Category Theory in Foundations of Computer Science
Exam assignment 2024/25

Concepts, terminology and notation:

We rely on the standard definitions of *algebraic signature* Σ, Σ-*algebra* and Σ-*homomorphism*, the category $\mathbf{Alg}(\Sigma)$ of Σ-algebras and their homomorphisms, and on the related notation, as introduced during the course.

A *hierarchy* is a poset $\mathcal{D} = \langle D, \leq \rangle$ (i.e., $\leq \subseteq D \times D$ is a reflexive, antisymmetric and transitive binary relation on D). A hierarchy $\mathcal{D} = \langle D, \leq \rangle$ is *directed* if all pairs of elements in D have an upper bound in $\mathcal{D}$ (i.e., for all $d_1, d_2 \in D$ there is $d \in D$ such that $d_1 \leq d$ and $d_2 \leq d$).

Consider any many-sorted signature $\Sigma = \langle S, \Omega \rangle$ and hierarchy $\mathcal{D} = \langle D, \leq \rangle$. A *hierarchical* $(\Sigma, \mathcal{D})$-*algebra* is a family $\mathcal{A} = \langle \langle A_d \rangle_{d \in D}, \langle f_{d \leq d'} \rangle_{d \leq d'} \rangle$ of Σ-algebras $A_d \in |\mathbf{Alg}(\Sigma)|$, $d \in D$, and for $d \leq d'$ in $\mathcal{D}$, Σ-homomorphisms $f_{d \leq d'} \colon A_d \to A_{d'}$ such that for all $d \in D$, $f_{d \leq d} = id_{A_d}$ and for all $d \leq d' \leq d''$ in $\mathcal{D}$, $f_{d \leq d''} = f_{d \leq d'} ; f_{d' \leq d''}$. Given two such hierarchical $(\Sigma, \mathcal{D})$-algebras $\mathcal{A} = \langle \langle A_d \rangle_{d \in D}, \langle f_{d \leq d'} \rangle_{d \leq d'} \rangle$ and $\mathcal{B} = \langle \langle B_d \rangle_{d \in D}, \langle g_{d \leq d'} \rangle_{d \leq d'} \rangle$, a *hierarchical* $(\Sigma, \mathcal{D})$-*homomorphisms* between them $h \colon \mathcal{A} \to \mathcal{B}$ is a family $h = \langle h_d \colon A_d \to B_d \rangle_{d \in D}$ of Σ-homomorphisms $h_d \colon A_d \to B_d$, $d \in D$, such that for all $d \leq d'$ in $\mathcal{D}$, $h_d ; g_{d \leq d'} = f_{d \leq d'} ; h_{d'}$. (i.e., for all $s \in S$ and $a \in |A_d|_s$, $h_{d'}(f_{d \leq d'}(a)) = g_{d \leq d'}(h_d(a))$). With the usual, component-wise composition of hierarchical homomorphisms, this defines the *category* $\mathbf{HAlg}(\Sigma, \mathcal{D})$ *of hierarchical* $(\Sigma, \mathcal{D})$-*algebras.*

The category $\mathbf{HSet}^S(\mathcal{D})$ of $\mathcal{D}$-*hierarchical* S-*sorted sets* is $\mathbf{HAlg}(\langle S, \emptyset \rangle, \mathcal{D})$. In other words: a $\mathcal{D}$-*hierarchical* S-*sorted set* $\mathcal{X}$ is a family $\mathcal{X} = \langle \langle X_d \rangle_{d \in D}, \langle f_{d \leq d'} \rangle_{d \leq d'} \rangle$ of S-sorted sets X_d, $d \in D$, and S-sorted functions $f_{d \leq d'} \colon X_d \to X_{d'}$, $d \leq d'$ in $\mathcal{D}$, such that for $d \in D$, $f_{d \leq d} = id_{X_d}$ and for $d \leq d' \leq d''$ in $\mathcal{D}$, $f_{d \leq d''} = f_{d \leq d'} ; f_{d' \leq d''}$. Then, given two such $\mathcal{D}$-hierarchical S-sorted sets $\mathcal{X} = \langle \langle X_d \rangle_{d \in D}, \langle f_{d \leq d'} \rangle_{d \leq d'} \rangle$ and $\mathcal{Y} = \langle \langle Y_d \rangle_{d \in D}, \langle g_{d \leq d'} \rangle_{d \leq d'} \rangle$, a $\mathcal{D}$-*hierarchical* S-*sorted function* $h \colon \mathcal{X} \to \mathcal{Y}$ is a family $h = \langle h_d \colon X_d \to Y_d \rangle_{d \in D}$ of S-sorted functions such that for all $d \leq d'$ in $\mathcal{D}$, $h_d ; g_{d \leq d'} = f_{d \leq d'} ; h_{d'}$. With the obvious composition, this more explicitly defines the category $\mathbf{HSet}^S(\mathcal{D})$.

Recall also that $\mathbf{Set}^S$ denotes the usual category of S-sorted sets.

For any hierarchy $\mathcal{D}$ and signature $\Sigma = \langle S, \Omega \rangle$, we have the following functors:

- $\mathcal{G}_{\Sigma, \mathcal{D}} \colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{HSet}^S(\mathcal{D})$ maps any hierarchical $(\Sigma, \mathcal{D})$-algebra to the family of the carriers of the algebras and functions between them it consists of:

$$\mathcal{G}_{\Sigma, \mathcal{D}}(\langle \langle A_d \rangle_{d \in D}, \langle f_{d \leq d'} \colon A_d \to A_{d'} \rangle_{d \leq d'} \rangle) = \langle \langle |A_d| \rangle_{d \in D}, \langle f_{d \leq d'} \colon |A_d| \to |A_{d'}| \rangle_{d \leq d'} \rangle$$

  and any hierarchical $(\Sigma, \mathcal{D})$-homomorphism to the family of (S-sorted) functions it in fact is.
- $\mathcal{P}_{\Sigma, \mathcal{D}} \colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{Set}^S$ maps any hierarchical $(\Sigma, \mathcal{D})$-algebra to the disjoint union of the carriers of the algebras it consists of:

$$\mathcal{P}_{\Sigma, \mathcal{D}}(\langle \langle A_d \rangle_{d \in D}, \langle f_{d \leq d'} \colon A_d \to A_{d'} \rangle_{d \leq d'} \rangle) = \bigcup_{d \in D} |A_d| \times \{d\}$$

  and any hierarchical $(\Sigma, \mathcal{D})$-homomorphism to the disjoint union of S-sorted functions it consists of, namely, given $\mathcal{A} = \langle \langle A_d \rangle_{d \in D}, \langle f_{d \leq d'} \rangle_{d \leq d'} \rangle$, $\mathcal{B} = \langle \langle B_d \rangle_{d \in D}, \langle g_{d \leq d'} \rangle_{d \leq d'} \rangle$ and $h = \langle h_d \colon A_d \to B_d \rangle_{d \in D} \colon \mathcal{A} \to \mathcal{B}$, the S-sorted function $\mathcal{P}_{\Sigma, \mathcal{D}}(h) \colon \mathcal{P}_{\Sigma, \mathcal{D}}(\mathcal{A}) \to \mathcal{P}_{\Sigma, \mathcal{D}}(\mathcal{B})$ is defined as follows: for $s \in S$, $d \in D$, and $a \in |A_d|_s$, $(\mathcal{P}_{\Sigma, \mathcal{D}}(h))_s(\langle a, d \rangle) = \langle (h_d)_s(a), d \rangle$.

Then, for any hierarchy $\mathcal{D}$ and signature $\Sigma = \langle S, \Omega \rangle$, the category $\mathbf{IAlg}(\Sigma, \mathcal{D})$ is the full subcategory of $\mathbf{HAlg}(\Sigma, \mathcal{D})$ with objects that are *inclusive* $(\Sigma, \mathcal{D})$*-algebras*: a hierarchical $(\Sigma, \mathcal{D})$-algebra $\mathcal{A} = \langle\langle A_d\rangle_{d\in D}, \langle f_{d\leq d'}\rangle_{d\leq d'}\rangle$ is *inclusive* if all the homomorphisms $f_{d\leq d'}\colon A_d \to A_{d'}$, $d \leq d'$ in $\mathcal{D}$, are inclusions.

This may be simplified as follows. An *inclusive* $(\Sigma, \mathcal{D})$*-algebra* is a family of Σ-algebras $\mathcal{A} = \langle A_d\rangle_{d\in D}$ such that for all $d \leq d'$ in $\mathcal{D}$, A_d is a subalgebra of $A_{d'}$. Given two such inclusive $(\Sigma, \mathcal{D})$-algebras $\mathcal{A} = \langle A_d\rangle_{d\in D}$ and $\mathcal{B} = \langle B_d\rangle_{d\in D}$, an *inclusive* $(\Sigma, \mathcal{D})$*-homomorphism* between them $h\colon \mathcal{A} \to \mathcal{B}$ is a family $h = \langle h_d\colon A_d \to B_d\rangle_{d\in D}$ of Σ-homomorphisms $h_d\colon A_d \to B_d$, $d \in D$, such that for all $d \leq d'$ in $\mathcal{D}$, $h_{d'}$ extends h_d (i.e., for $s \in S$ and $a \in |A_d|_s \subseteq |A_{d'}|_s$, $(h_{d'})_s(a) = (h_d)_s(a)$). With the usual, component-wise composition of inclusive $(\Sigma, \mathcal{D})$-homomorphisms, this defines $\mathbf{IAlg}(\Sigma, \mathcal{D})$.

Analogously: a $\mathcal{D}$*-inclusive* S*-sorted set* is a family $\mathcal{X} = \langle X_d\rangle_{d\in D}$ of S-sorted sets X_d, $d \in D$, such that for all $d \leq d'$ in $\mathcal{D}$, $X_d \subseteq X_{d'}$, and a $\mathcal{D}$*-inclusive* S*-sorted function* $f\colon \mathcal{X} \to \mathcal{Y}$ between two such $\mathcal{D}$-inclusive S-sorted sets $\mathcal{X} = \langle X_d\rangle_{d\in D}$ and $\mathcal{Y} = \langle Y_d\rangle_{d\in D}$ is a family $f = \langle f_d\colon X_d \to Y_d\rangle_{d\in D}$ of S-sorted functions such that for all $d \leq d'$ in $\mathcal{D}$, $f_{d'}$ extends f_d — with the expected composition of $\mathcal{D}$-inclusive S-sorted functions, this defines the category $\mathbf{ISet}^S(\mathcal{D})$.

Furthermore, by a $(\Sigma, \mathcal{D})$*-classification statement* we mean a sentence of the form $\forall Y.t{:}d$, where Y is a finite S-sorted set of variables, $t \in |T_\Sigma(Y)|_s$ is a Σ-term of a sort $s \in S$, and $d \in D$. An inclusive $(\Sigma, \mathcal{D})$-algebra $\mathcal{A} = \langle A_d\rangle_{d\in D}$ *satisfies* $\forall Y.t{:}d$, written $\mathcal{A} \models \forall Y.t{:}d$, if for all $d' \in D$ and all valuations $v\colon Y \to |A_{d'}|$, the value of t in $A_{d'}$ under v is in A_d, i.e., $t_{A_{d'}}[v] \in |A_d|_s$. Given a set Φ of such $(\Sigma, \mathcal{D})$-classification statements, $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$ is the full subcategory of $\mathbf{IAlg}(\Sigma, \mathcal{D})$ with objects that are all inclusive $(\Sigma, \mathcal{D})$-algebras that satisfy all classification statement in Φ.

For any directed hierarchy $\mathcal{D}$, signature $\Sigma = \langle S, \Omega\rangle$ and set Φ of $(\Sigma, \mathcal{D})$-classification statements, we have the following functors:

- $\mathcal{J}_{\Sigma,\mathcal{D}}\colon \mathbf{IAlg}(\Sigma, \mathcal{D}) \to \mathbf{ISet}^S(\mathcal{D})$ maps any inclusive $(\Sigma, \mathcal{D})$-algebra to the family of the carriers of the algebras it consists of, $\mathcal{J}_{\Sigma,\mathcal{D}}(\langle A_d\rangle_{d\in D}) = \langle |A_d|\rangle_{d\in D}$, and any hierarchical $(\Sigma, \mathcal{D})$-homomorphism to the family of functions it in fact is.
- $\mathcal{J}_{\Sigma,\mathcal{D},\Phi}\colon \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{ISet}^S(\mathcal{D})$ restricts $\mathcal{J}_{\Sigma,\mathcal{D}}$ to $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$.
- $\mathcal{U}_{\Sigma,\mathcal{D}}\colon \mathbf{IAlg}(\Sigma, \mathcal{D}) \to \mathbf{Set}^S$ maps any inclusive $(\Sigma, \mathcal{D})$-algebra to the union of the carriers of the algebras it consists of, $\mathcal{U}_{\Sigma,\mathcal{D}}(\langle A_d\rangle_{d\in D}) = \bigcup_{d\in D} |A_d|$, and any hierarchical $(\Sigma, \mathcal{D})$-homomorphism to the union of S-sorted functions it consists of (this is well-defined for directed $\mathcal{D}$).
- $\mathcal{U}_{\Sigma,\mathcal{D},\Phi}\colon \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{Set}^S$ restricts $\mathcal{U}_{\Sigma,\mathcal{D}}$ to $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$

To do:

Prove a positive answer or give a counterexample to the following questions:

1. Is the category $\mathbf{HAlg}(\Sigma, \mathcal{D})$
   - (C) complete
   - (CC) cocomplete

   for all signatures Σ and
   - (H) all hierarchies $\mathcal{D}$?
   - (D) all directed hierarchies $\mathcal{D}$?
2. Do the functors
   - (a) $\mathcal{G}_{\Sigma,\mathcal{D}} \colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{HSet}^S(\mathcal{D})$
   - (b) $\mathcal{P}_{\Sigma,\mathcal{D}} \colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{Set}^S$

   have left adjoints for all signatures Σ and
   - (H) all hierarchies $\mathcal{D}$?
   - (D) all directed hierarchies $\mathcal{D}$?
3. Consider categories:
   - (a) $\mathbf{IAlg}(\Sigma, \mathcal{D})$
   - (b) $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$

   Are these categories
   - (C) complete
   - (CC) cocomplete

   for all signatures Σ, all directed hierarchies $\mathcal{D}$, and where relevant, all sets Φ of $(\Sigma, \mathcal{D})$-classification statements?
4. Do the functors:
   - (a) $\mathcal{J}_{\Sigma,\mathcal{D}} \colon \mathbf{IAlg}(\Sigma, \mathcal{D}) \to \mathbf{ISet}^S(\mathcal{D})$
   - (b) $\mathcal{J}_{\Sigma,\mathcal{D},\Phi} \colon \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{ISet}^S(\mathcal{D})$
   - (c) $\mathcal{U}_{\Sigma,\mathcal{D}} \colon \mathbf{IAlg}(\Sigma, \mathcal{D}) \to \mathbf{Set}^S$
   - (d) $\mathcal{U}_{\Sigma,\mathcal{D},\Phi} \colon \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{Set}^S$

   have left adjoints for all signatures Σ, all directed hierarchies $\mathcal{D}$, and where relevant, all sets Φ of $(\Sigma, \mathcal{D})$-classification statements?

Notes:

- All constructions and facts presented during the course may be used without proofs (but their use should be explicitly mentioned). This applies in particular to the existence and constructions of limits and colimits in $\mathbf{Alg}(\Sigma)$, and in functor categories $\mathbf{K}^{\mathbf{K}'}$ (given the existence and constructions of limits and colimits in $\mathbf{K}$).
- There are 16 questions above: **1.{C,CC}.{H,D}**, **2.{a,b}.{H,D}**), **3.{a,b}.{C,CC}**, and **4.{a,b,c,d}**. However, the answers to these questions are not independent. For instance, a proof of **1.C.H** implies the positive answer to **1.C.D** as well, a counterexample to **1.CC.D** is a counterexample to **1.CC.H**, a proof for **3.b.CC** proves **3.a.CC**, a counterexample for **4.d** is a counterexample for **4.c**, etc. No need to repeat detailed arguments in such cases, indicating the dependency is enough.
- Still, there are quite a few questions: deal with as many of them as you can...

Sketch of a solution:

The “hierarchical” case:

Fact 1 *Given two categories* $\mathbf{K}$ *and* $\mathbf{K}'$*, if* $\mathbf{K}$ *is complete and cocomplete then so is the functor category* $\mathbf{K}^{\mathbf{K}'}$*.* $\square$

Fact 2 *Let* $F\colon \mathbf{K}' \to \mathbf{K}''$ *be left adjoint to* $G\colon \mathbf{K}'' \to \mathbf{K}'$ *with unit* $\eta\colon \mathbf{Id}_{\mathbf{K}'} \to F;G$*. Then the functor* $\_;G\colon (\mathbf{K}'')^{\mathbf{K}} \to (\mathbf{K}')^{\mathbf{K}}$ *has a left adjoint: a free object over a functor* $\mathcal{A}\colon \mathbf{K} \to \mathbf{K}'$ *w.r.t.* $\_;G$ *is* $\mathcal{A};F\colon \mathbf{K} \to \mathbf{K}''$ *with unit* $\mathcal{A}\cdot\eta\colon \mathcal{A} \to \mathcal{A};F;G$*.*

Proof: Unlike Fact 1, this might not have been proved explicitly at the course. Consider any functor $\mathcal{B}\colon \mathbf{K} \to \mathbf{K}'$ and natural transformation $\delta\colon \mathcal{A} \to \mathcal{B};G$. For $k \in |\mathbf{K}|$, $F(\mathcal{A}(k))$ with unit $\eta_{\mathcal{A}(k)}\colon \mathcal{A}(k) \to G(F(\mathcal{A}(k)))$ is free over $\mathcal{A}(k)$ w.r.t. $G\colon \mathbf{K}'' \to \mathbf{K}'$. Hence, there is a unique $\delta^{\#}_k\colon F(\mathcal{A}(k)) \to \mathcal{B}(k)$ such that $\eta_{\mathcal{A}(k)};G(\delta^{\#}_k) = \delta_k$,

Now, $\delta^{\#}\colon (\mathcal{A};F) \to \mathcal{B}$ is a natural transformation: to see that for any $f\colon k \to k'$ in $\mathbf{K}$, $\delta^{\#}_k;\mathcal{B}(f) = F(\mathcal{A}(f));\delta^{\#}_{k'}$, by the characterisation of equalities between morphisms going out of a free object, it is enough to calculate $\eta_{\mathcal{A}(k)};G(\delta^{\#}_{\mathcal{A}(k)};\mathcal{B}(f)) = \eta_{\mathcal{A}(k)};G(\delta^{\#}_{\mathcal{A}(k)});G(\mathcal{B}(f)) = \delta_k;G(\mathcal{B}(f)) = \mathcal{A}(f);\delta_{k'} = \mathcal{A}(f);\eta_{\mathcal{A}(k')};G(\delta^{\#}_{k'}) = \eta_{\mathcal{A}(k)};G(F(\mathcal{A}(f)));G(\delta^{\#}_{k'}) = \eta_{\mathcal{A}(k)};G(F(\mathcal{A}(f));\delta^{\#}_{k'})$.

Moreover, by the construction we have $(\mathcal{A}\cdot\eta);(\delta^{\#}\cdot G) = \delta$, and $\delta^{\#}$ is the only natural transformation between $\mathcal{A};F\colon \mathbf{K} \to \mathbf{K}''$ and $\mathcal{B}\colon \mathbf{K} \to \mathbf{K}'$ with this property — which completes the proof. $\square$

Consider any signature Σ and hierarchy $\mathcal{D} = \langle D, \leq\rangle$. Let $\mathbf{D}$ be the thin category determined by $\mathcal{D}$ (i.e., $|\mathbf{D}| = D$ and $\mathbf{D}(d, d')$ is a singleton set if $d \leq d'$ in $\mathcal{D}$ and is empty otherwise).

Then the category $\mathbf{HAlg}(\Sigma, \mathcal{D})$ is (equivalent to) the functor category $\mathbf{Alg}(\Sigma)^{\mathbf{D}}$. In particular, $\mathbf{HSet}^S(\mathcal{D})$ is $(\mathbf{Set}^S)^{\mathbf{D}}$. Moreover, $\mathcal{G}_{\Sigma,\mathcal{D}}\colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{HSet}^S(\mathcal{D})$ coincides with $\_;G\colon \mathbf{Alg}(\Sigma)^{\mathbf{D}} \to (\mathbf{Set}^S)^{\mathbf{D}}$, where $G\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$ is the usual forgetful functor that maps any Σ-algebra to its S-sorted carrier.

Since $\mathbf{Alg}(\Sigma)$ is complete and cocomplete, so is $\mathbf{Alg}(\Sigma)^{\mathbf{D}}$ by Fact 1. Moreover, since $G\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$ has a left adjoint, so does $\mathcal{G}_{\Sigma,\mathcal{D}}\colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{HSet}^S(\mathcal{D})$ by Fact 2.

YES: 1.{C,CC}.{H,D}, 2.a.{H,D}

The terminal object in $\mathbf{HAlg}(\Sigma, \mathcal{D})$ is a hierarchical $(\Sigma, \mathcal{D})$-algebra $\mathbf{1}_{\Sigma,\mathcal{D}}$ which consists of singleton algebras. For any hierarchy $\mathcal{D}$ with more than one element, by definition, $\mathcal{P}_{\Sigma,\mathcal{D}}(\mathbf{1}_{\Sigma,\mathcal{D}})$ is not a singleton set, hence $\mathcal{P}_{\Sigma,\mathcal{D}}\colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{Set}^S$ is not continuous and therefore does not have a left adjoint.

NO: 2.b.{H,D}

The “inclusive” case:

Consider any signature $\Sigma = \langle S, \Omega\rangle$ and directed hierarchy $\mathcal{D} = \langle D, \leq\rangle$. The arguments below use the functor $\mathsf{T}_{\Sigma,\mathcal{D}}\colon \mathbf{HAlg}(\Sigma, \mathcal{D}) \to \mathbf{Alg}(\Sigma)$ defined as follows:

- for any inclusive $(\Sigma, \mathcal{D})$-algebra $\mathcal{A} = \langle A_d\rangle_{d\in D}$, $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A})$ is the *directed union* of $\mathcal{A}$, i.e., a Σ-algebra A such that for $s \in S$, $|A|_s = \bigcup_{d\in D} |A_d|_s$ and for any $f\colon s_1 \times \ldots \times s_n \to s$ in Σ and $a_1 \in |A|_{s_1}, \ldots, a_n \in |A|_{s_n}$, we put $f_A(a_1, \ldots, a_n) = a$ whenever for some $d \in D$ such that $a_1 \in |A_d|_{s_1}, \ldots, a_n \in |A_d|_{s_n}$ we have $f_{A_d}(a_1, \ldots, n_n) = a$ — this is well-defined since $\mathcal{D}$ is directed and $\mathcal{A}$ is inclusive.

- for any inclusive $(\Sigma,\mathcal{D})$-homomorphism $h = \langle h_d{:}\, A_d \to B_d\rangle_{d\in D}{:}\, \mathcal{A} \to \mathcal{B}$ between inclusive $(\Sigma,\mathcal{D})$-algebras $\mathcal{A} = \langle A_d\rangle_{d\in D}$ and $\mathcal{B} = \langle B_d\rangle_{d\in D}$, $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}) = \bigcup_{d\in D} h_d{:}\, \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}) \to \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{B})$ — again, this is a well-defined Σ-homomorphism since $\mathcal{D}$ is directed and h is inclusive.

As a special case, for signatures $\Sigma = \langle S,\emptyset\rangle$, we get functors $\mathsf{T}_{S,\mathcal{D}}{:}\, \mathbf{ISet}^S(\mathcal{D}) \to \mathbf{Set}^S$.

For any inclusive $(\Sigma,\mathcal{D})$-algebra $\mathcal{A} = \langle A_d\rangle_{d\in D}$, its components A_d, $d \in D$, are subalgebras of $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A})$. Moreover, inclusive $(\Sigma,\mathcal{D})$-homomorphisms $h, h'{:}\, \mathcal{A} \to \mathcal{B}$ coincide whenever $\mathsf{T}_{\Sigma,\mathcal{D}}(h) = \mathsf{T}_{\Sigma,\mathcal{D}}(h')$.

For any signature Σ, directed hierarchy $\mathcal{D} = \langle D,\leq\rangle$ and set Φ of $(\Sigma,\mathcal{D})$-classification statements, we show a construction of coproducts and coequalisers in $\mathbf{IAlg}(\Sigma,\mathcal{D},\Phi)$, thus showing that it is cocomplete.

coequalisers: Consider inclusive $(\Sigma,\mathcal{D})$-algebras $\mathcal{A} = \langle A_d\rangle_{d\in D}$ and $\mathcal{B} = \langle B_d\rangle_{d\in D}$ and two inclusive $(\Sigma,\mathcal{D})$-homomorphisms $h, h'{:}\, \mathcal{A} \to \mathcal{B}$, where $h = \langle h_d{:}\, A_d \to B_d\rangle_{d\in D}$ and $h' = \langle h'_d{:}\, A_d \to B_d\rangle_{d\in D}$.
Let $c{:}\, \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{B}) \to C$ be the coequaliser of $\mathsf{T}_{\Sigma,\mathcal{D}}(h), \mathsf{T}_{\Sigma,\mathcal{D}}(h'){:}\, \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}) \to \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{B})$ in $\mathbf{Alg}(\Sigma)$ given by the standard quotient construction (as presented at the course).
For $d \in D$, let now $C_d = c(B_d)$ be the subalgebra of C that is the image of B_d w.r.t. Σ-homomorphism c. Then $\mathcal{C} = \langle C_d\rangle_{d\in D}$ is an inclusive $(\Sigma,\mathcal{D})$-algebra, and c "lifts" to an inclusive $(\Sigma,\mathcal{D})$-homomorphism $\widetilde{c} = \langle c_d\rangle_{d\in D}{:}\, \mathcal{C} \to \mathcal{C}$, where $c_d{:}\, B_d \to C_d$ is the restriction of c to B_d ($\widetilde{c}{:}\, \mathcal{B} \to \mathcal{C}$ is the unique inclusive $(\Sigma,\mathcal{D})$-homomorphism such that $\mathsf{T}_{\Sigma,\mathcal{D}}(\widetilde{c}) = c$). Moreover, $h;\widetilde{c} = h';\widetilde{c}$.
Now, given any inclusive $(\Sigma,\mathcal{D})$-homomorphism $f{:}\, \mathcal{B} \to \mathcal{C}'$, where $\mathcal{C}' = \langle C'_d\rangle_{d\in D}$, such that $h;f = h';f$, we have a unique Σ-homomorphism $g{:}\, C \to \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{C}')$ such that $c;g = \mathsf{T}_{\Sigma,\mathcal{D}}(f)$, which "lifts" to a unique inclusive $(\Sigma,\mathcal{D})$-homomorphism $\widetilde{g} = \langle g_d{:}\, C_d \to C'_d\rangle_{d\in D}{:}\, \mathcal{C} \to \mathcal{C}'$, where $g_d{:}\, C_d \to C'_d$ is the restriction of g to C_d. We have then $\widetilde{c};\widetilde{g} = f$, and $\widetilde{g}$ is the only inclusive $(\Sigma,\mathcal{D})$-homomorphism with this property.
Moreover, if $\mathcal{B}$ satisfies a $(\Sigma,\mathcal{D})$-classification statement then so does $\mathcal{C}$.
Thus, $\widetilde{c}{:}\, \mathcal{B} \to \mathcal{C}$ is a coequaliser of $h, h'{:}\, \mathcal{A} \to \mathcal{B}$ in $\mathbf{HAlg}(\Sigma,\mathcal{D},\Phi)$.

coproducts: Consider inclusive $(\Sigma,\mathcal{D})$-algebras $\mathcal{A}_j = \langle A_{j,d}\rangle_{d\in D}$, $j \in J$.
Let C be the coproduct in $\mathbf{Alg}(\Sigma)$ of $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}_j)$, $j \in J$, with injections $\iota_j{:}\, \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}_j) \to C$, given as the quotient by the appropriate congruence of the algebra of terms with the elements of the carriers of $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A}_j)$, $j \in J$, as "variables" (as sketched during the course) .
For $d \in D$, let $Z_d \subseteq |C|$ be the (S-sorted) set of the elements of C that are values of the terms in the classifications statements in Φ that classify them to d:

$$Z_d = \{t_C[v] \mid d' \leq d, \forall Y.t{:}d' \in \Phi, v{:}\, Y \to |C|\}$$

Let $C_d = \langle C\rangle_{Z_d \cup \bigcup_{j\in J} \iota_j(A_{j,d})}$ be the least subalgebra of C that contains Z_d as well as the union of all the images of the appropriate component algebra in $\mathcal{A}_j$, $j \in J$. Clearly, for $d \leq d'$, C_d is a subalgebra of $C_{d'}$, and so $\mathcal{C} = \langle C_d\rangle_{d\in D}$ is an inclusive $(\Sigma,\mathcal{D})$-algebra, $\mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{C}) = C$, and for $j \in J$, we have inclusive $(\Sigma,\mathcal{D})$-homomorphism $\widetilde{\iota_j} = \langle \iota_{j,d}{:}\, A_{j,d} \to C_d\rangle_{d\in D}{:}\, \mathcal{A}_j \to \mathcal{C}$ such that $\mathsf{T}_{\Sigma,\mathcal{D}}(\widetilde{\iota_j}) = \iota_j$. Moreover, $\mathcal{C} \models \Phi$.
Now, given any inclusive $(\Sigma,\mathcal{D},\Phi)$-algebra $\mathcal{C}' = \langle C'_d\rangle_{d\in D}$ with inclusive $(\Sigma,\mathcal{D})$-homomorphisms $f_j{:}\, \mathcal{A}_J \to \mathcal{C}'$, $f_j = \langle f_{j,d}{:}\, A_{j,d} \to C'_d\rangle_{d\in D}$, we have a unique Σ-homomorphism $h{:}\, C \to \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{C}')$ such that $\iota_j;h = \mathsf{T}_{\Sigma,\mathcal{D}}(f_j)$. Since for $d \in D$, $h(Z_d) \subseteq |C'_d|$ (because $\mathcal{C}' \models \Phi$) and $h(\iota_j(|A_{j,d}|)) = \mathsf{T}_{\Sigma,\mathcal{D}}(f_j)(|A_{j,d}|) = f_{j,d}(|A_{j,d}|) \subseteq |C'_d|$, we have $h(|C_d|) \subseteq |C'_d|$. Thus, we get an inclusive $(\Sigma,\mathcal{D})$-homomorphism $\widetilde{h} = \langle h_d{:}\, C_d \to C'_d\rangle_{d\in D}{:}\, \mathcal{C} \to \mathcal{C}'$, where h_d restricts h to C_d, $d \in D$, with $\widetilde{\iota_j};\widetilde{h} = f_j$, for all $j \in J$. Moreover, $\widetilde{h}$ is unique with this property (otherwise h would not be unique either).

YES: 3.{a,b}.CC

Products and equalisers in $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$ can be built component-wise:

equalisers: Consider inclusive $(\Sigma, \mathcal{D})$-algebras $\mathcal{A} = \langle A_d \rangle_{d \in D}$ and $\mathcal{B} = \langle B_d \rangle_{d \in D}$ satisfying Φ and two inclusive $(\Sigma, \mathcal{D})$-homomorphisms $h, h' : \mathcal{A} \to \mathcal{B}$, where $h = \langle h_d : A_d \to B_d \rangle_{d \in D}$ and $h' = \langle h'_d : A_d \to B_d \rangle_{d \in D}$. For $d \in D$, let E_d with inclusion $e_d : E_d \to A_d$ be the largest subalgebra of A_d on which h_d and h'_d coincide. Then $\mathcal{E} = \langle E_d \rangle_{d \in D}$ is an inclusive $(\Sigma, \mathcal{D})$-algebra, which clearly satisfies Φ when $\mathcal{A}$ does so, and $e = \langle e_d : E_d \to A_d \rangle_{d \in D}$ is an inclusive $(\Sigma, \mathcal{D})$-homomorphisms which is an equaliser of h and h' in $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$.

products: Consider inclusive $(\Sigma, \mathcal{D})$-algebras $\mathcal{A}_j = \langle A_{j,d} \rangle_{d \in D}$ satisfying Φ, $j \in J$. For $d \in D$, let $P_d = \Pi_{j \in J} A_{j,d}$ be the usual Cartesian product of Σ-algebras $A_{j,d}$, $j \in J$, with projections $\pi_{j,d} : P_d \to A_{j,d}$, $j \in J$. Then $\mathcal{P} = \langle P_d \rangle_{d \in D}$ is an inclusive $(\Sigma, \mathcal{D})$-algebra, which clearly satisfies Φ when all $\mathcal{A}_j$, $j \in J$, do so, and $\mathcal{P}$ with inclusive $(\Sigma, \mathcal{D})$-homomorphisms $\widetilde{\pi_j} = \langle \pi_{j,d} \rangle_{d \in D} : \mathcal{P} \to \mathcal{A}_j$, $j \in J$, is a product of $\mathcal{A}_j$, $j \in J$, in $\mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi)$.

YES: 3.{a,b}.C

We show now that for any $\mathcal{D}$-inclusive S-sorted set $\mathcal{X} = \langle X_d \rangle_{d \in D}$, there is an inclusive $(\Sigma, \mathcal{D})$-algebra satisfying Φ that is free over $\mathcal{X}$ w.r.t. $\mathcal{J}_{\Sigma,\mathcal{D},\Phi} : \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{ISet}^S(\mathcal{D})$.

Let $X = \mathsf{T}_{S,\mathcal{D}}(\mathcal{X}) = \bigcup_{d \in D} X_d$; consider the usual term algebra $T_\Sigma(X)$. Similarly as in the construction of coproducts, define:

$$Z_d = \{ t_{T_\Sigma(X)}[v] \mid d' \leq d, \forall Y.t{:}d' \in \Phi, v : Y \to |T_\Sigma(X)| \}$$

For $d \in D$, let $T_d = \langle T_\Sigma(X) \rangle_{Z_d \cup X_d}$ be the least subalgebra of the term algebra that contains all the variables in X_d and all the terms that the classification statements in Φ classify to d (resulting by substituting arbitrary terms for variables in the terms that occur in the appropriate classification statements). Clearly, we get an inclusive $(\Sigma, \mathcal{D})$-algebra $\mathcal{T} = \langle T_d \rangle_{d \in D}$, which satisfies Φ, and a family of inclusions $\iota_d : X_d \to |T_d|$ which form a $\mathcal{D}$-inclusive S-sorted function $\iota = \langle \iota_d \rangle_{d \in D} : \mathcal{X} \to \mathcal{J}_{\Sigma,\mathcal{D},\Phi}(\mathcal{T})$.

Moreover, for any inclusive $(\Sigma, \mathcal{D})$-algebra $\mathcal{A} = \langle A_d \rangle_{d \in D}$ satisfying Φ and $\mathcal{D}$-inclusive S-sorted function $f = \langle f_d \rangle_{d \in D} : \mathcal{X} \to \mathcal{J}_{\Sigma,\mathcal{D},\Phi}(\mathcal{A})$, there is a unique Σ-homomorphism $h : T_\Sigma(X) \to \mathsf{T}_{\Sigma,\mathcal{D}}(\mathcal{A})$ such that $\mathsf{T}_{S,\mathcal{D}}(\iota);h = \mathsf{T}_{S,\mathcal{D}}(f)$. For each $d \in D$ it follows now that $h(X_d) = f_d(X_d) \subseteq |A_d|$ and since $\mathcal{A} \models \Phi$, also $h(Z_d) \subseteq |A_d|$. Consequently, $h(|T_d|) \subseteq |A_d|$, and the family $h_d : T_d \to A_d$ of restrictions of h to T_d, $\widetilde{h} = \langle h_d \rangle_{d \in D} : \mathcal{T} \to \mathcal{A}$, is an inclusive $(\Sigma, \mathcal{D})$-homomorphism such that $\iota;\mathcal{J}_{\Sigma,\mathcal{D},\Phi}(\widetilde{h}) = f$. Moreover, $\widetilde{h}$ is the only inclusive $(\Sigma, \mathcal{D})$-homomorphism with this property, which shows that $\mathcal{T}$ with unit $\iota : \mathcal{X} \to \mathcal{J}_{\Sigma,\mathcal{D},\Phi}(\mathcal{T})$ is free over $\mathcal{X}$ w.r.t. $\mathcal{J}_{\Sigma,\mathcal{D},\Phi} : \mathbf{IAlg}(\Sigma, \mathcal{D}, \Phi) \to \mathbf{ISet}^S(\mathcal{D})$.

YES: 4.{a,b}

For a signature with no operations $\mathcal{U}_{\langle S,\emptyset \rangle,\mathcal{D}} = \mathsf{T}_{S,\mathcal{D}} : \mathbf{ISet}^S(\mathcal{D}) \to \mathbf{Set}^S$ on any object $\mathcal{X} = \langle X_d \rangle_{d \in D}$ is given by $\mathsf{T}_{S,\mathcal{D}}(\mathcal{X}) = \bigcup_{d \in D} X_d$. Now, by the construction above, a product of a family $\mathcal{X}_j = \langle X_{j,d} \rangle_{d \in D}$, $j \in J$, in $\mathbf{ISet}^S(\mathcal{D})$ is $\mathcal{P} = \langle \Pi_{j \in J} X_{j,d} \rangle_{d \in D}$, and $\mathsf{T}_{S,\mathcal{D}}(\mathcal{P}) = \bigcup_{d \in D} \Pi_{j \in J} X_{j,d}$ is in general properly smaller than $\Pi_{j \in J} \mathsf{T}_{S,\mathcal{D}}(\mathcal{X}_j) = \Pi_{j \in J}(\bigcup_{d \in D} X_{j,d})$.[1] This shows that functors $\mathsf{T}_{S,\mathcal{D}} : \mathbf{ISet}^S(\mathcal{D}) \to \mathbf{Set}^S$ need not be continuous, and so in general do not have left adjoints.

NO: 4.{c,d}

□

[1] This may happen for infinite J even when $\mathcal{D}$ is directed and $\mathcal{X}_j$, $j \in J$, are inclusive, as it is the case here: let $\mathcal{D} = \langle \mathbb{N}, \leq \rangle$, $J = \mathbb{N}$ and $X_{j,n} = \{0, \ldots, n\}$, $j, n \in \mathbb{N}$. Then $\bigcup_{n \in \mathbb{N}} \Pi_{j \in \mathbb{N}} X_{j,n}$ is countable, while $\Pi_{j \in \mathbb{N}}(\bigcup_{n \in \mathbb{N}} X_{j,n})$ is not.