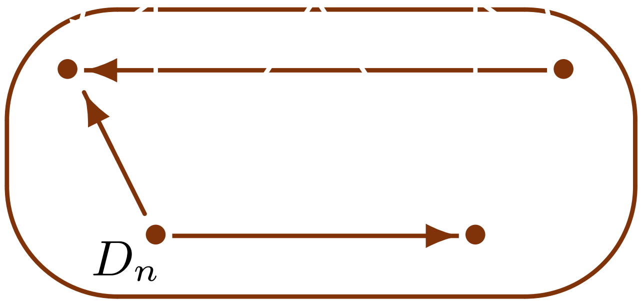

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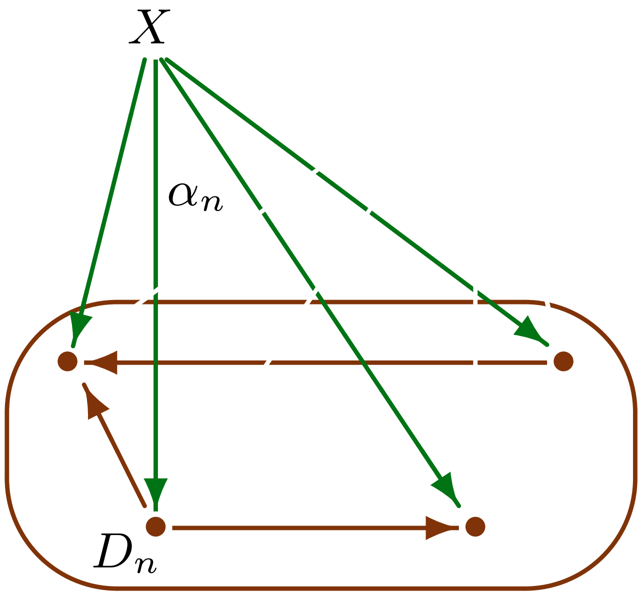

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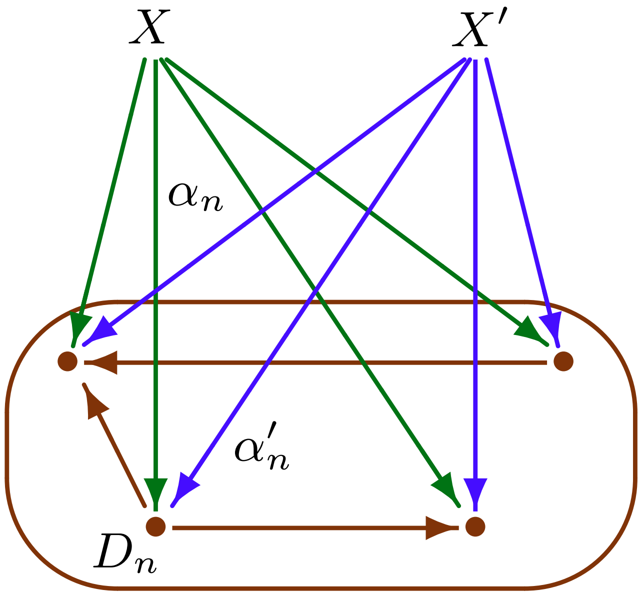

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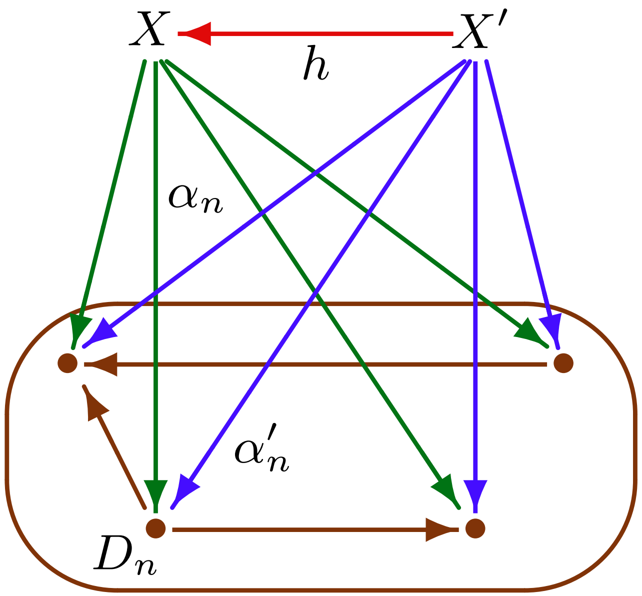

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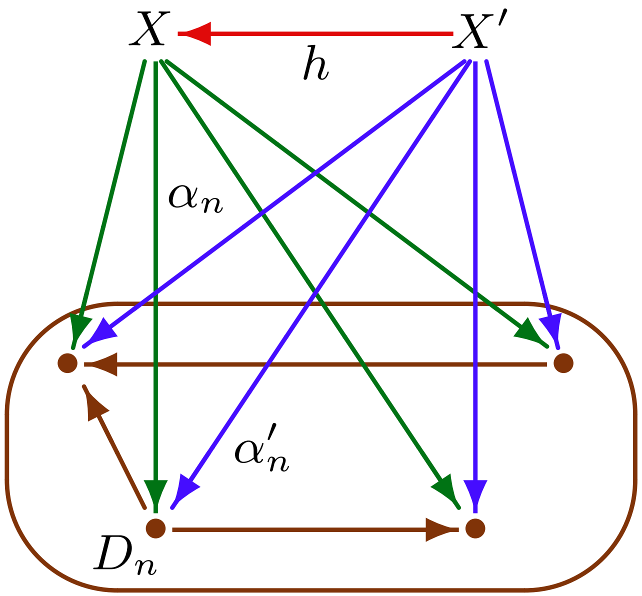

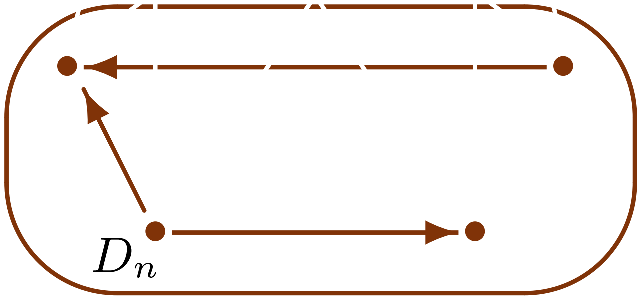

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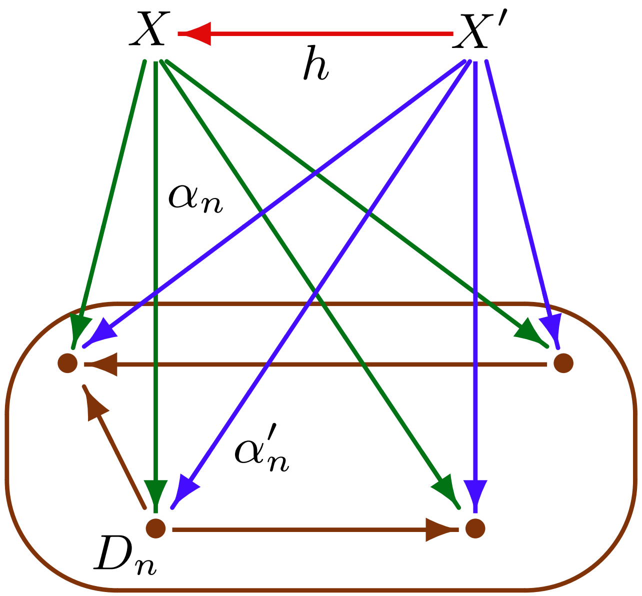

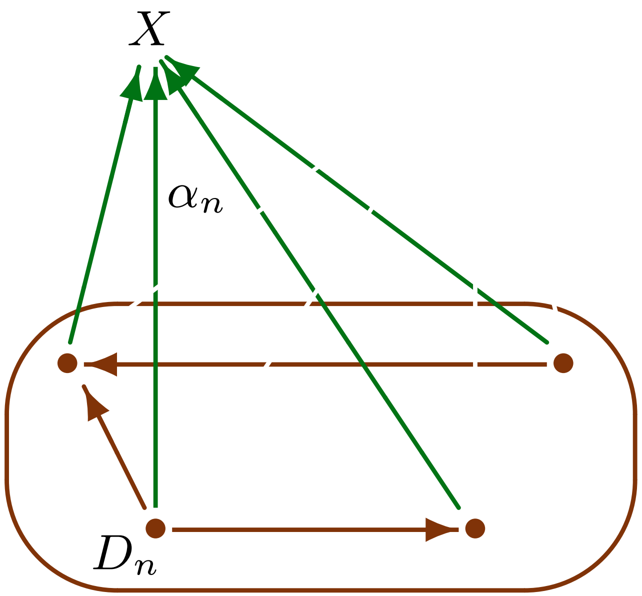

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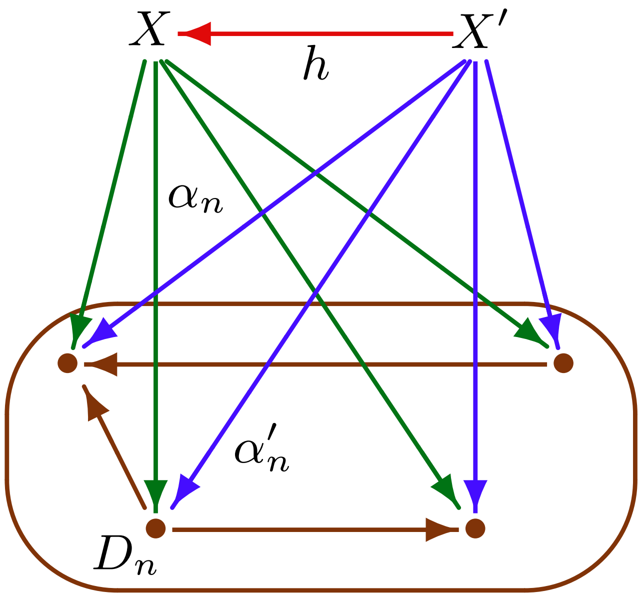

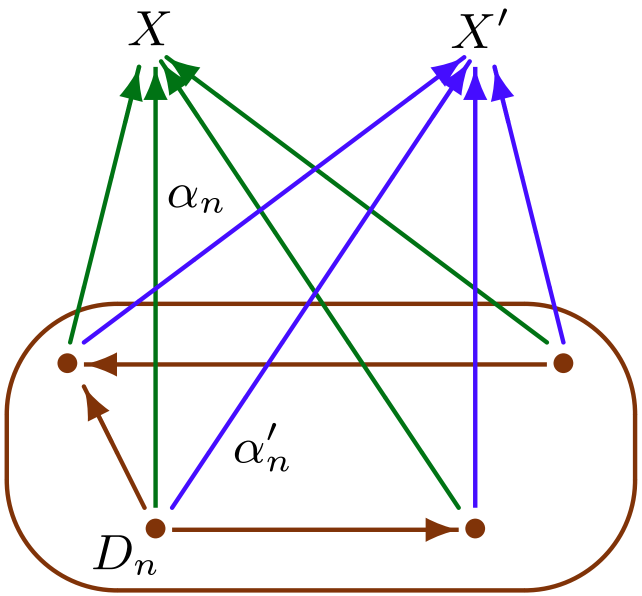

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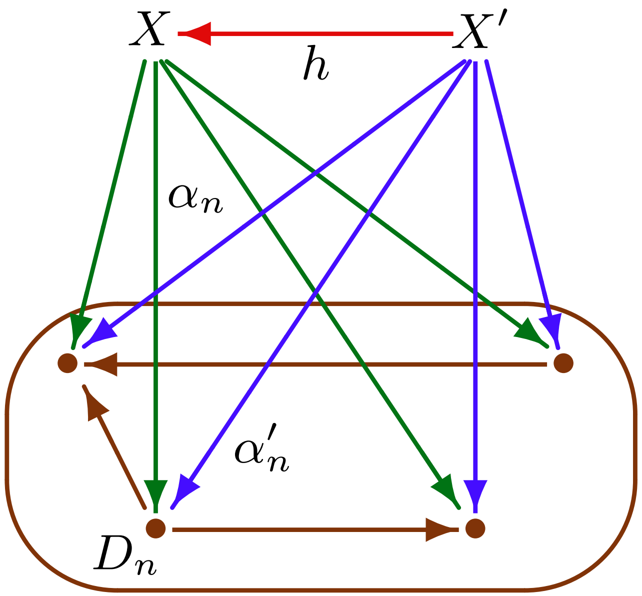

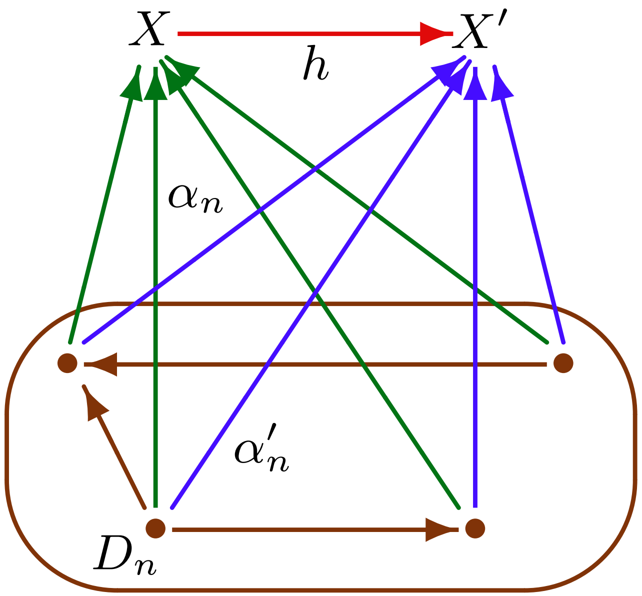

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Some limits

diagram	limit	in **Set**

Some limits

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(empty)	*terminal object*	$\{*\}$

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(empty)	*terminal object*	$\{*\}$
$A \quad B$	*product*	$A \times B$

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(empty)	*terminal object*	$\{*\}$
$A \qquad B$	*product*	$A \times B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*equaliser*	$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A$

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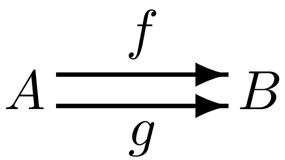

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Cones $X \xrightarrow{\alpha_A} A \overset{f}{\underset{g}{\rightrightarrows}} B$, $X \xrightarrow{\alpha_B} B$ where $\alpha_A;f = \alpha_B$ and $\alpha_A;g = \alpha_B$

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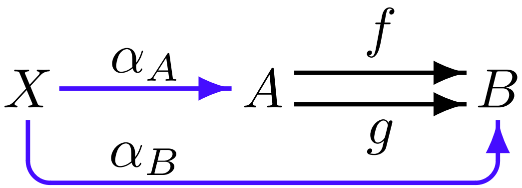
Cones 

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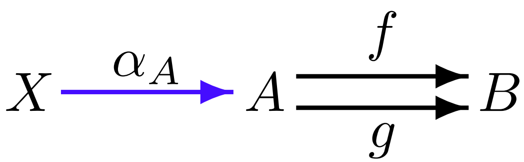
coincide with morphisms 

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Some limits

diagram	limit	in $\mathbf{Set}$
(empty)	*terminal object*	$\{*\}$
$A \qquad B$	*product*	$A \times B$
$A \underset{g}{\overset{f}{\rightrightarrows}} B$	*equaliser*	$\{a \in A \mid f(a) = g(a)\} \hookrightarrow A$
$A \xrightarrow{f} C \xleftarrow{g} B$	*pullback*	$\{(a, b) \in A \times B \mid f(a) = g(b)\}$

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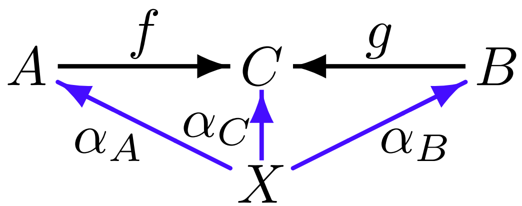
Cones 

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Some limits

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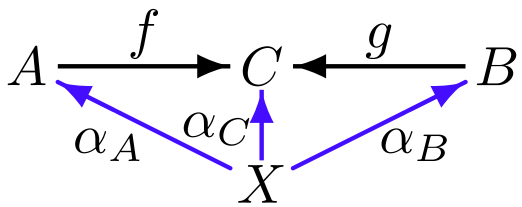
Cones 

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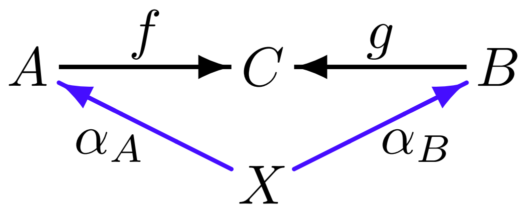
coincide with pairs of morphisms 

where $\alpha_A;f = \alpha_B;g$.

. . . & colimits

diagram	colimit	in **Set**
(empty)	*initial object*	$\emptyset$
$A \qquad B$	*coproduct*	$A \uplus B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*coequaliser*	$B \longrightarrow B/\!\equiv$ where $f(a) \equiv g(a)$ for all $a \in A$
$A \overset{f}{\longleftarrow} C \overset{g}{\longrightarrow} B$	*pushout*	$(A \uplus B)/\!\equiv$ where $f(c) \equiv g(c)$ for all $c \in C$

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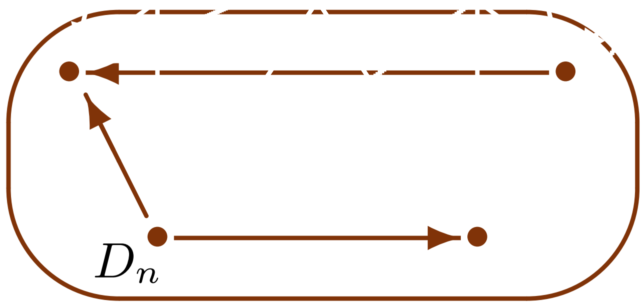

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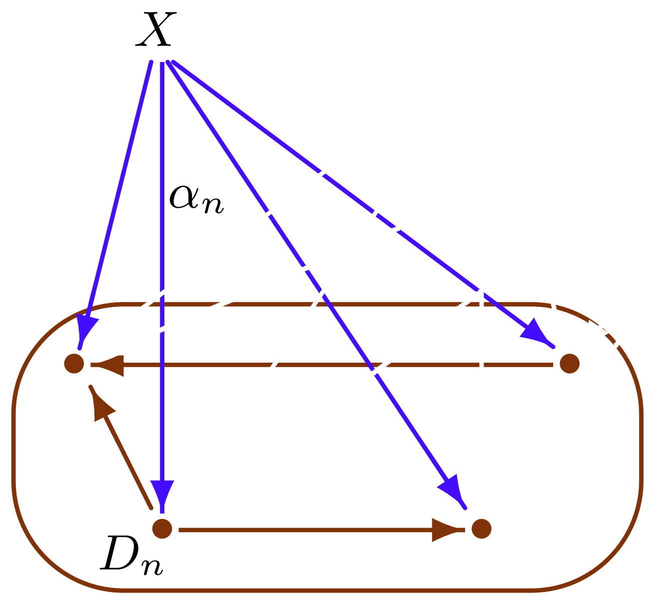

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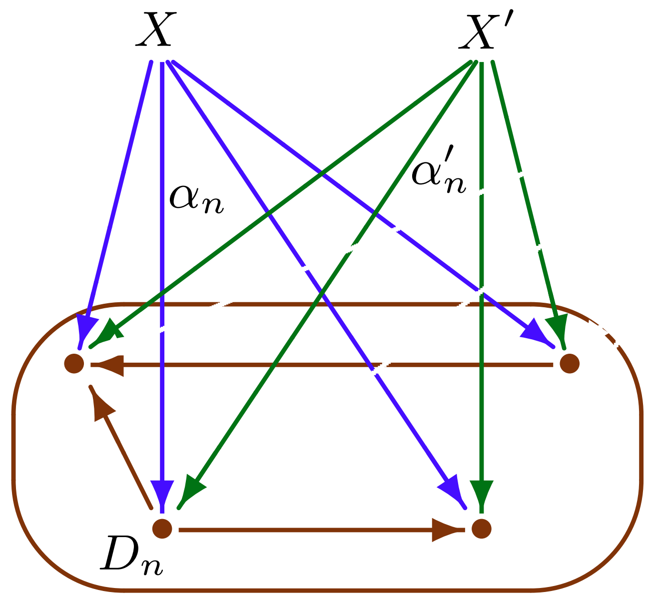

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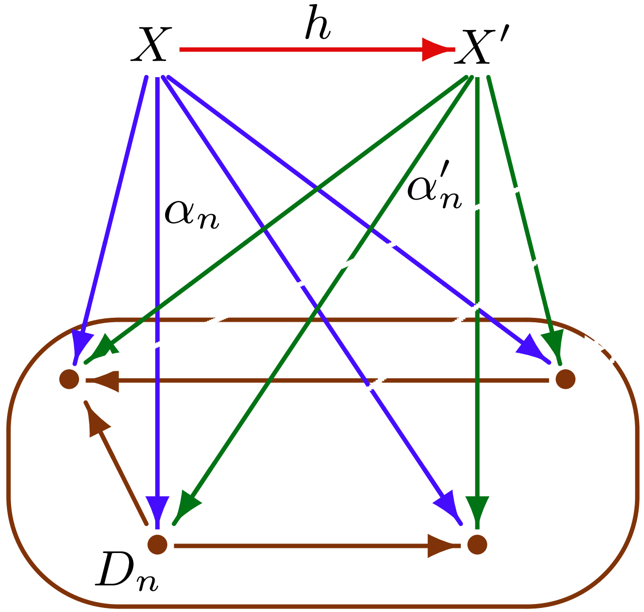

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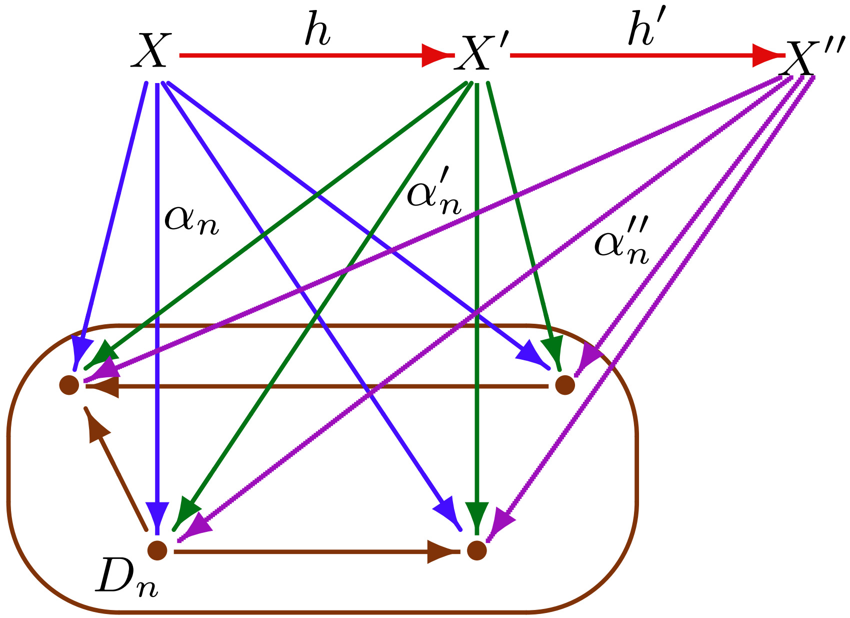

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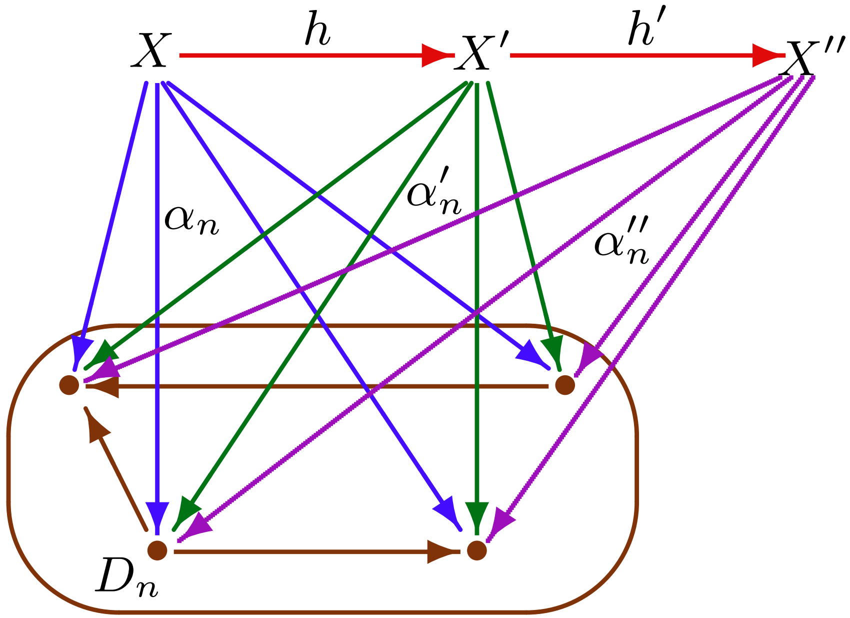

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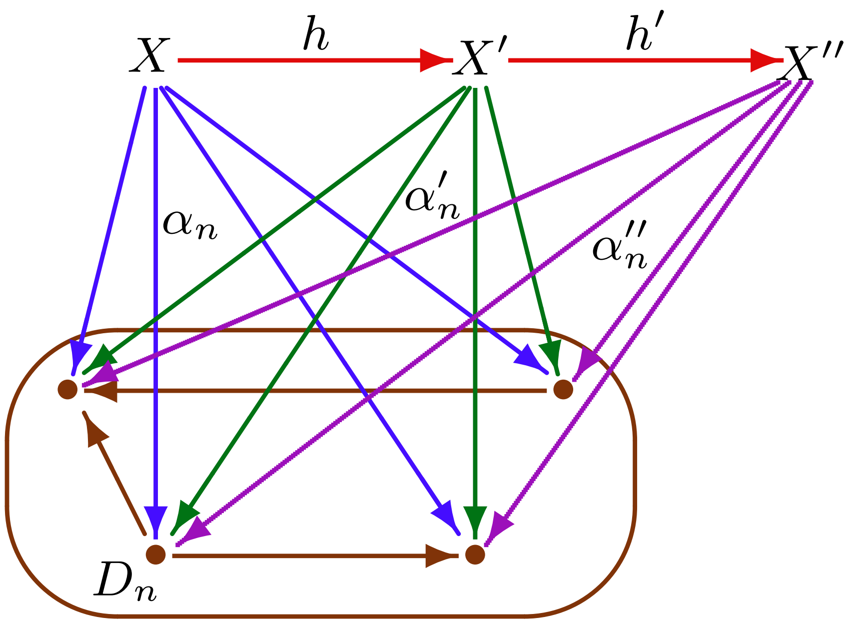

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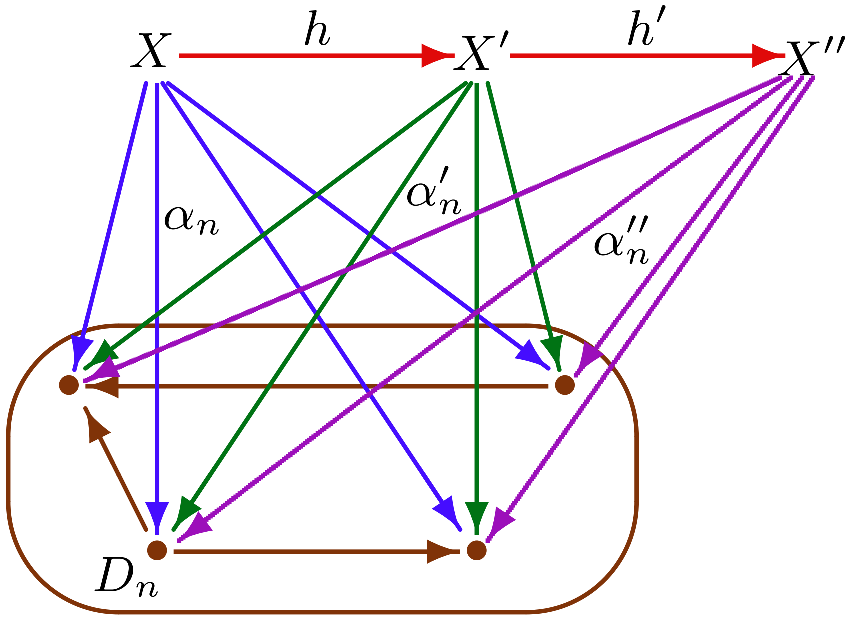

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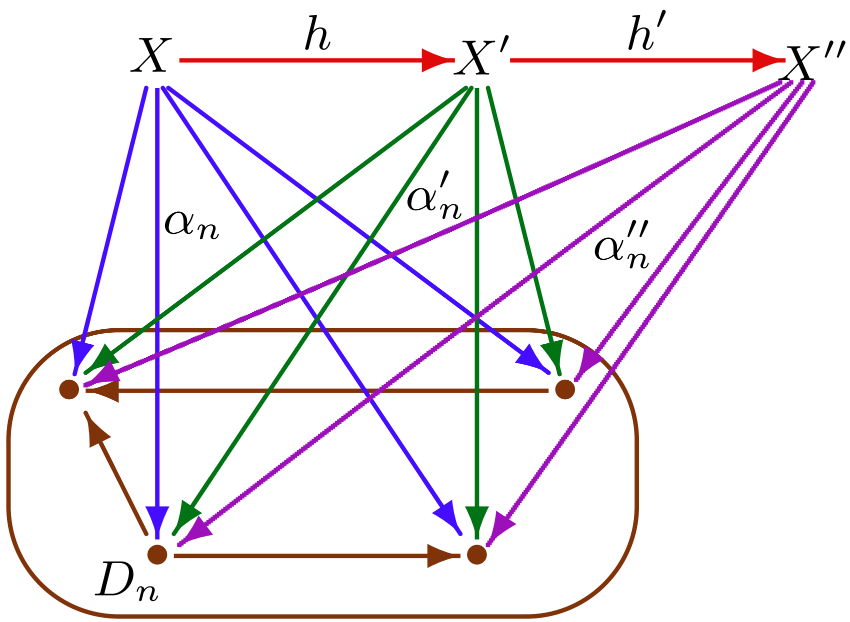

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Hint: $\{\langle a_i \rangle_{i \geq 0} \mid$ for $i \geq 0, a_i \in A_i$ and $f_i(a_{i+1}) = a_i\}$

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Dualise all the exercises above!

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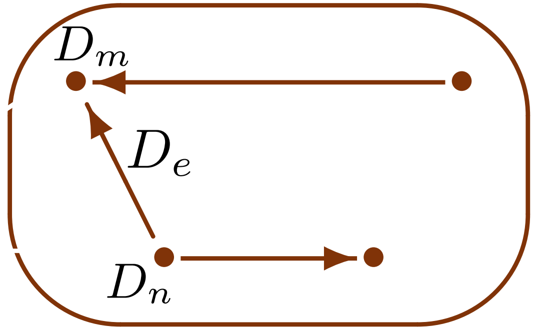

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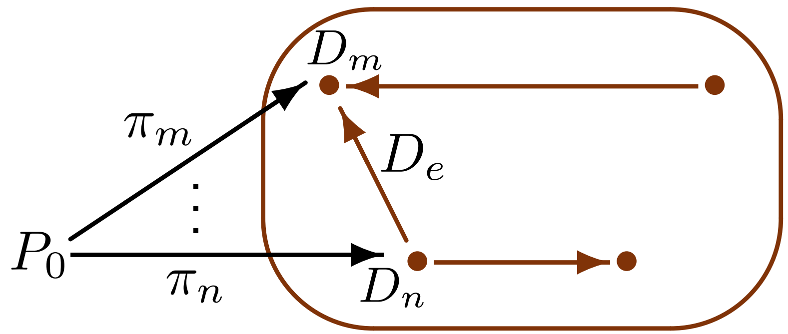

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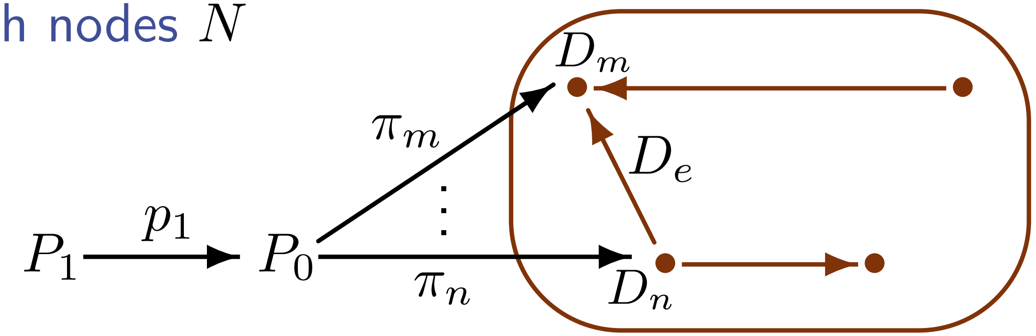

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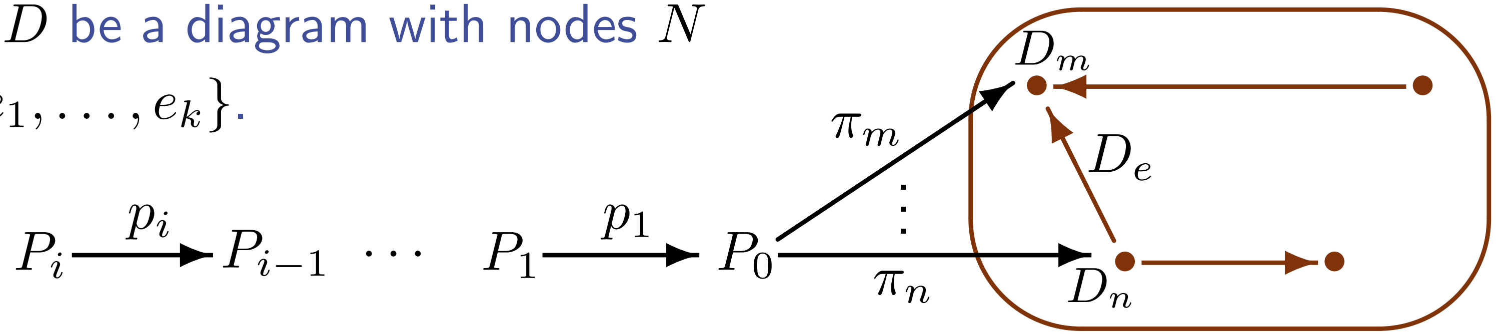

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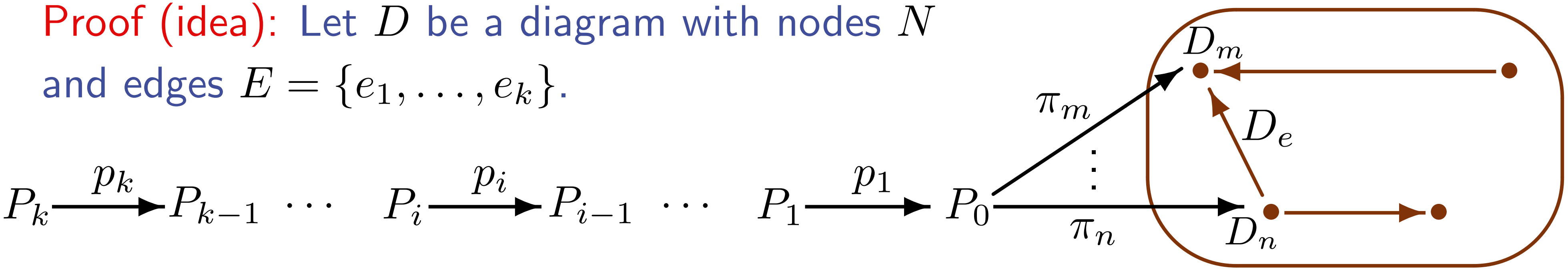

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  - P_k with projections $p_k;\cdots;p_1;\pi_n : P_k \to D_n$, $n \in N$, is the limit of D.

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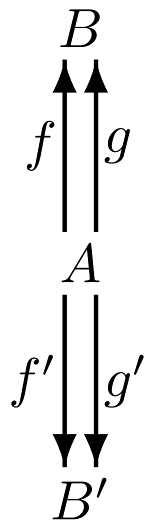

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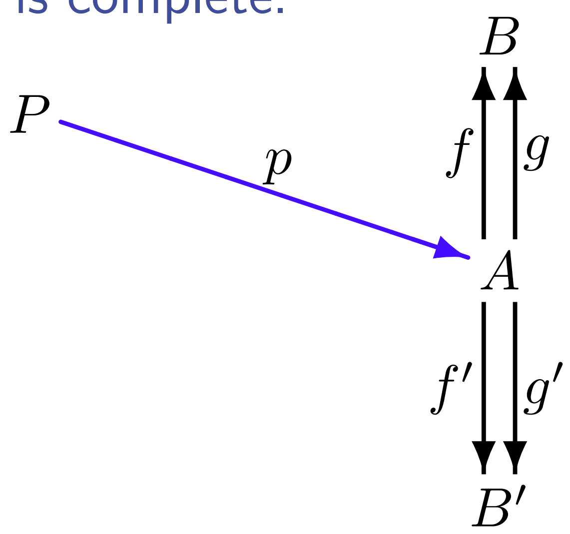

$p = eql(f, g)$

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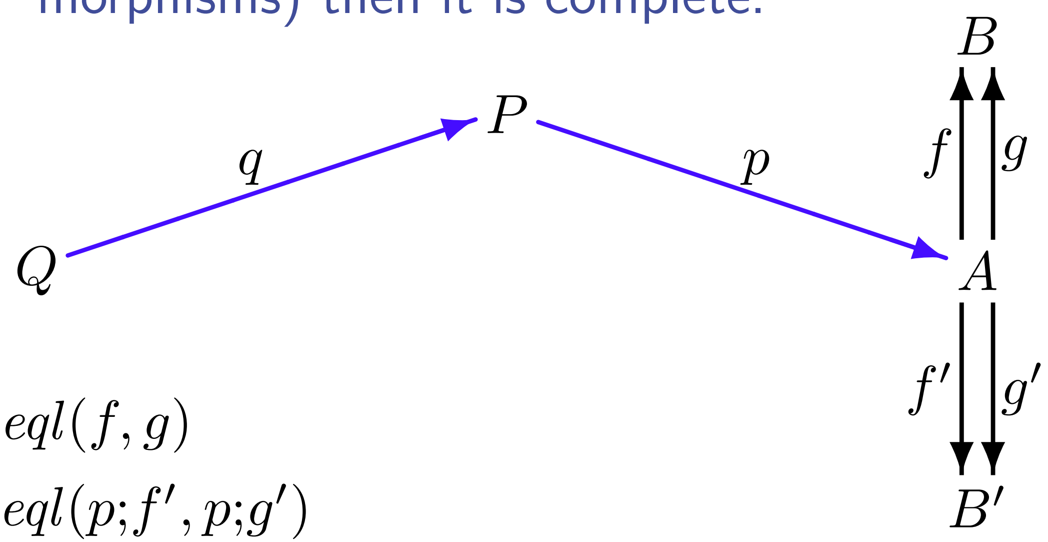

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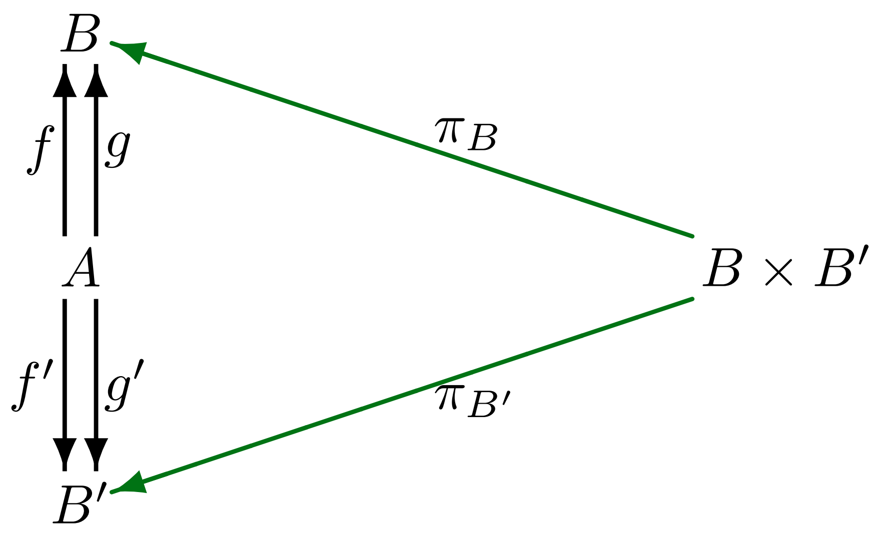

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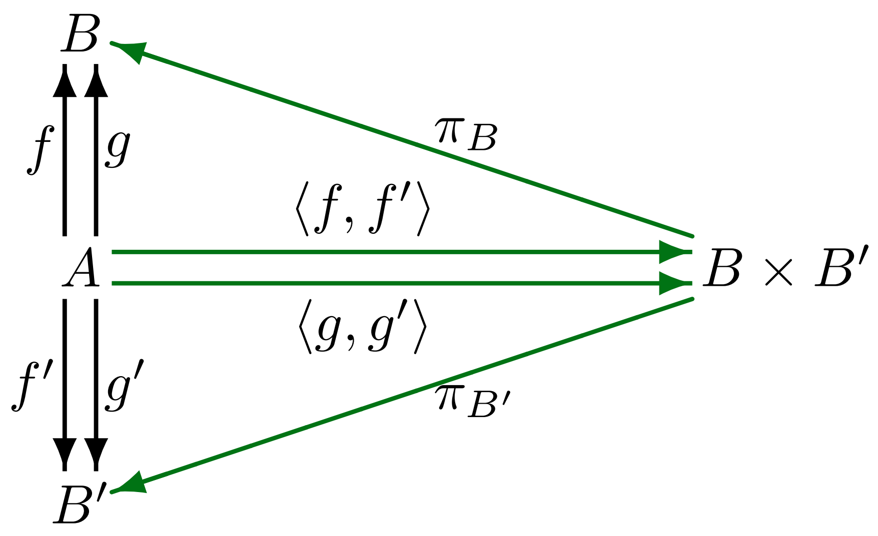

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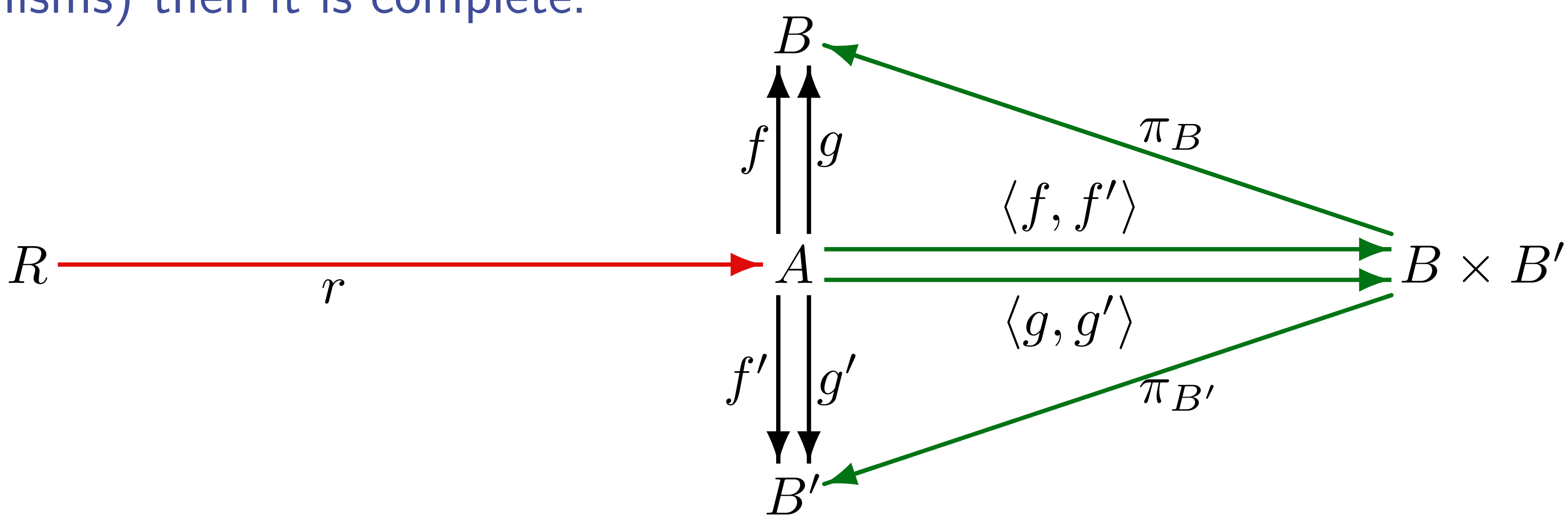

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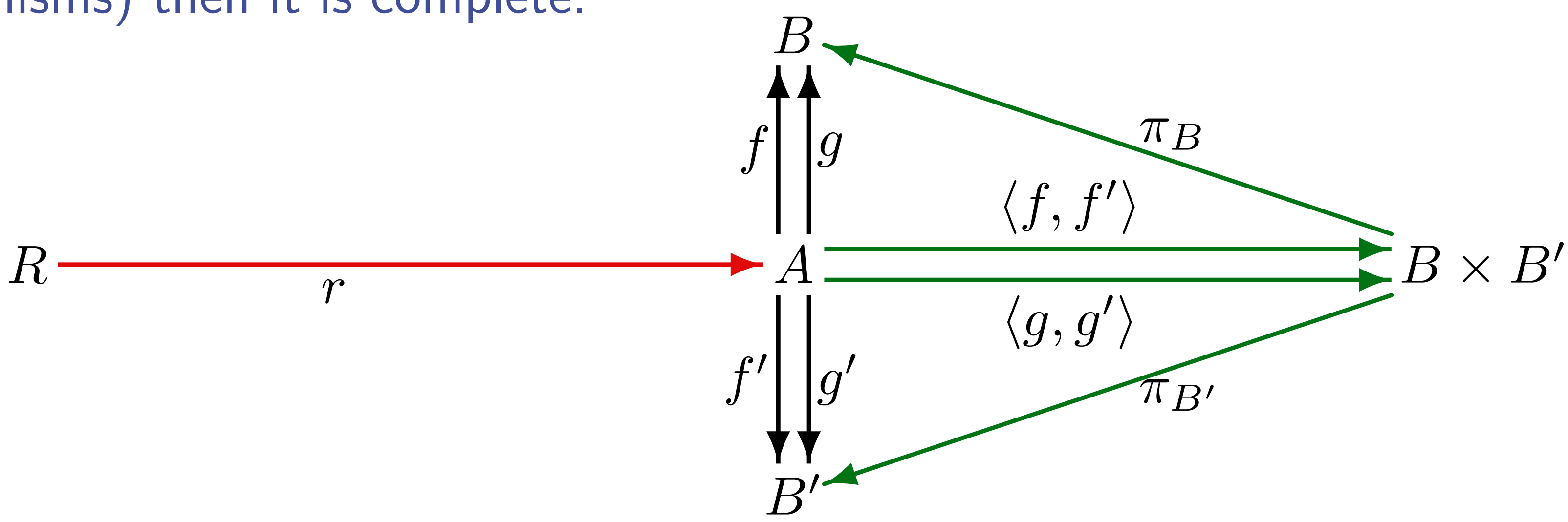

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Hint: $r;\langle f, f' \rangle = r;\langle g, g' \rangle$ iff $r;f = r;g$ and $r;f' = r;g'$

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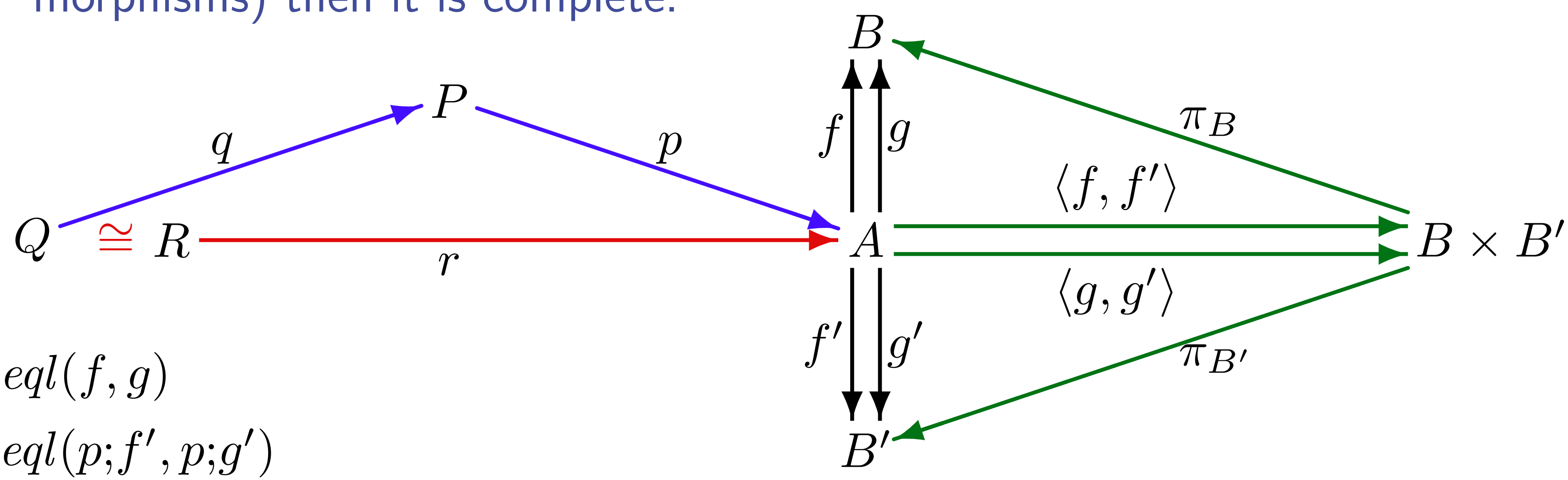

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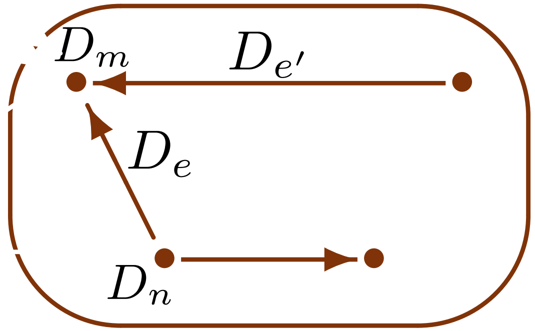

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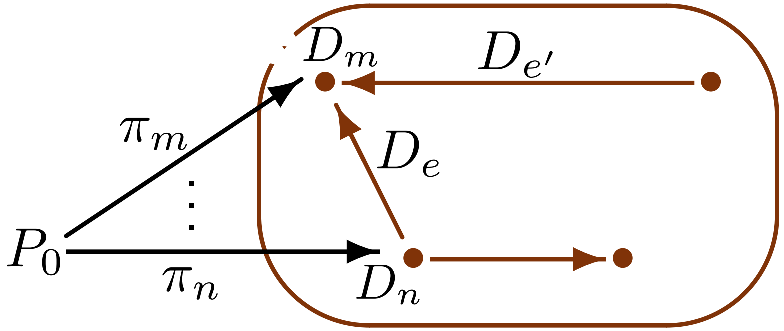

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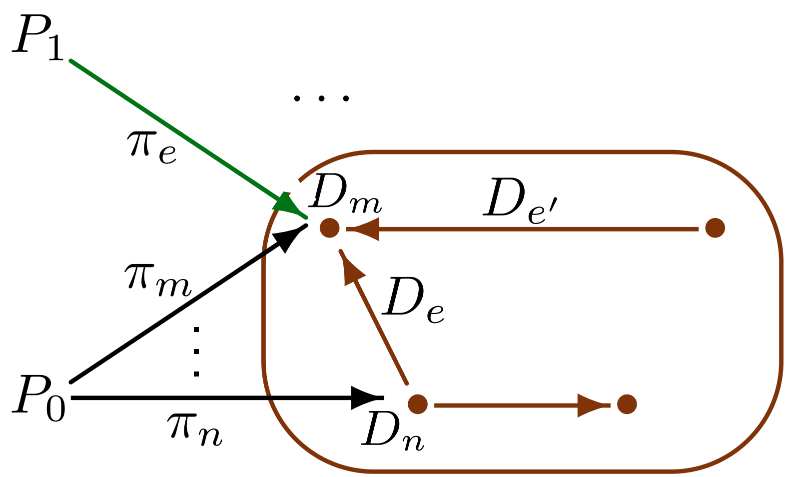

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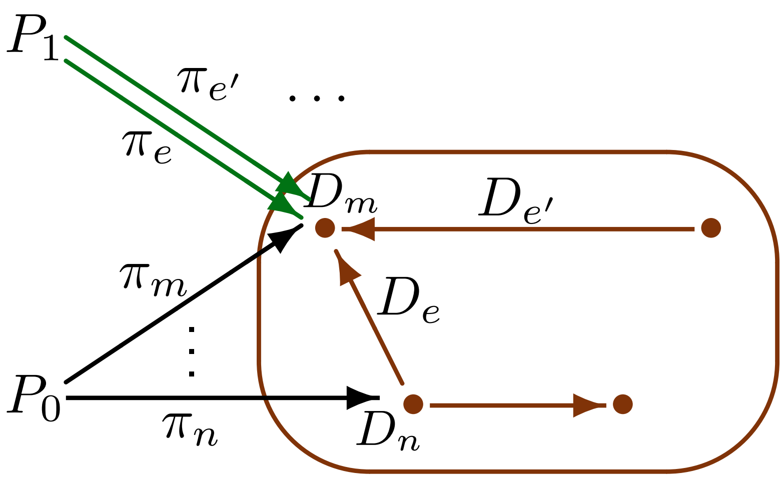

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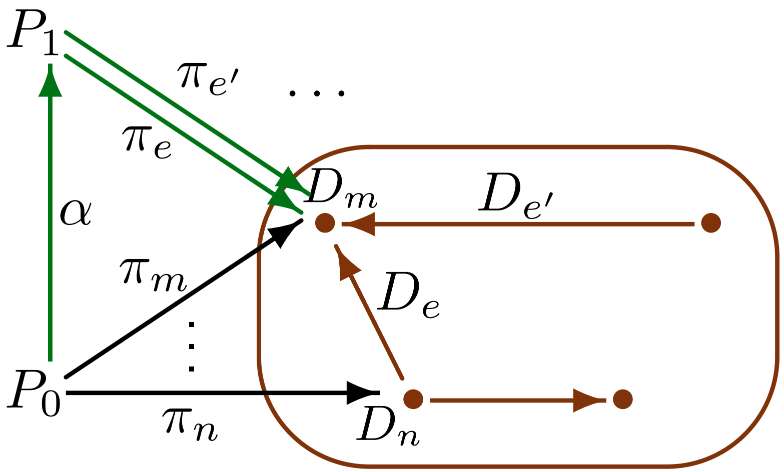

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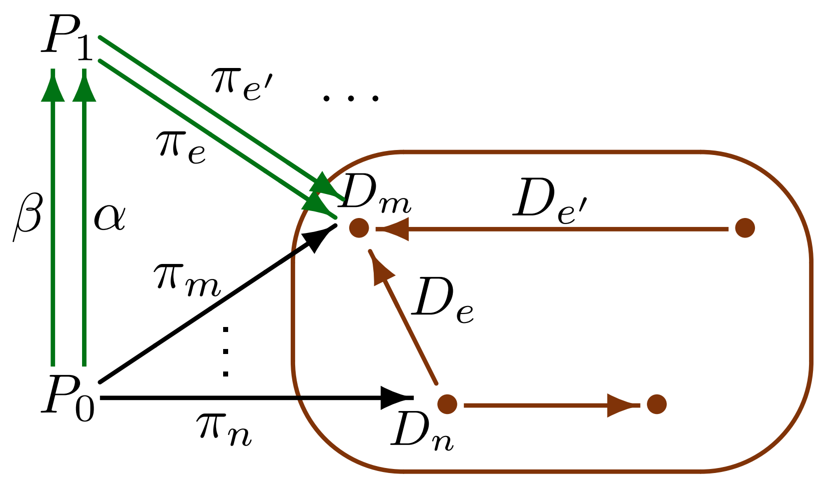

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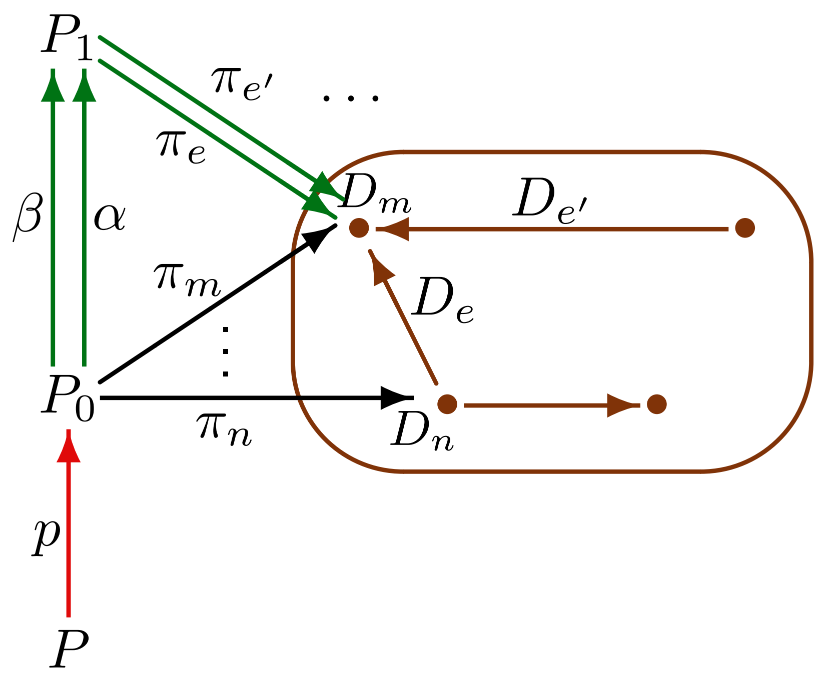

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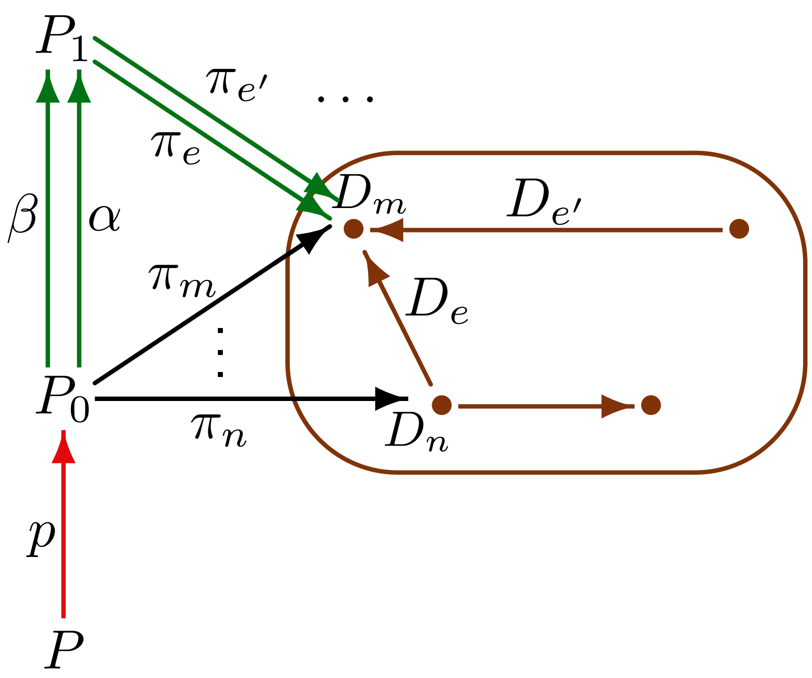
– P with projections $\langle p;\pi_n \rangle_{n \in N}$ is the limit of D.



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Proof: Consider a small category $\mathbf{K}$, with $card(\mathbf{K}) = \kappa$, and let $f, g \colon A \to B$ (in $\mathbf{K}$).

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