Monads

Monads

monoids: a categorical generalisation
algebras: a categorical generalisation

Monads

category theory for
effects in typed functional programming

monoids: a categorical generalisation
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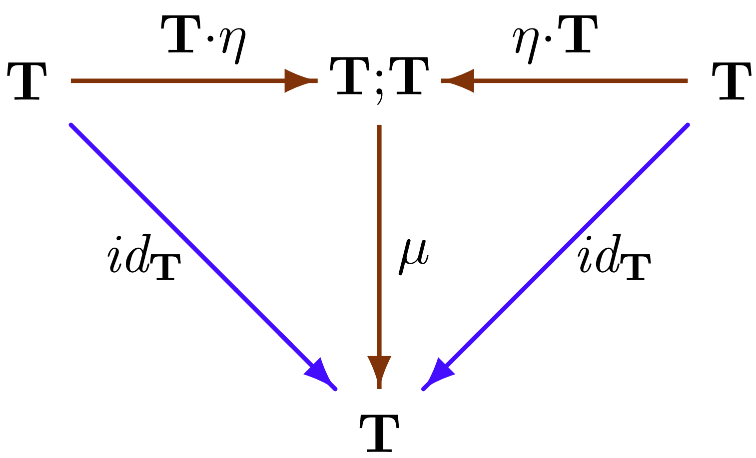
- $\eta_{\mathbf{T}(X)};\mu_X = id_{\mathbf{T}(X)} = \mathbf{T}(\eta_X);\mu_X$



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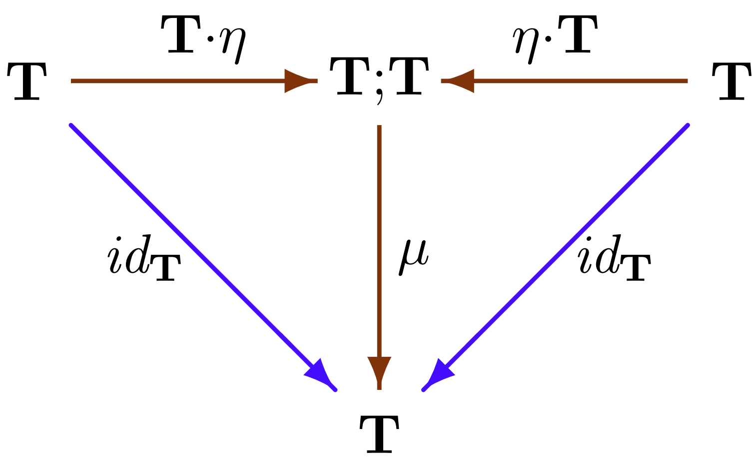

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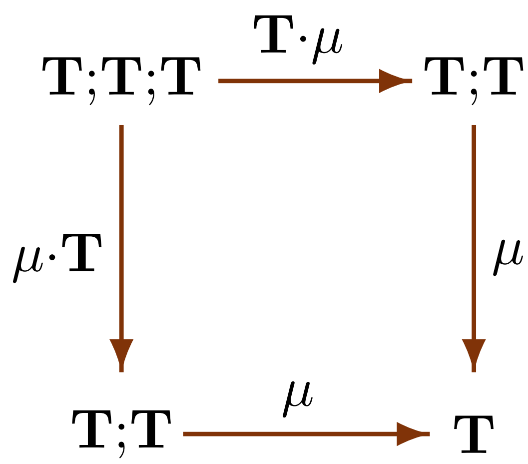

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Simple Examples

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- *Partiality* monad:
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  - $\eta^{\mathbf{P}}_X(x) = x$;

$\eta^{\mathbf{P}}_X : X \to X + \{\bot\}$

Simple Examples

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  - $\mathbf{P}(X) = X + \{\bot\}$; $\qquad \eta^{\mathbf{P}}_X \colon X \to X + \{\bot\}$
  - $\eta^{\mathbf{P}}_X(x) = x$; $\qquad \mu^{\mathbf{P}}_X \colon (X + \{\bot\}) + \{\bot\} \to X + \{\bot\}$

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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.

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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \perp$ for $x \notin X$.
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  - $\mathcal{E}(X) = X + E$;

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  - $\mathcal{E}(X) = X + E$; $\eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$;

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- *Exceptions* monad;
  - $\mathcal{E}(X) = X + E$; $\qquad \eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$; $\qquad \mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$

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  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.

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- *Nondeterminism* monad:

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- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$;

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  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$; $\quad \eta^{\mathcal{P}}_X : X \to 2^X$
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$;

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- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$; $\qquad \eta^{\mathcal{P}}_X \colon X \to 2^X$
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$; $\qquad \mu^{\mathcal{P}}_X \colon 2^{2^X} \to 2^X$

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  - $\eta^{\mathcal{P}}_X(x) = \{x\}$; $\quad \mu^{\mathcal{P}}_X \colon 2^{2^X} \to 2^X$
  - $\mu^{\mathcal{P}}_X(U) = \bigcup U$ for $U \in 2^{2^X}$.

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  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.

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- *Term* monad:
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  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$;

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- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\qquad \eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$; $\qquad \mu^{\mathcal{T}_\Sigma}_X : T_\Sigma(T_\Sigma(X)) \to T_\Sigma(X)$

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  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$; $\mu^{\mathcal{T}_\Sigma}_X : T_\Sigma(T_\Sigma(X)) \to T_\Sigma(X)$
  - $\mu^{\mathcal{T}_\Sigma}_X(t) = t[id_{T_\Sigma(X)}]$ for $t \in T_\Sigma(T_\Sigma(X))$.

Difficult(?) examples

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  - $\mu^{\mathcal{S}}_X(f)(s) = g(s')$ where $f(s) = \langle g, s'\rangle$, for $f \in ((X \times S)^S \times S)^S$.

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  - $\eta^{\mathcal{K}}_X(x)(k) = k(x)$; $\quad \mu^{\mathcal{K}}_X \colon A^{(A^{(A^{(A^X)})})} \to A^{(A^X)}$

Difficult(?) examples

- *Side-effects* monad:
  - $\mathcal{S}(X) = (X \times S)^S$; $\eta^{\mathcal{S}}_X \colon X \to (X \times S)^S$
  - $\eta^{\mathcal{S}}_X(x)(s) = \langle x, s\rangle$; $\mu^{\mathcal{S}}_X \colon ((X \times S)^S \times S)^S \to (X \times S)^S$
  - $\mu^{\mathcal{S}}_X(f)(s) = g(s')$ where $f(s) = \langle g, s'\rangle$, for $f \in ((X \times S)^S \times S)^S$.
- *Continuation* monad:
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  - $\mu^{\mathcal{K}}_X(f)(k) = f(\lambda g \in A^{(A^X)}.g(k))$, for $f \in A^{(A^{(A^{(A^X)})})}$.

Instead of more examples

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Adjunctions give rise to monads

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Theorem: *For any adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$,

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$\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$
$\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$
$\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$
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  (i.e. $\mu^{\mathbf{T}}_X = \mathbf{G}(\varepsilon_{\mathbf{F}(X)})\colon \mathbf{G}(\mathbf{F}(\mathbf{G}(\mathbf{F}(X)))) \to \mathbf{G}(\mathbf{F}(X))$*)*

Proof

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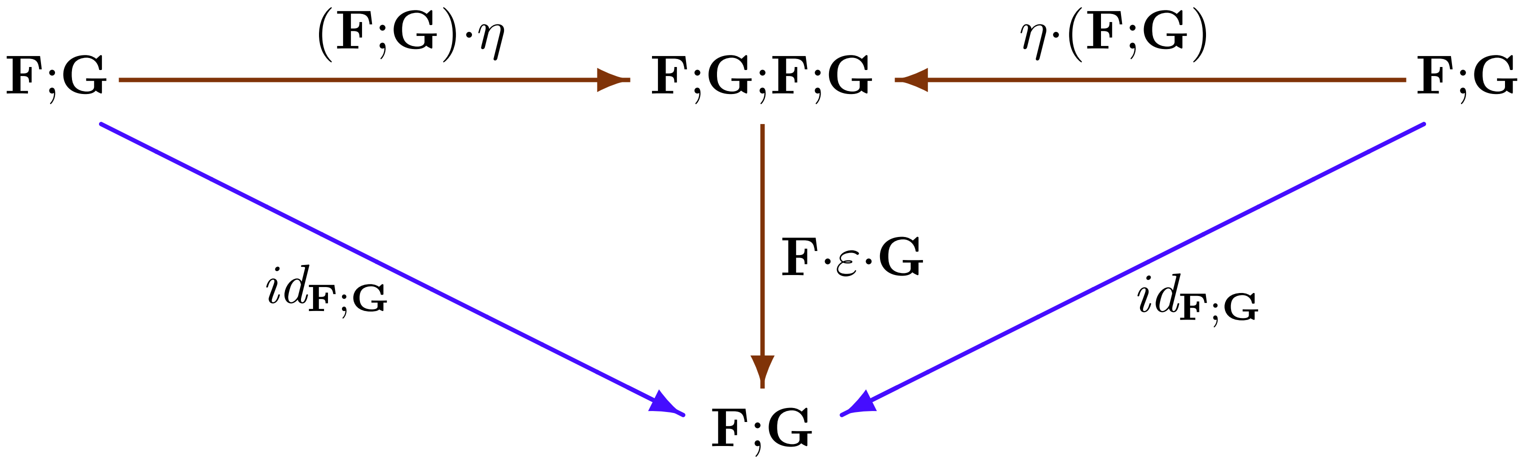

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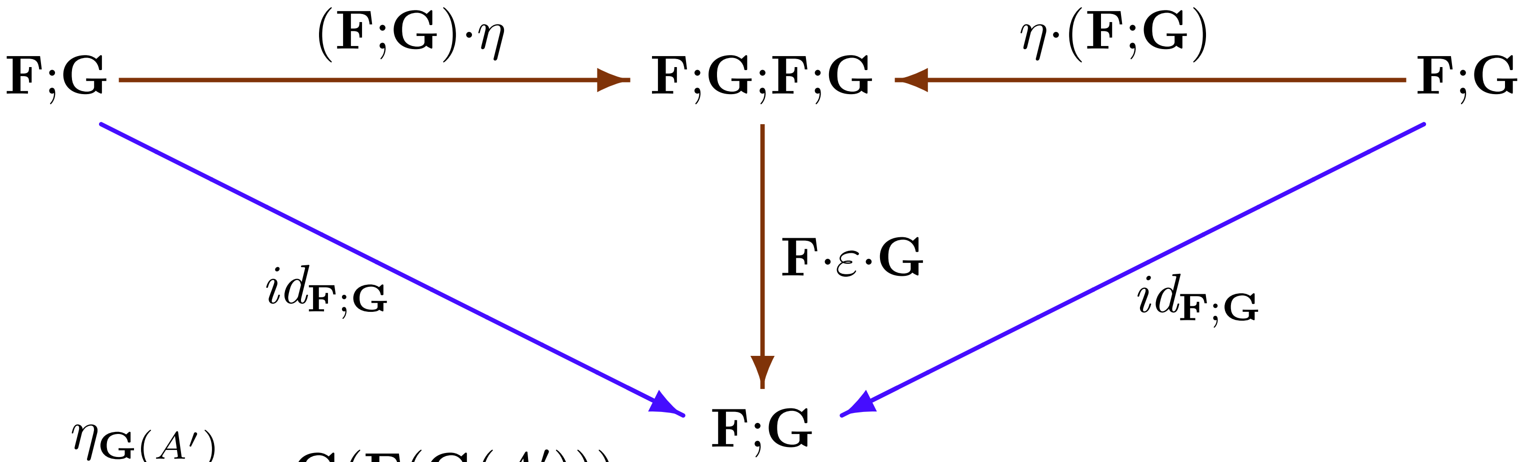

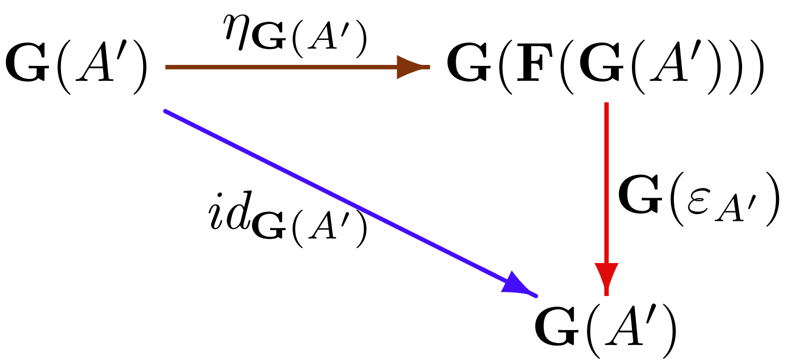

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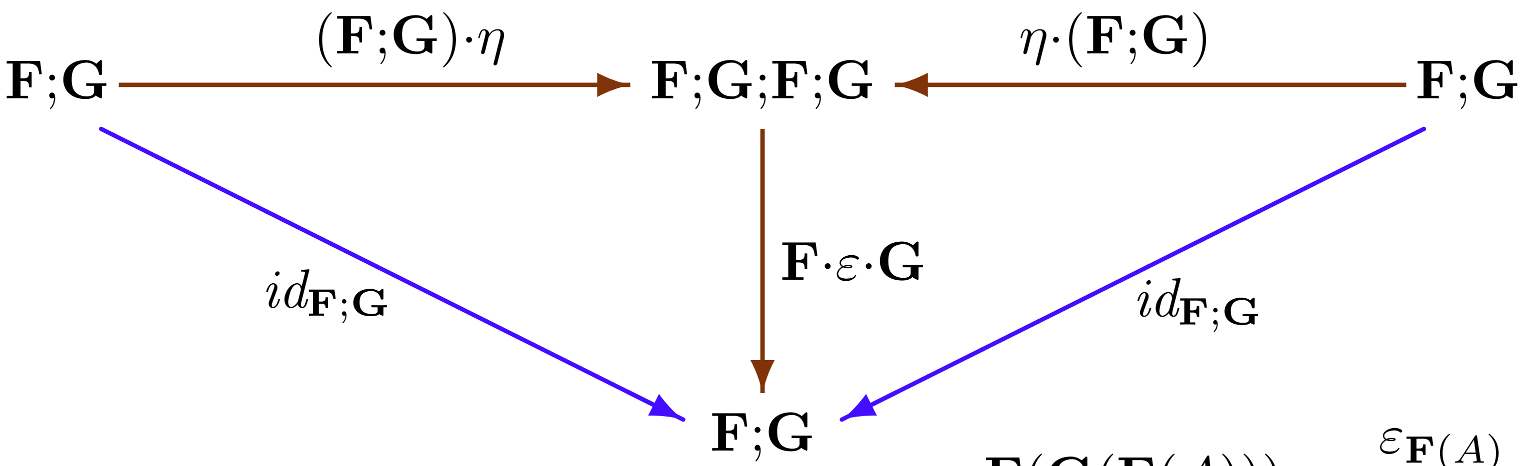

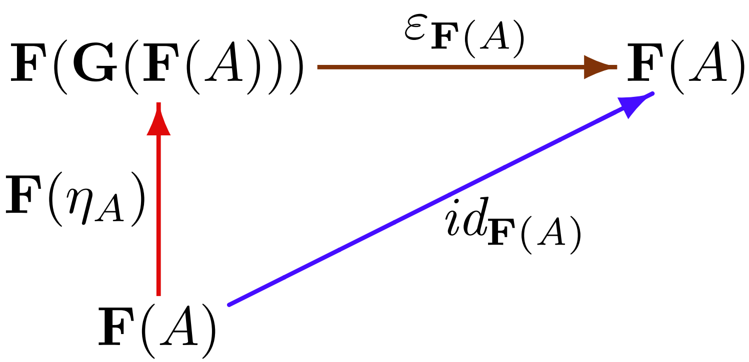

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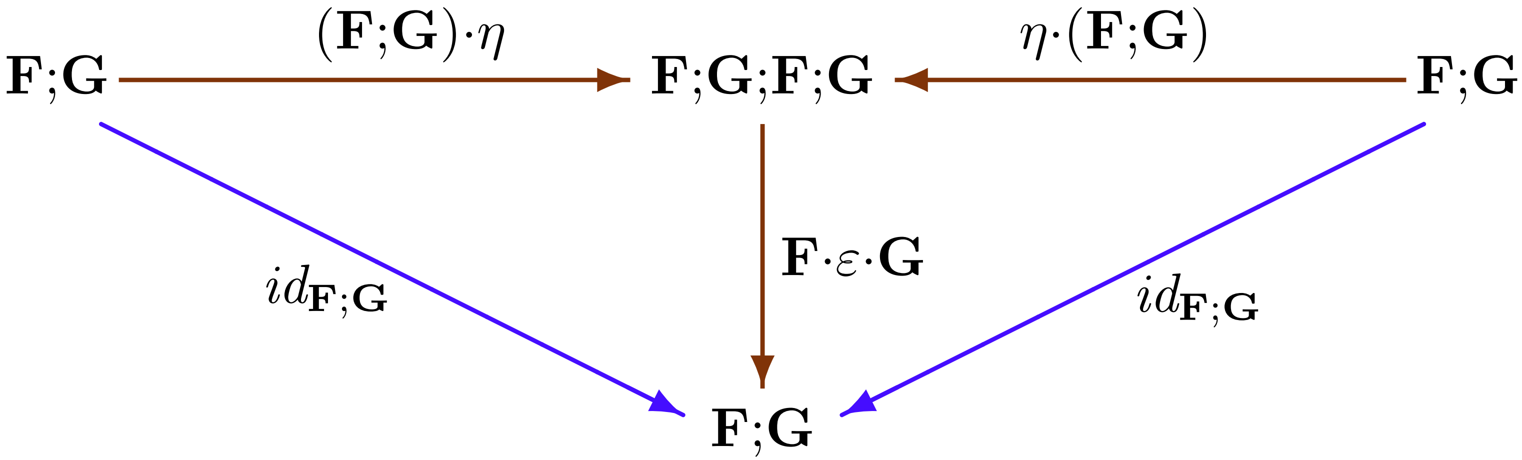

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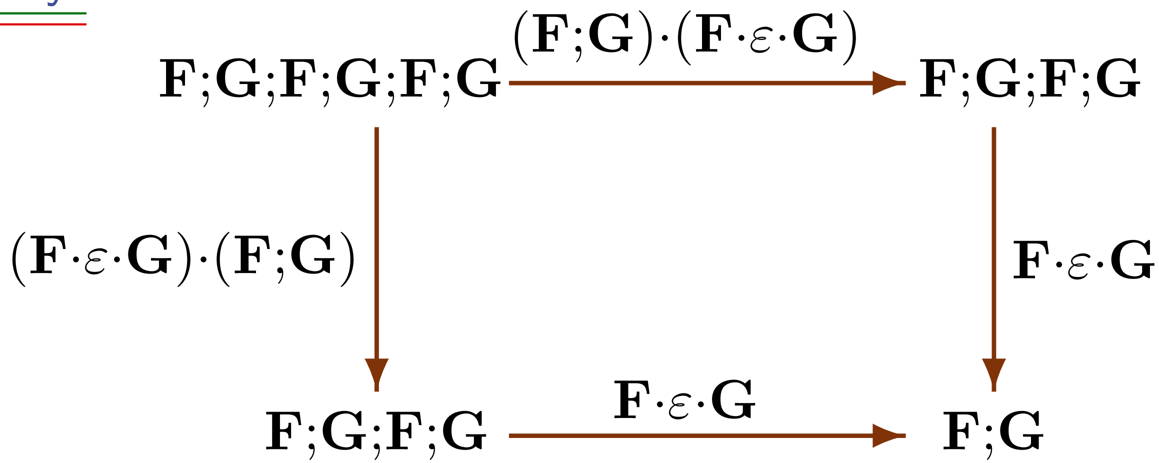

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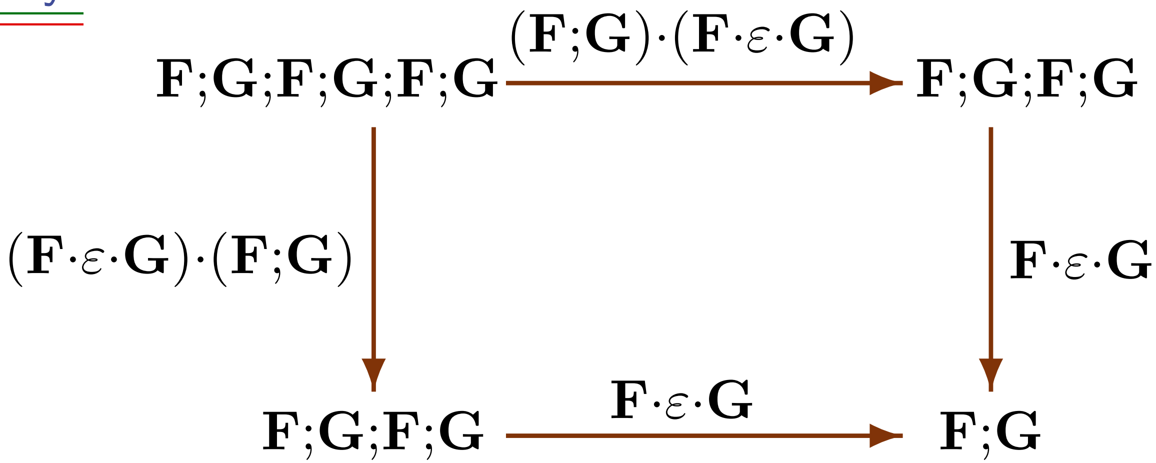

Follows by the commutativity of the diagrams below:

$$\begin{array}{l}\mathbf{T} = \mathbf{F};\mathbf{G} \colon \mathbf{K} \to \mathbf{K} \\ \eta^{\mathbf{T}} = \eta \colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{T} \\ \mu^{\mathbf{T}} = \mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G} \colon \mathbf{T};\mathbf{T} \to \mathbf{T}\end{array}$$

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$$\begin{array}{ccc} \mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} & \xrightarrow{(\mathbf{F};\mathbf{G})\cdot(\mathbf{F}\cdot\varepsilon\cdot\mathbf{G})} & \mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} \\ {\scriptstyle (\mathbf{F}\cdot\varepsilon\cdot\mathbf{G})\cdot(\mathbf{F};\mathbf{G})}\downarrow & & \downarrow{\scriptstyle \mathbf{F}\cdot\varepsilon\cdot\mathbf{G}} \\ \mathbf{F};\mathbf{G};\mathbf{F};\mathbf{G} & \xrightarrow{\mathbf{F}\cdot\varepsilon\cdot\mathbf{G}} & \mathbf{F};\mathbf{G} \end{array}$$

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$$\begin{array}{ccc} \mathbf{F}(\mathbf{G}(\mathbf{F}(\mathbf{G}(X')))) & \xrightarrow{\varepsilon_{\mathbf{F}(\mathbf{G}(X'))}} & \mathbf{F}(\mathbf{G}(X')) \\ {\scriptstyle \mathbf{F}(\mathbf{G}(\varepsilon_{X'}))}\downarrow & & \downarrow{\scriptstyle \varepsilon_{X'}} \\ \mathbf{F}(\mathbf{G}(X')) & \xrightarrow{\varepsilon_{X'}} & X' \end{array}$$

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$$\begin{array}{ccc}
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{\scriptstyle (\varepsilon\cdot\mathbf{G})\cdot\mathbf{F}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\
\mathbf{G};\mathbf{F} & \xrightarrow{\varepsilon} & \mathbf{Id}_{\mathbf{K}}
\end{array}
\qquad
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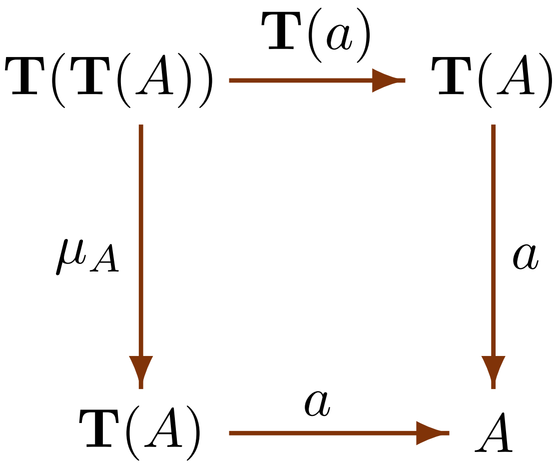

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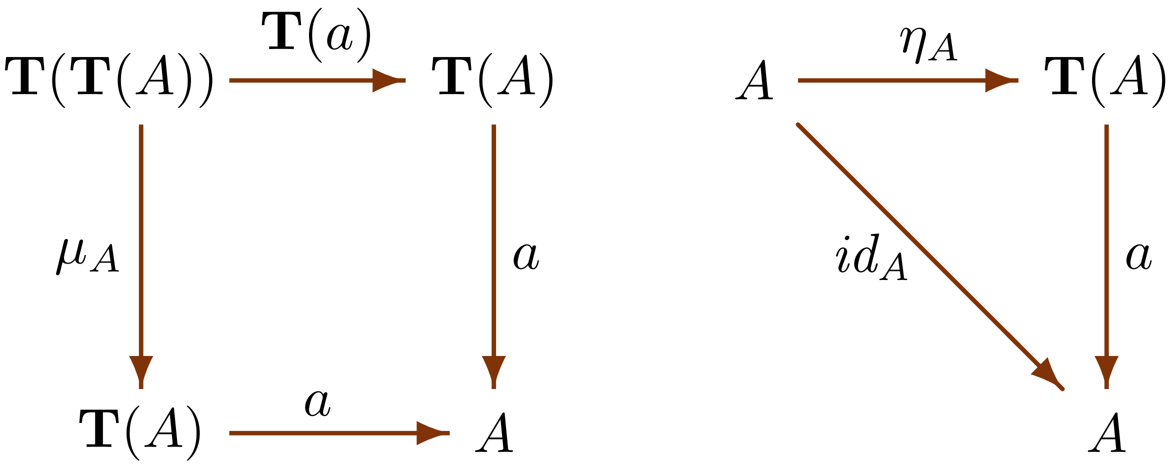

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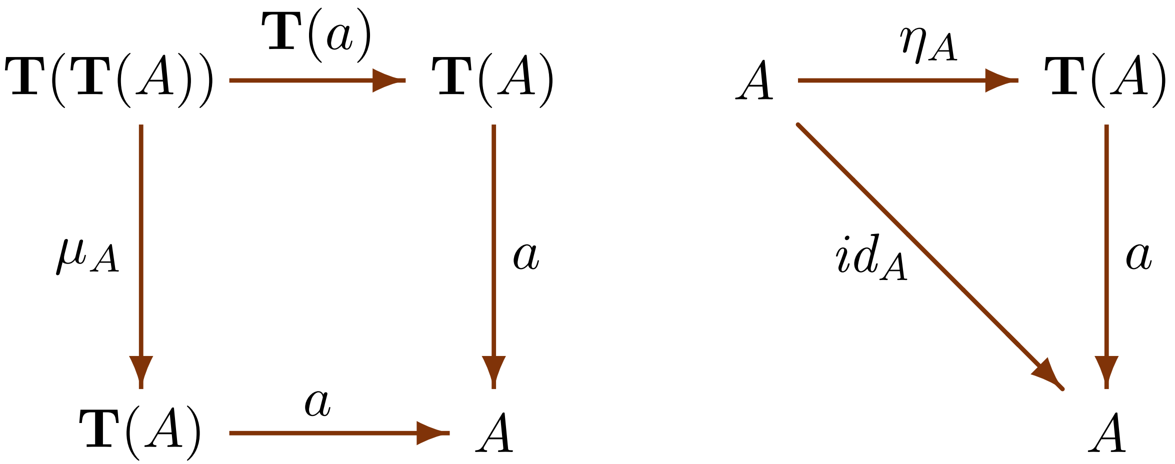
- $\mathbf{T}$-*homomorphism* from $\langle A, a\colon \mathbf{T}(A) \to A \rangle$ to $\langle B, b\colon \mathbf{T}(B) \to B \rangle$:



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Check this out for the term monad

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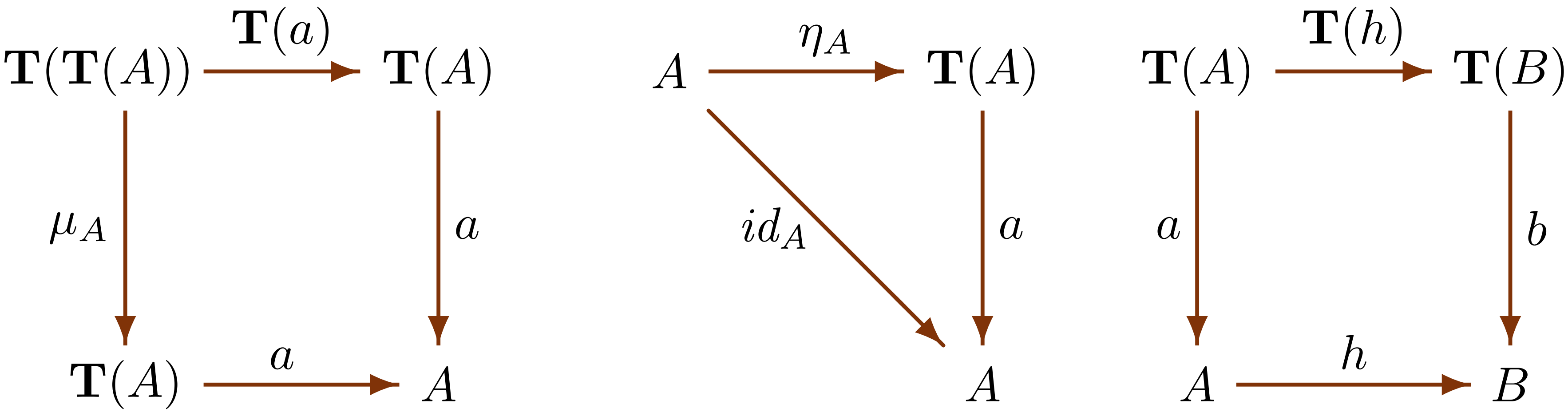
The category $\mathbf{Alg(T)}$ of $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms:

- $\mathbf{T}$-*algebras*:

  $\langle A \in |\mathbf{K}|, a \colon \mathbf{T}(A) \to A \rangle$ such that $\mathbf{T}(a);a = \mu_A;a$ and $\eta_A;a = id_A$

- $\mathbf{T}$-*homomorphism* from $\langle A, a \colon \mathbf{T}(A) \to A \rangle$ to $\langle B, b \colon \mathbf{T}(B) \to B \rangle$:

  $h \colon A \to B$ such that $\mathbf{T}(h);b = a;h$



Monadic adjunction

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Let $\mathbf{G^T}\colon \mathbf{Alg(T)} \to \mathbf{K}$ be the obvious projection: $\mathbf{G^T}(\langle A, a\rangle) = A$, ...

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$$X \xrightarrow{\eta_X} \mathbf{T}(X) \qquad\qquad \mathbf{T}(\mathbf{T}(X)) \xrightarrow{\mu_X} \mathbf{T}(X)$$

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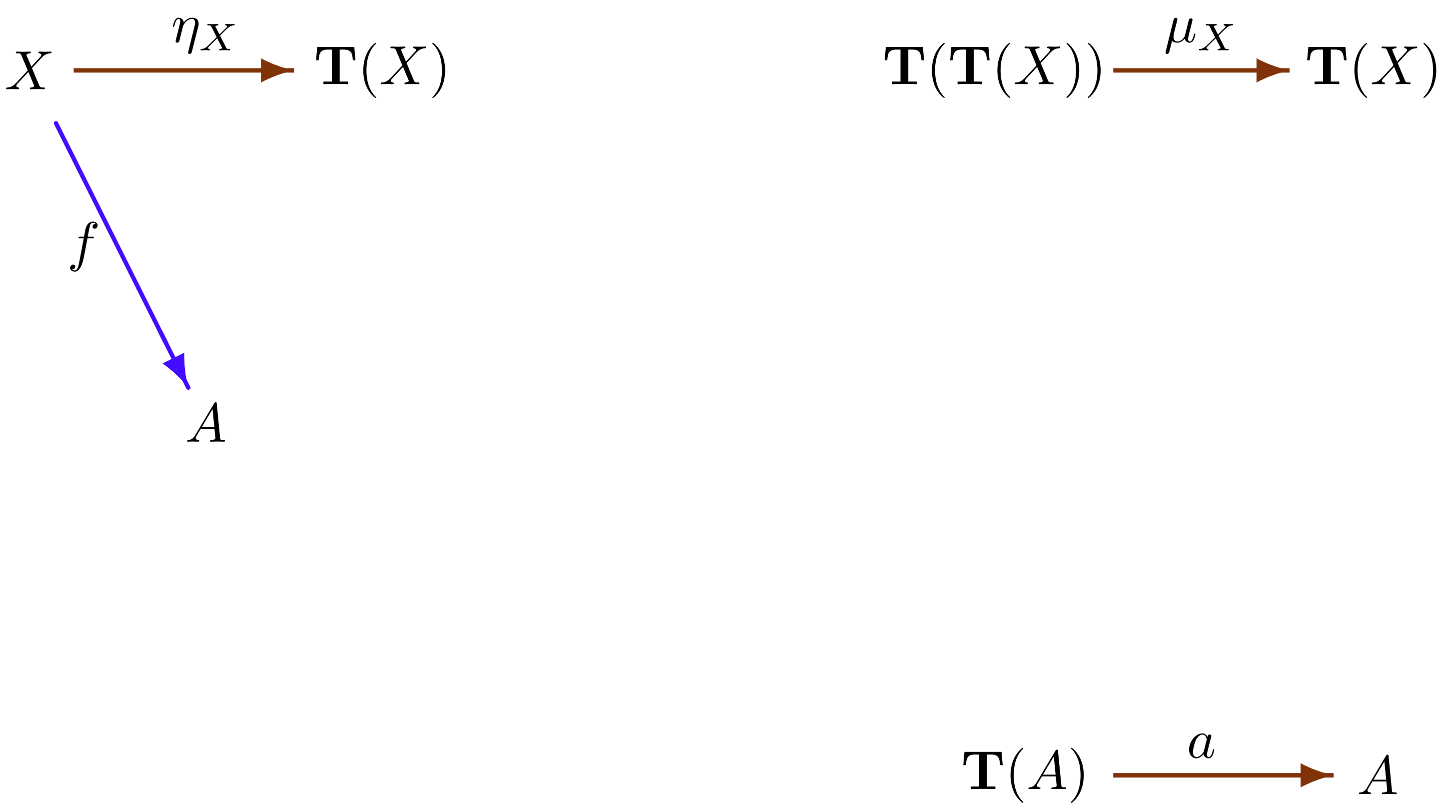

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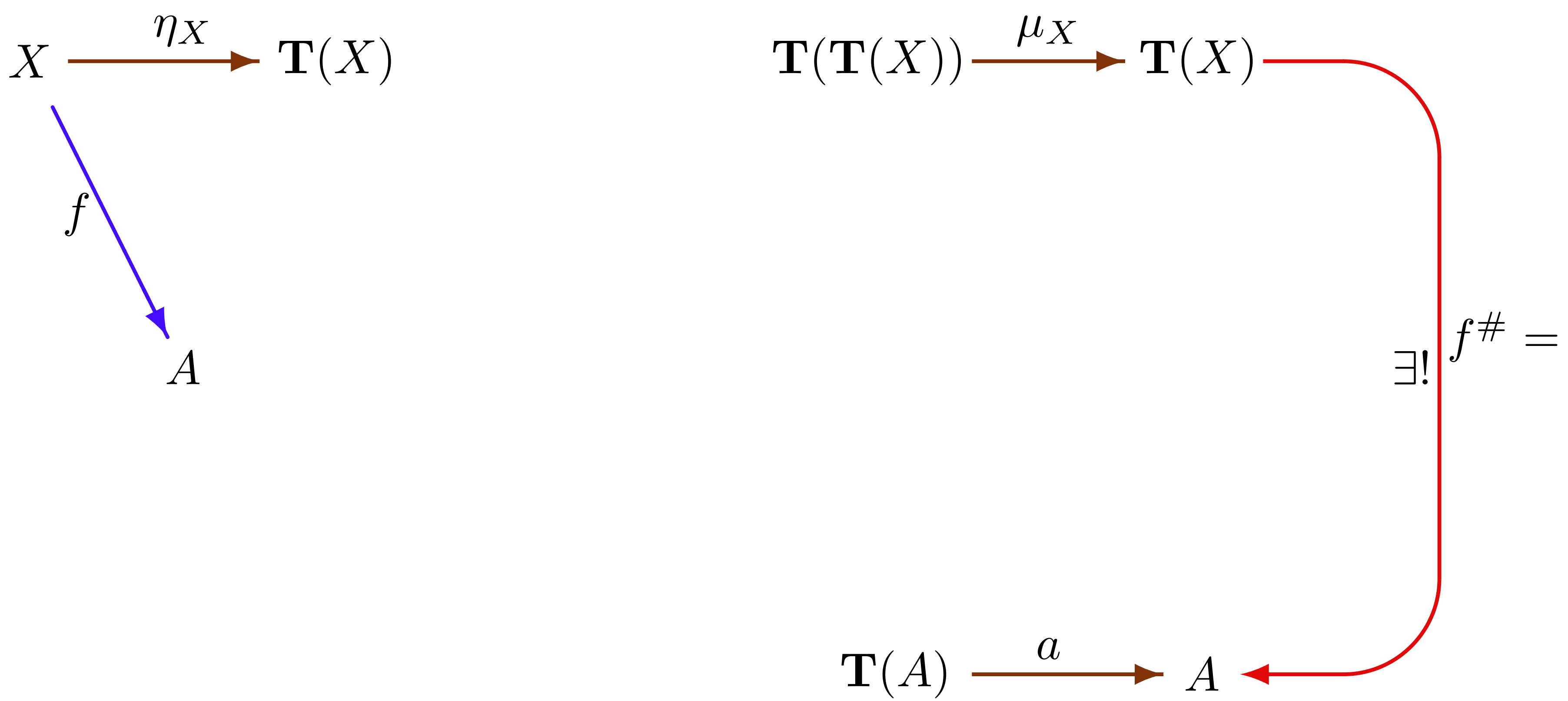

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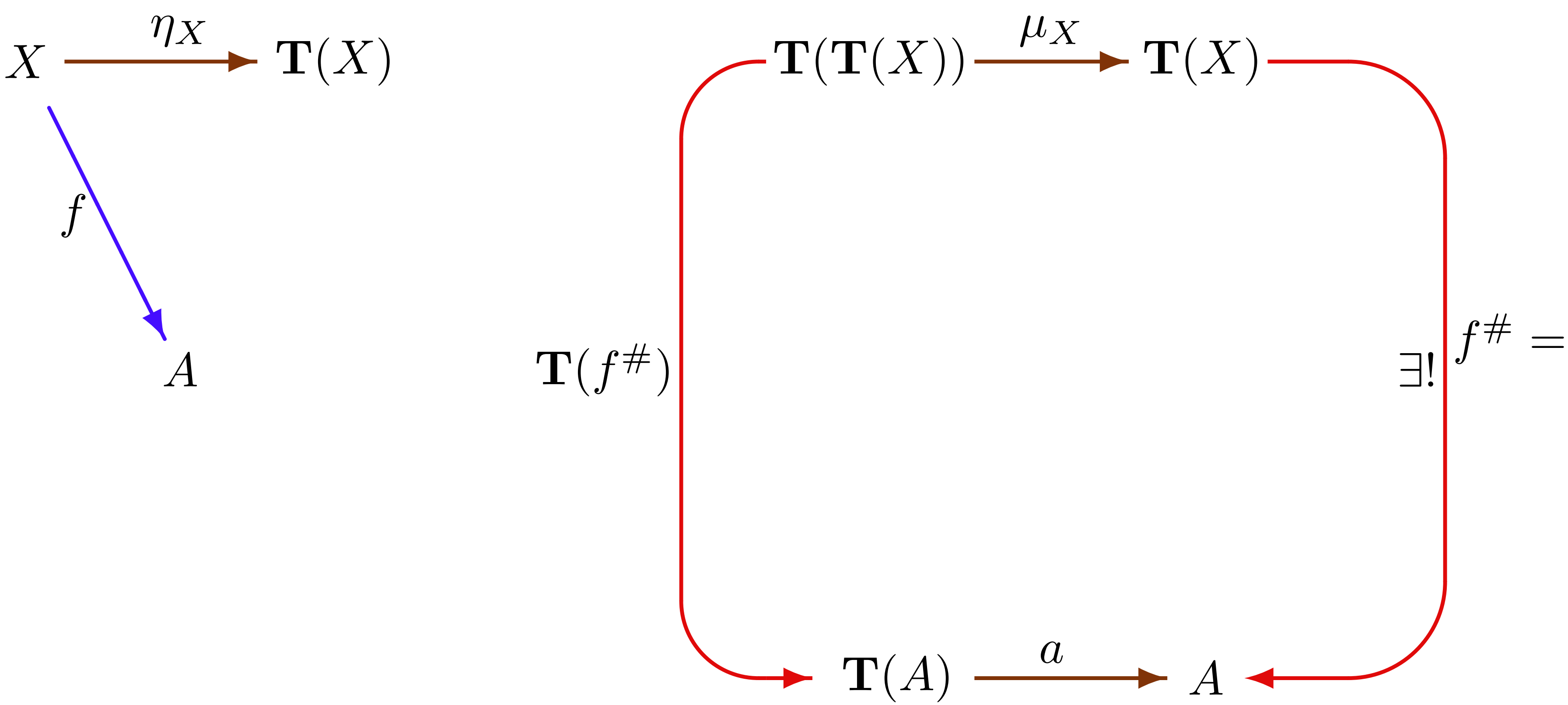

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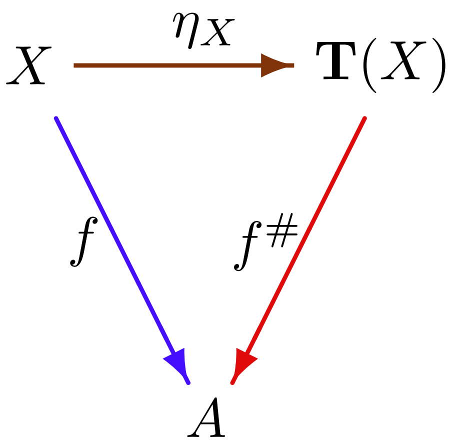

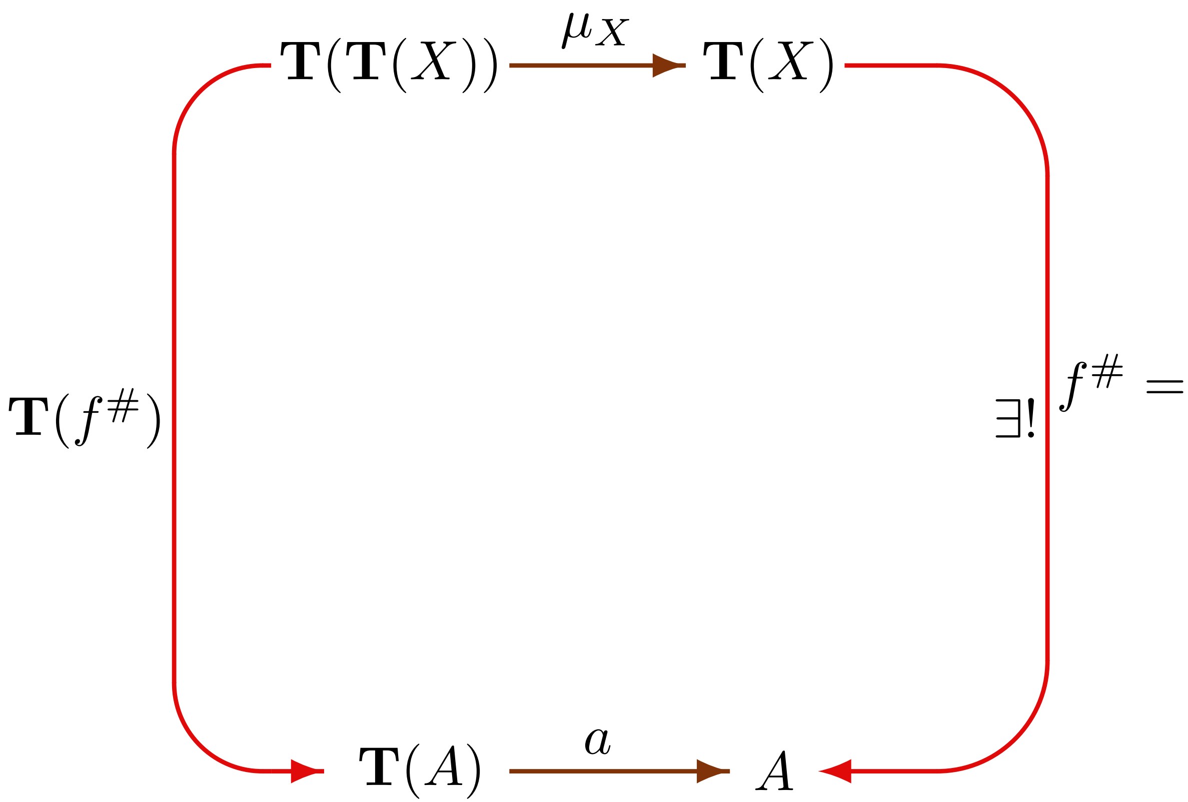

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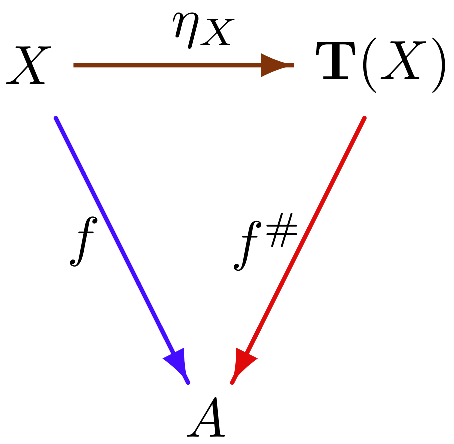

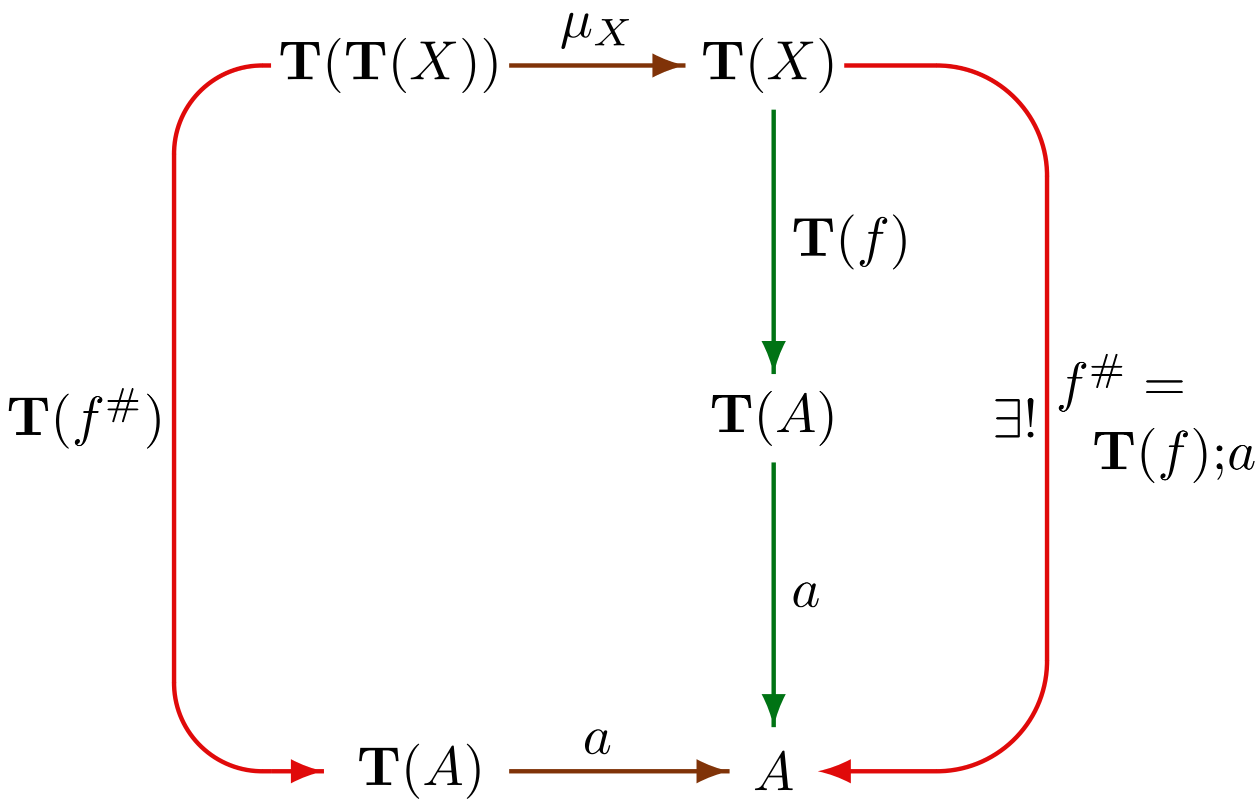

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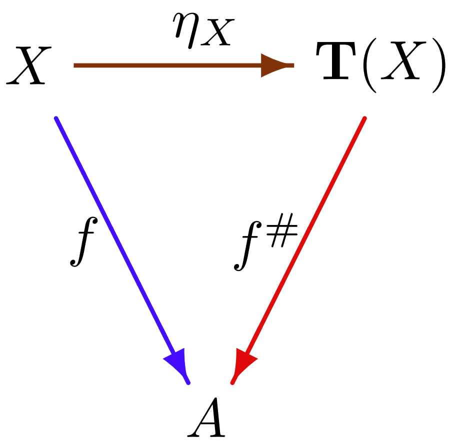

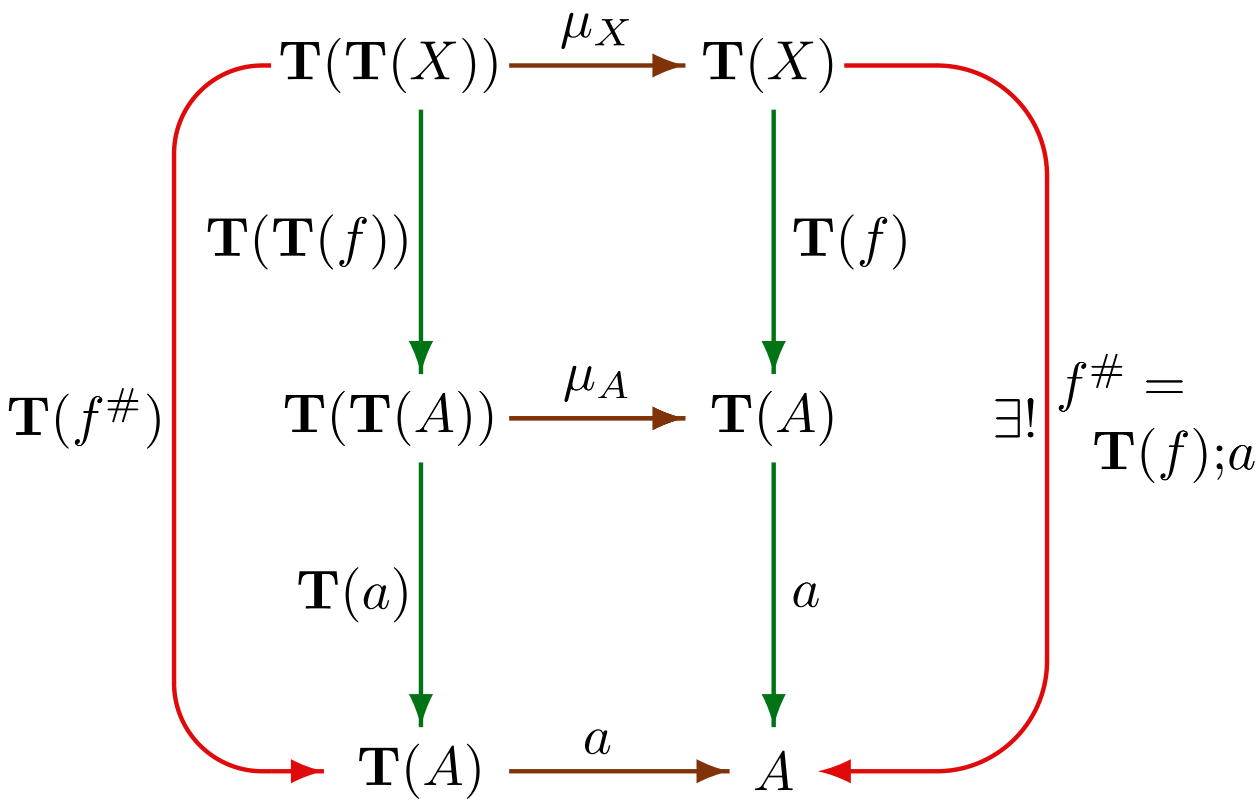

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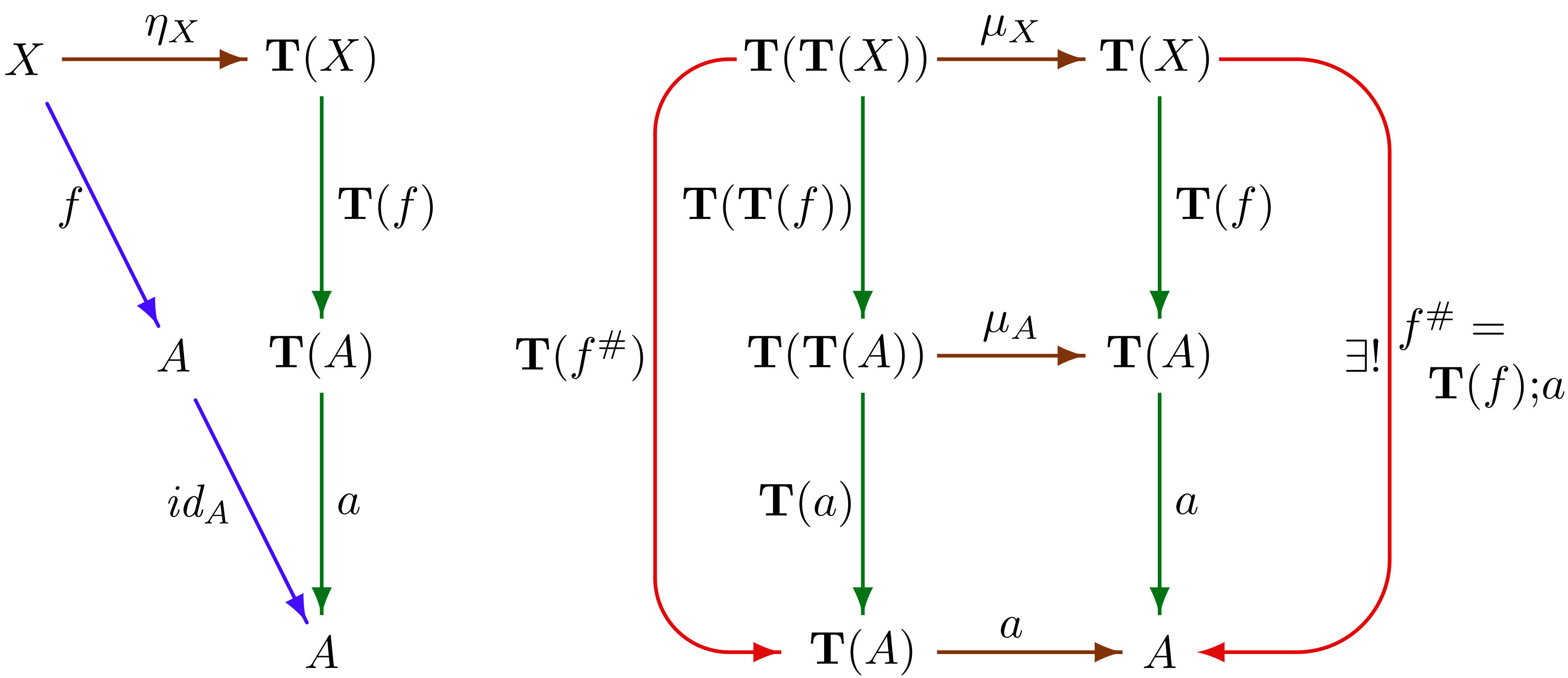

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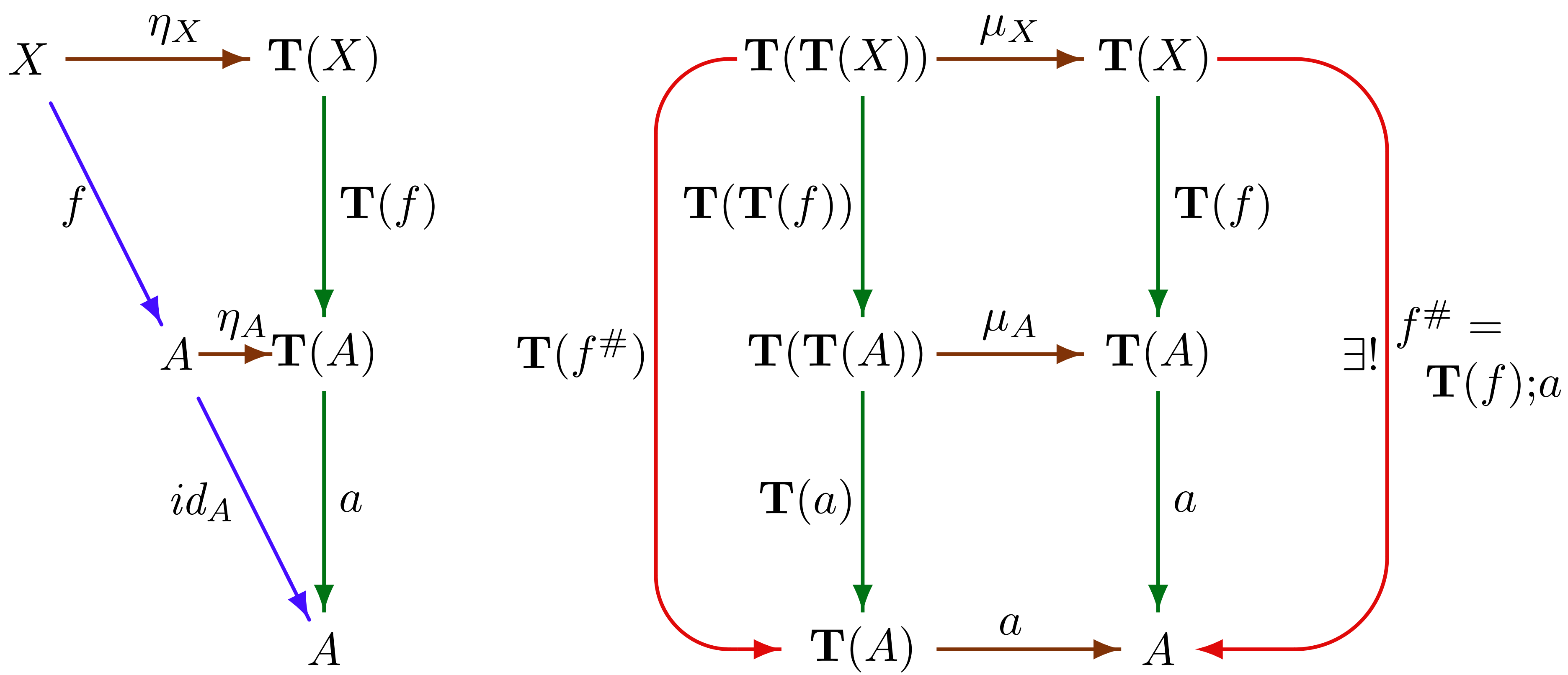

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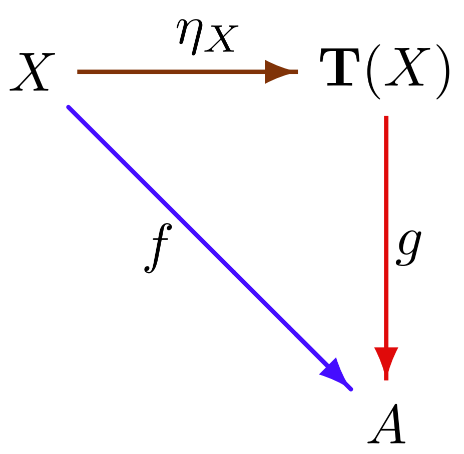

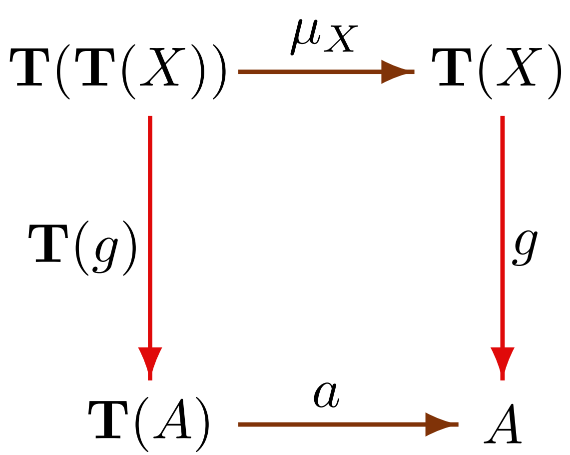

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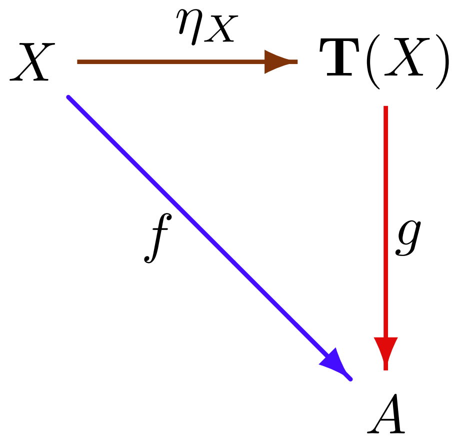

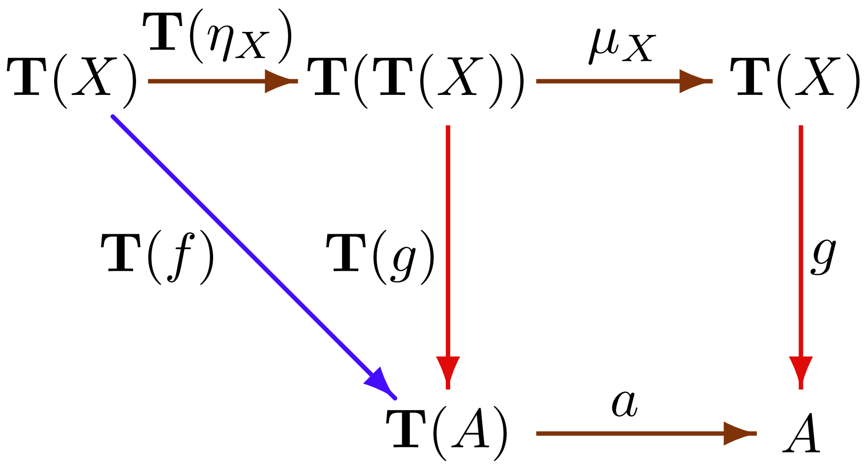

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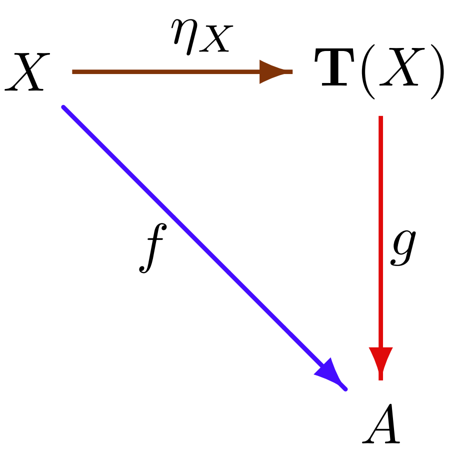

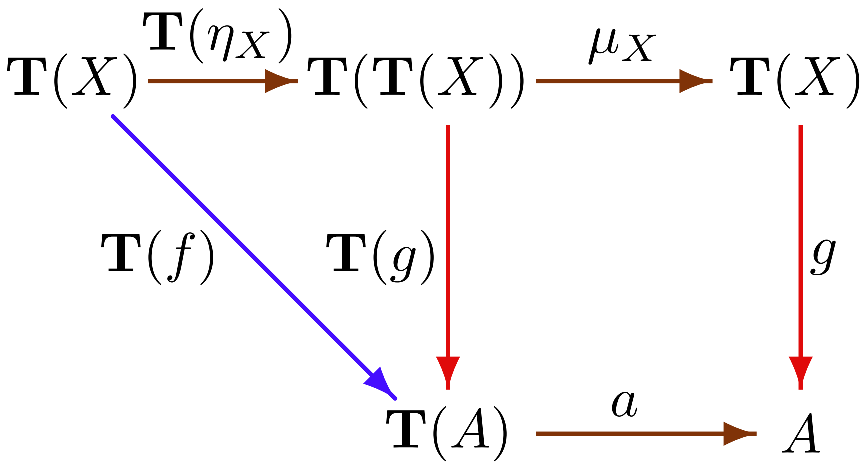

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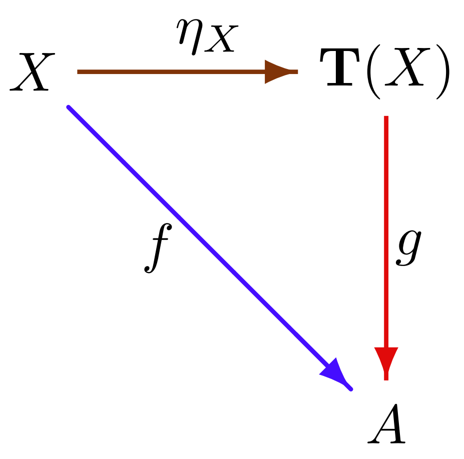

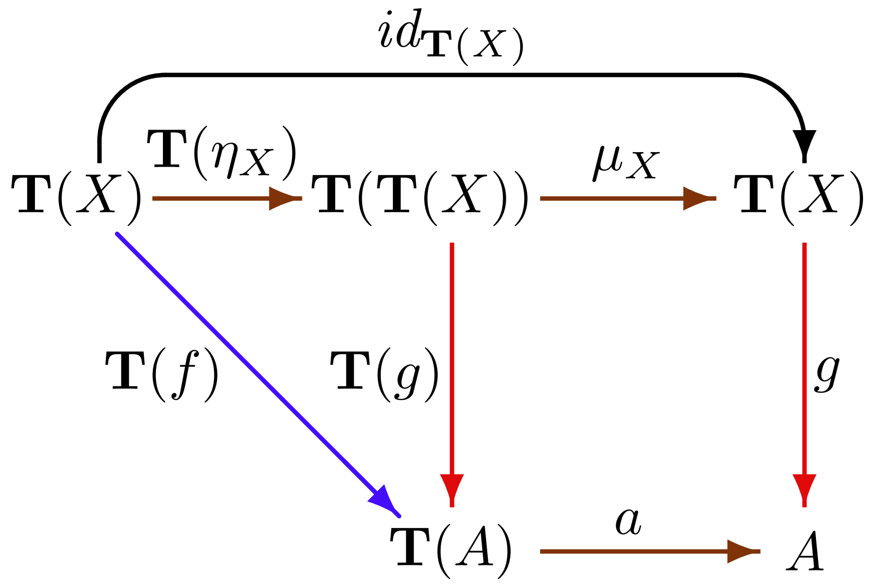

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BTW: $$\begin{aligned}\mathbf{F}^{\mathbf{T}}(f \colon X \to Y) &= (f;\eta_Y)^{\#} \\ &= \mathbf{T}(f;\eta_Y);\mu_Y \\ &= \mathbf{T}(f);\mathbf{T}(\eta_Y);\mu_Y \\ &= \mathbf{T}(f) \colon \langle \mathbf{T}(X), \mu_X\rangle \to \langle \mathbf{T}(Y), \mu_Y\rangle\end{aligned}$$

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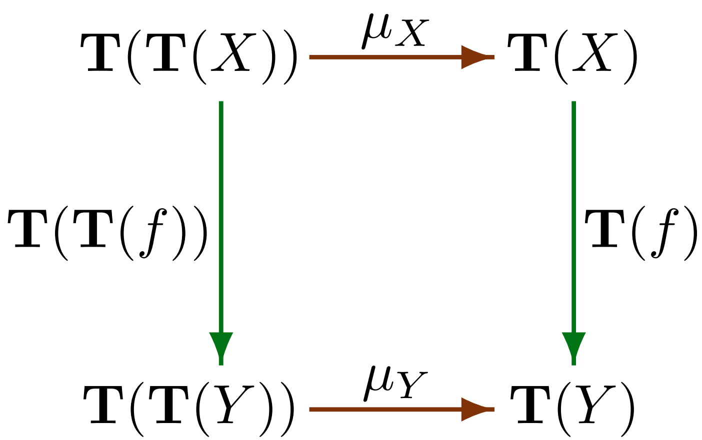

All monads arise from adjunctions

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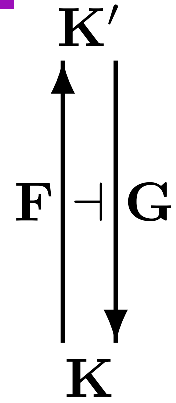

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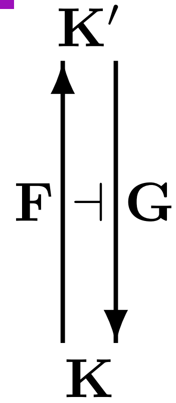

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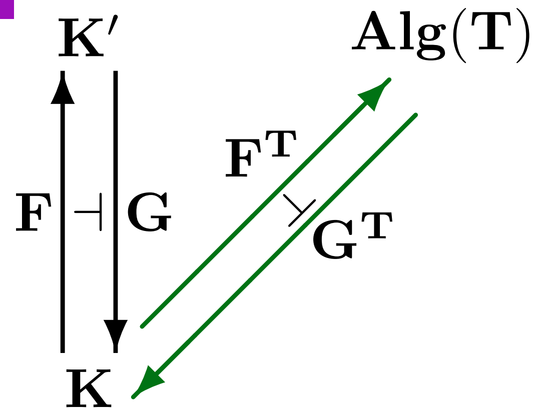
Theorem: *Given an adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$, *let* $\langle \mathbf{T} = \mathbf{F};\mathbf{G}, \eta, \mu^{\mathbf{T}} = \mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G} \rangle$ *be the monad it yields.*

All monads arise from adjunctions



Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$ we have an adjunction

$$\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle \colon \mathbf{K} \to \mathbf{Alg}(\mathbf{T})$$

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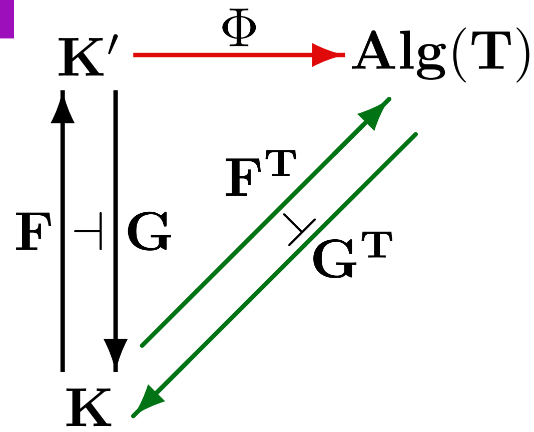

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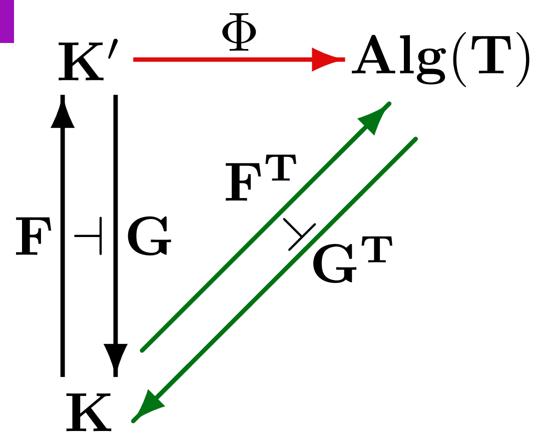
Then there is a unique comparison functor $\Phi \colon \mathbf{K}' \to \mathbf{Alg}(\mathbf{T})$ *such that* $\Phi;\mathbf{G}^{\mathbf{T}} = \mathbf{G}$ *and* $\mathbf{F};\Phi = \mathbf{F}^{\mathbf{T}}$.

All monads arise from adjunctions



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Theorem: *Given an adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$, *let* $\langle \mathbf{T} = \mathbf{F};\mathbf{G}, \eta, \mu^{\mathbf{T}} = \mathbf{F} \cdot \varepsilon \cdot \mathbf{G} \rangle$ *be the monad it yields. Let then* $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle : \mathbf{K} \to \mathbf{Alg}(\mathbf{T})$ *be the adjunction for* $\mathbf{T}$. *Then there is a unique comparison functor* $\Phi : \mathbf{K}' \to \mathbf{Alg}(\mathbf{T})$ *such that* $\Phi;\mathbf{G}^{\mathbf{T}} = \mathbf{G}$ *and* $\mathbf{F};\Phi = \mathbf{F}^{\mathbf{T}}$.

- $\Phi(A') = \langle \mathbf{G}(A'), \mathbf{G}(\varepsilon_{A'}) : \mathbf{G}(\mathbf{F}(\mathbf{G}(A'))) \to \mathbf{G}(A') \rangle$
- $\Phi(f : A' \to B') = \mathbf{G}(f) : \langle \mathbf{G}(A'), \mathbf{G}(\varepsilon_{A'}) \rangle \to \langle \mathbf{G}(B'), \mathbf{G}(\varepsilon_{B'}) \rangle$

Free algebras

Free algebras

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Free algebras

Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$:

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Free algebras

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Again: there is an adjunction $\langle \mathbf{F}_\mathbf{T}, \mathbf{G}_\mathbf{T}, \eta, \varepsilon_\mathbf{T} \rangle\colon \mathbf{K} \to \mathbf{Kl}(\mathbf{T})$

- $\mathbf{F}_\mathbf{T}(X) = X$ for $X \in |\mathbf{K}|$; $\mathbf{F}_\mathbf{T}(f) = f;\eta_Y$ for $f\colon X \to Y$ in $\mathbf{K}$.
- $\mathbf{G}_\mathbf{T}(X) = \mathbf{T}(X)$ for $X \in |\mathbf{Kl}(\mathbf{K})|$, i.e. $X \in |\mathbf{K}|$;
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- $\eta_X = \eta_X : X \to \mathbf{T}(X)$

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Free algebras

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- $\eta_X = \eta_X : X \to \mathbf{T}(X)$
- Then for $f: X \to \mathbf{G_T}(Y) = \mathbf{T}(Y)$, $f^\# = f: \mathbf{F_T}(X) = X \to Y$ in $\mathbf{Kl}(\mathbf{T})$

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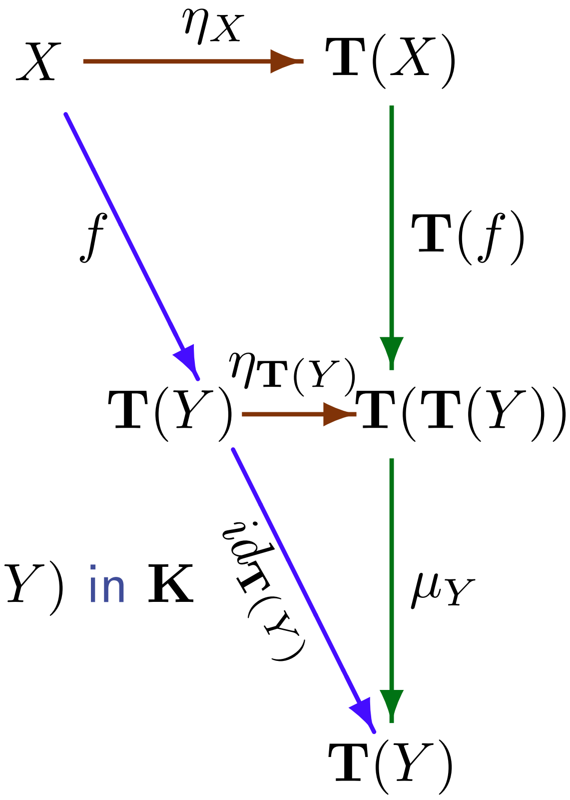
Free algebras



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  $\mathbf{G_T}(f) = \mathbf{T}(f);\mu_Y$ for $f \colon X \to Y$ in $\mathbf{Kl}(\mathbf{T})$, i.e. $f \colon X \to \mathbf{T}(Y)$ in $\mathbf{K}$
- $\eta_X = \eta_X \colon X \to \mathbf{T}(X)$
- Then for $f \colon X \to \mathbf{G_T}(Y) = \mathbf{T}(Y)$, $f^{\#} = f \colon \mathbf{F_T}(X) = X \to Y$ in $\mathbf{Kl}(\mathbf{T})$

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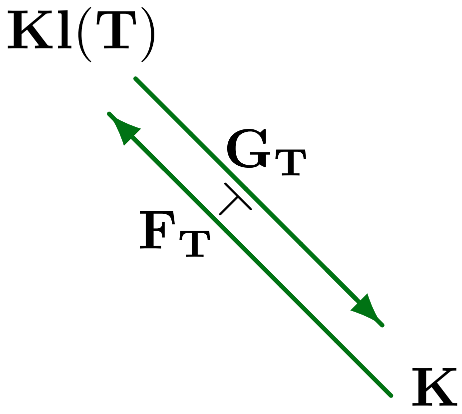
Free algebras



Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$:

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- objects: $|\mathbf{Kl}(\mathbf{T})| = |\mathbf{K}|$
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Again: there is an adjunction $\langle \mathbf{F_T}, \mathbf{G_T}, \eta, \varepsilon_\mathbf{T} \rangle\colon \mathbf{K} \to \mathbf{Kl}(\mathbf{T})$ which gives rise to the monad $\langle \mathbf{T}, \eta, \mu \rangle$,

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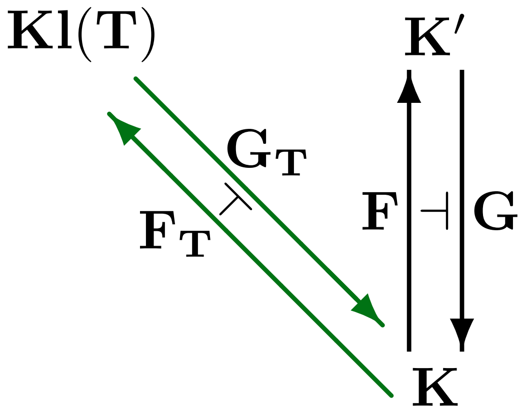
Free algebras



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Again: there is an adjunction $\langle \mathbf{F_T}, \mathbf{G_T}, \eta, \varepsilon_\mathbf{T} \rangle : \mathbf{K} \to \mathbf{Kl}(\mathbf{T})$ which gives rise to the monad $\langle \mathbf{T}, \eta, \mu \rangle$, and for any adjunction $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$ which also gives rise to this monad,

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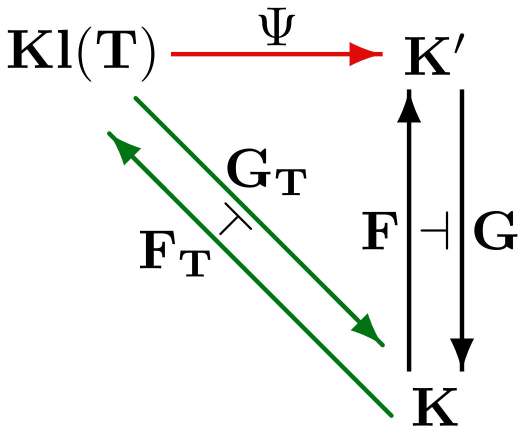
Free algebras



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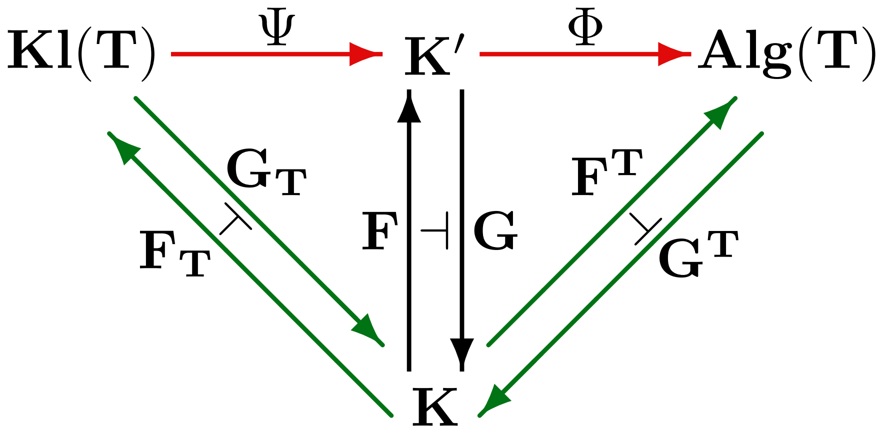
Free algebras



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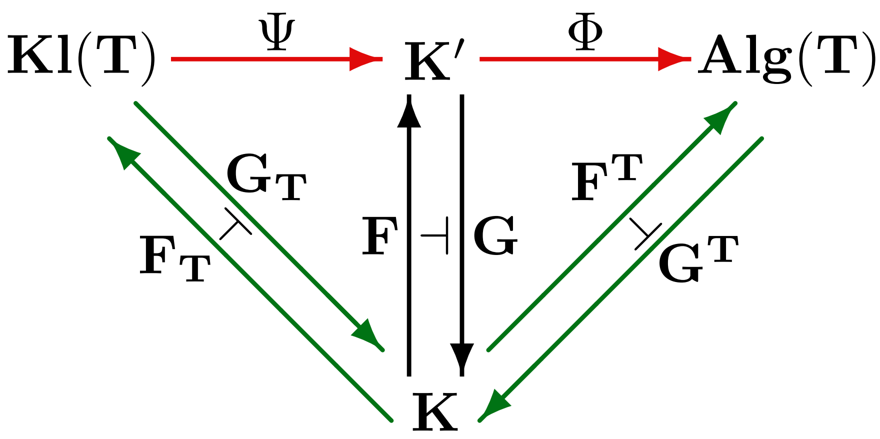

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- composition: given $f : X \to Y$, $g : Y \to Z$ in $\mathbf{Kl(T)}$,
  $f;g : X \to Z$ in $\mathbf{Kl(K)}$ is $f;\mathbf{T}(g);\mu_Z : X \to \mathbf{T}(Z)$ in $\mathbf{K}$.

$$X \xrightarrow{f} \mathbf{T}(Y) \xrightarrow{\mathbf{T}(g)} \mathbf{T}(\mathbf{T}(Z)) \xrightarrow{\mu_Z} \mathbf{T}(Z)$$

Again: there is an adjunction $\langle \mathbf{F_T}, \mathbf{G_T}, \eta, \varepsilon_\mathbf{T} \rangle : \mathbf{K} \to \mathbf{Kl(T)}$ which gives rise to the monad $\langle \mathbf{T}, \eta, \mu \rangle$, and for any adjunction $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$ which also gives rise to this monad, we have a comparison functor $\Psi : \mathbf{Kl(T)} \to \mathbf{K}'$ such that $\Psi;\mathbf{G} = \mathbf{G_T}$ and $\mathbf{F_T};\Psi = \mathbf{F}$.

View $\mathbf{Kl(T)}$ as the image of $\mathbf{F^T}$ in $\mathbf{Alg(T)}$

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Triples and monads are the same concepts

Triples as monads, monads as triples

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